Solved Examples for Chapter 9

Example for Section 9.1

Suppose that England, France, and Spain produce all the wheat, barley, and oats in the world. The world demand for wheat requires 125 million acres of land devoted to wheat production. Similarly, 60 million acres of land are required for barley and 75 million acres of land for oats. The total amount of land available for these purposes in England, France, and Spain is 70 million acres, 110 million acres, and 80 million acres, respectively. The number of hours of labor needed in England, France and Spain to produce an acre of wheat is 18, 13, and 16, respectively. The number of hours of labor needed in England, France, and Spain to produce an acre of barley is 15, 12, and 12, respectively. The number of hours of labor needed in England, France, and Spain to produce an acre of oats is 12, 10, and 16, respectively. The labor cost per hour in producing wheat is $9.00, $7.20, and $9.90 in England, France, and Spain, respectively. The labor cost per hour in producing barley is $8.10, $9.00, and $8.40 in England, France, and Spain respectively. The labor cost per hour in producing oats is $6.90, $7.50, and $6.30 in England, France, and Spain, respectively. The problem is to allocate land use in each country so as to meet the world food requirement and minimize the total labor cost.

(a) Formulate this problem as a transportation problem by constructing the appropriate parameter table.

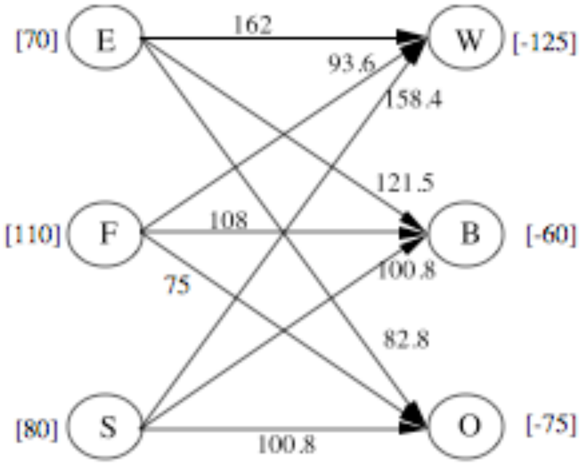
Let England, France, and Spain be the three sources, where their supplies are the millions of acres of land that are available for growing these crops. Let Wheat, Barley, and Oats be the three destinations, where their demands are the millions of acres of land that are needed to fulfill the world demand for these respective crops. The unit cost (in millions of dollars) is the labor cost per million acres, so the number of hours of labor needed is multiplied by the cost per hour. The parameter table is as follows.

Unit Cost ($ million)

		Destination			
		Wheat	Barley	Oats	Supply
	England	162	121.5	82.8	70
Source	France	93.6	108	75	110
	Spain	158.4	100.8	100.8	80
Demand		125	60	75	

(b) Draw the network representation of this problem.

The network presentation of this problem is given below.



(c) Obtain an optimal solution.
We can use Solver to solve this problem and obtain the following solution.

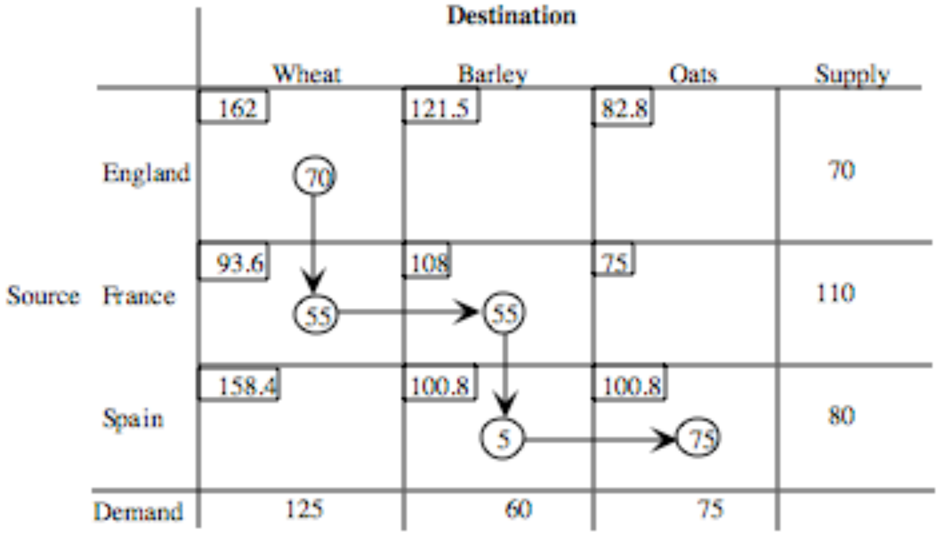
Allocation Quantities

			Destination				
		Wheat	Barley	Oats	Totals		Supply
	England	0	0	70	70	=	70
Source	France	110	0	0	110	=	110
	Spain	15	60	5	80	=	80
Totals		125	60	75			
		=	=	=	Total cost = $25.02 billion		
Demand		125	60	75			

Example for Section 9.2

Reconsider the problem in the preceding example. Starting with the northwest corner rule, interactively apply the transportation simplex method to obtain an optimal solution for this problem

For this problem, the initial BF solution obtained by the northwest corner rule is shown below.



Optimality Test:

Since $c_{ij} - u_i - v_j = 0$ if x_{ij} is a basic variable,

$c_{ij} = u_i + v_j$ for each (i, j) such that x_{ij} is basic.

Because the number of unknowns (the u_i and v_j) exceed the number of these equations by one, we can set one unknown equal to an arbitrary value, say 0. These equations can then be solved as outlined below.

x_{21}: $93.6 = u_2 + v_1$. Set $u_2 = 0$, so $v_1 = 93.6$,

x_{22}: $108 = u_2 + v_2$. $v_2 = 108$.

x_{11}: $162 = u_1 + v_1$. Know $v_1 = 93.6$, so $u_1 = 68.4$.

x_{32}: $100.8 = u_3 + v_2$. Know $v_2 = 108$, so $u_3 = -7.2$.

x_{33}: $100.8 = u_3 + v_3$. Know $u_3 = -7.2$, so $v_3 = 108$.

Since $c_{ij} - u_i - v_j$ represents the rate at which the objective function will change as a nonbasic variable x_{ij} is increased, we now can check whether increasing any nonbasic variable will decrease the total cost Z.

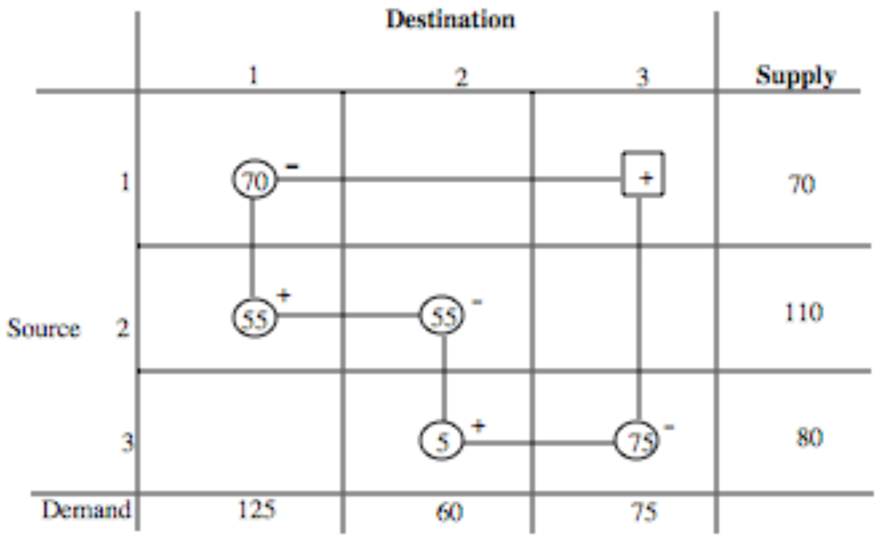
Nonbasic variable	$c_{ij} - u_i - v_j$
x_{12}	121.5 - 68.4 - 108 = -54.9
x_{13}	82.8 - 68.4 - 108 = -93.6
x_{23}	75 - 0 - 108 = -33
x_{31}	158.4 -(-7.2) - 93.6 = 72

Because some of these ($c_{ij} - u_{ij} - v_j$) values are negative, the initial BF solution is not optimal.

Iteration 1:

We select the nonbasic variable x_{13} to be the entering basic variable because it has the largest negative value of ($c_{ij} - u_i - v_j$).

When x_{13} is increased from 0 by any particular amount, a chain reaction is set off that requires alternately decreasing and increasing current basic variables by the same amount in order to continue satisfying the supply and demand constraints. This chain reaction is depicted in the next figure, where the + sign inside a box in cell (1, 3) indicates that the entering basic variable is being increased there and the + or - sign next to other circles indicate that a basic variable is being increased or decreased there.



Each *donor cell* (indicated by a minus sign) decreases its allocation by exactly the same amount as the entering basic variable and each *recipient cell* (indicated by a plus sign) is increased. The entering basic variable will be increased as far as possible until the allocation for one of the donor cells drops all the way down to 0. Since the original allocations for the donor cells are

$$x_{11} = 70, \qquad x_{22} = 55, \qquad x_{33} = 75,$$

x_{22} will be the one that drops to 0 as x_{13} is increased (by 55). Therefore, x_{22} is the leaving basic variable.

Since each of the basic variables is being increased or decreased by 55, the values of the basic variables in the new BF solution are

$$x_{11} = 15, \qquad x_{13} = 55, \qquad x_{21} = 110, \qquad x_{32} = 60, \qquad x_{33} = 20.$$

Optimality Test After Iteration 1:

Since Source 1 now has two basic variables (tied for the maximum number), let us set $u_1 = 0$ this time. The $c_{ij} = u_i + v_j$ equations then would be solved as follows.

x_{11}:	$162 = u_1 + v_1$.	Set $u_1 = 0$,	so v_1	$= 162$,
x_{13}:	$82.8 = u_1 + v_3$.		v_3	$= 82.8$.
x_{21}:	$93.6 = u_2 + v_1$.	Know $v_1 = 162$,	so u_2	$= -68.4$.
x_{33}:	$100.8 = u_3 + v_3$.	Know $v_3 = 82.8$,	so u_3	$= 18$.
x_{32}:	$100.8 = u_3 + v_2$.	Know $u_3 = 18$,	so v_2	$= 82.8$.

We next calculate $(c_{ij} - u_i - v_j)$ for the nonbasic variables.

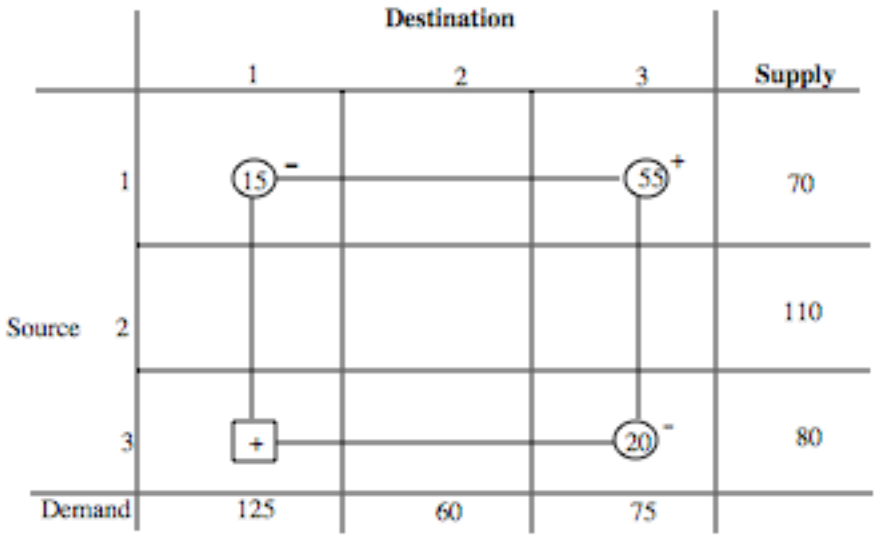
Nonbasic variable	$c_{ij} - u_i - v_j$
x_{12}	$121.5 - 0 \quad - 82.8 \quad = \quad 38.7$
x_{22}	$108 - (-68.4) - 82.8 = \quad 93.6$
x_{23}	$75 - (-68.4) - 82.8 \quad = \quad 60.6$
x_{31}	$158.4 - 18 - 162 \quad = -21.6$

We still have one negative value of (c_{ij} - u_i - v_j), so the current BF solution is not optimal.

Iteration 2:

Since x_{31} is the one nonbasic variable with a negative value of (c_{ij} - u_i - v_j), x_{31} becomes the entering basic variable.

The resulting chain reaction is depicted next.



The donor cells have allocations of $x_{11} = 15$ and $x_{33} = 20$. Because $15 < 20$, the leaving basic variable is x_{11}.

Since the basic variables x_{21} and x_{32} were not part of this chain reaction, their values do not change. However, x_{31} and x_{13} increase by 15 while x_{11} and x_{33} decrease by 15. Therefore, the values of the basic variables in the new BF solution are

$$x_{13} = 70, \qquad x_{21} = 110, \qquad x_{31} = 15, \qquad x_{32} = 60, \qquad x_{33} = 5$$

Optimality Test After Iteration 2:

Because Source 3 now has the largest number of basic variables, we set $u_3 = 0$ this time. The resulting calculations are shown below.

x_{31}: $158.4 = u_3 + v_1$. Set $u_3 = 0$, so $v_1 = 158.4$,

x_{32}: $100.8 = u_3 + v_2$. $v_2 = 100.8$.

x_{33}: $100.8 = u_3 + v_3$. $v_3 = 100.8$.

x_{13}: $82.8 = u_1 + v_3$. Know $v_3 = 100.8$, so $u_1 = -18$.

x_{21}: $93.6 = u_2 + v_1$. Know $v_1 = 158.4$, so $u_2 = -64.8$.

Nonbasic variable	$c_{ij} - u_i - v_j$
x_{11}	162 - (-18) - 158.4 = 21.6
x_{12}	121.5 - (-18) - 100.8 = 38.7
x_{22}	108 - (-68.4) - 100.8 = 72
x_{23}	75 - (-64.8) - 100.8 = 39

Since all of these values of ($c_{ij} - u_i - v_j$) are nonnegative, the current BF solution is optimal.

Thus, the optimal allocation of land to crops is

70 million acres in England for oats,
110 million acres in France for wheat,
15 million acres in Spain for wheat,
60 million acres in Spain for barley,
5 million acres in Spain for oats.

The total cost of this grand enterprise would be

$Z = \$25.02$ billion.

Example for Section 9.3

A contractor, Susan Meyer, has to haul gravel to three building sites. She can purchase as much as 18 tons at a gravel pit in the north of the city and 14 tons at one in the south. She needs 10, 5, and 10 tons at sites 1, 2, and 3, respectively. The purchase price per ton at each gravel pit and the hauling cost per ton are given in the table below. Susan wishes to determine how much to haul from each pit to each site to minimize the total cost for purchasing and hauling gravel.

Pit	Hauling Cost per Ton at Site			Price per Ton
	1	2	3	
North	$30	$60	$50	$100
South	$60	$30	$40	$120

Now suppose that trucks (and their drivers) need to be hired to do the hauling, where each truck can only be used to haul gravel from a single pit to a single site. Each truck can haul 5 tons, and the cost per truck is five times the hauling cost per ton given above. Only full trucks would be used to supply each site.

(a) Formulate this problem as an assignment problem by constructing the appropriate cost table, including identifying the assignees and tasks.

The tasks are the loads needed at sites 1, 2, and 3. The assignees are the three trucks from the North pit and the two trucks from the South pit. Considering both the purchase price for the gravel and the hauling cost per truck, the cost table is constructed as follows.

		Task (Site)				
		1a	1b	2	3a	3b
	North 1	650	650	800	750	750
	North 2	650	650	800	750	750
Assignee	North 3	650	650	800	750	750
	South 1	900	900	750	800	800
	South 2	900	900	750	800	800

(b) Obtain an optimal solution.

We use Solver to obtain the following optimal solution with a minimum cost of $3500.

		Task (Site)				
		1a	1b	2	3a	3b
	North 1	X				
	North 2		X			
Assignee	North 3				X	
	South 1			X		
	South 2					X

(c) Reformulate this assignment problem as an equivalent transportation problem with two sources and three destinations by constructing the appropriate parameter table.

The parameter table for the formulation as an equivalent transportation problem is given below.

		Destination			
		1	2	3	Supply
Source	North	650	800	750	3
	South	650	800	750	2
	Demand	2	1	2	

(d) Obtain an optimal solution for the problem as formulated in part (c).

We use Solver to obtain the following optimal solution with a minimum cost of $3500.

		Destination			
		1	2	3	Supply
Source	North	2		1	3
	South		1	1	2
	Demand	2	1	2	