Solved Examples for Chapter 8

Example for Section 8.1

Consider the following linear programming problem.

Maximize $Z = -5\, x_1 + 5\, x_2 + 13\, x_3$,

subject to

$$-1\, x_1 + 1\, x_2 + 3\, x_3 \leq 20$$
$$12\, x_1 + 4\, x_2 + 10\, x_3 \leq 90$$

and

$$x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0.$$

After introducing x_4 and x_5 as the slack variables for the respective constraints and then applying the simplex method, the final simplex tableau is

Basic Variable	Eq	Coefficient of: Z	x_1	x_2	x_3	x_4	x_5	Right Side
Z	(0)	1	0	0	2	5	0	100
x_2	(1)	0	-1	1	3	1	0	20
x_5	(2)	0	16	0	-2	-4	1	10

Now suppose that the right-hand side of the second constraint is changed from $b_2 = 90$ to $b_2 = 70$. Using the sensitivity analysis procedure described in Sec. 7.2 for Case 1, the only resulting change in the above final simplex tableau is that the Right Side entry for Eq. (2) changes from 10 to - 10. This revised tableau is shown below, labeled as Iteration 0 (for the dual simplex method).

Application of the Dual Simplex Method

Because $x_5 = -10 < 0$, our basic solution that was optimal is no longer feasible. However, since all the coefficients in Eq. (0) still are nonnegative, we can quickly reoptimize by applying the dual simplex method, starting with the Iteration 0 tableau shown below.

Since x_5 is the only negative variable in this tableau, it is chosen as the leaving basic variable for the first iteration of the dual simplex method.

To select the entering basic variable, we consider x_3 and x_4, since they are the only nonbasic variables that have negative coefficients in Eq. (2). Taking the absolute values of the ratios of these coefficients to the corresponding coefficients in E. (0),

$$\frac{2}{2} < \frac{5}{4},$$

so x_3 is selected as the entering basic variable.

To use Gaussian elimination to solve for the new basic solution, we divide Eq. (2) by (-2) and then subtract 2 times this new Eq. (2) from Eq. (0) and also subtract 3 times this new Eq. (2) from Eq. (1). This yields the Iteration 1 tableau shown below.

The corresponding basic solution is $x_1 = 0$, $x_2 = 5$, $x_3 = 5$, $x_4 = 0$, $x_5 = 0$, with $Z = 90$, which is feasible and therefore optimal.

Iteration	Basic Variable	Eq	Coefficient of:						Right Side
			Z	x_1	x_2	x_3	x_4	x_5	
0	Z	(0)	1	0	0	2	5	0	100
	x_2	(1)	0	-1	1	3	1	0	20
	x_5	(2)	0	16	0	-2	-4	1	-10
1	Z	(0)	1	16	0	0	1	1	90
	x_2	(1)	0	-23	1	0	-5	3/2	5
	x_3	(2)	0	-8	0	1	2	-1/2	5

Example for Section 8.2

Consider the following problem (previously analyzed in the preceding example).

Maximize $\quad Z = -5x_1 + 5x_2 + 13x_3,$

subject to

$$-1x_1 + 1x_2 + 3x_3 \leq 20$$
$$12x_1 + 4x_2 + 10x_3 \leq 90$$

and

$$x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0.$$

Application of Parametric Linear Programming

Suppose that we now want to apply parametric linear programming analysis to this problem by changing the right-hand sides of the functional constraints to

$20 + 2\theta$ (for constraint 1)

and

$90 - \theta$ (for constraint 2),

where θ can be varied over the range $0 \le \theta \le 20$.

To start, we apply the simplex method to solve the problem with $\theta = 0$. Letting x_4 and x_5 be the slack variables for the respective constraints, the resulting final simplex tableau is the first one shown below (when setting $\theta = 0$). (This same tableau also is shown at the beginning of the preceding example.)

Next, we introduce θ into the problem by using the sensitivity analysis procedure for Case 1 presented in Sec. 7.2. Thus, the only changes in the tableau just obtained are that the Right Side values become

$$Z^* = \mathbf{y}^* \bar{\mathbf{b}} = [5 \quad 0]\begin{bmatrix} 20+2\theta \\ 90-\theta \end{bmatrix} = 100 + 10\theta,$$

$$\mathbf{b}^* = \mathbf{S}^* \bar{\mathbf{b}} = \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix}\begin{bmatrix} 20+2\theta \\ 90-\theta \end{bmatrix} = \begin{bmatrix} 20+2\theta \\ 10-9\theta \end{bmatrix},$$

as shown in the first tableau below. When θ is increased from 0, the basic solution given in this tableau (basic variables $x_2 = 20 + 2\theta$ and $x_5 = 10 - 9\theta$) will remain feasible, and therefore optimal, as long as $20 + 2\theta \ge 0$ and $10 - 9\theta \ge 0$. These inequalities hold for $0 \le \theta \le 10/9$.

If $\theta > 10/9$, then $x_5 < 0$, so the dual simplex method needs to be applied (as illustrated in the preceding example) with x_5 as the leaving basic variable. The entering basic variable is x_3 ($2/2 < 5/4$), which leads to the second tableau below. For $10/9 \le \theta \le 70/23$, the optimal basic solution $x_1 = 0$, $x_2 = 35-(23/2)\theta$, $x_3 = -5+(9/2)\theta$, $x_4 = 0$, $x_5 = 0$ with $Z(\theta) = 110+\theta$.

If $70/23 < \theta \le 20$, then $x_2 < 0$ and it will become the leaving basic variable. The entering basic variable is x_4 ($16/23 > 1/5$), which leads to the third tableau below. For $70/23 \le \theta \le 20$, the optimal basic solution is $x_1 = 0$, $x_2 = 0$, $x_3 = 9-(1/10)\theta$, $x_4 = -7 + (23/10)\theta$, $x_5 = 0$ with $Z(\theta) = 117-(13/10)\theta$.

Range of θ	Basic Variable	Eq	Coefficient of:						Right Side
			Z	x_1	x_2	x_3	x_4	x_5	
$0 \le \theta \le 10/9$	$Z(\theta)$	(0)	1	0	0	2	5	0	$100+10\theta$
	x_2	(1)	0	-1	1	3	1	0	$20+2\theta$
	x_5	(2)	0	16	0	-2	-4	1	$10-9\theta$
$10/9 \le \theta \le 70/23$	$Z(\theta)$	(0)	1	16	0	0	1	1	$110+\theta$
	x_2	(1)	0	-23	1	0	-5	3/2	$35-(23/2)\theta$
	x_3	(2)	0	-8	0	1	2	-1/2	$-5+(9/2)\theta$
$70/23 \le \theta \le 20$	$Z(\theta)$	(0)	1	103/5	1/5	0	0	13/10	$117-(13/10)\theta$
	x_4	(1)	0	-23/5	-1/5	0	1	-3/10	$-7+(23/10)\theta$
	x_3	(2)	0	6/5	2/5	1	0	1/10	$9-(1/10)\theta$

Example for Section 8.3

We will illustrate the upper bound technique by applying it to solve the Wyndor Glass Co. problem presented in Sec. 3.1.

Recall that the model for the Wyndor Glass Co. problem is

Maximize $Z = 3x_1 + 5x_2$,

subject to

$$1x_1 \le 4$$
$$2x_2 \le 12$$
$$3x_1 + 2x_2 \le 18$$

and

$$x_1 \ge 0, \quad x_2 \ge 0.$$

Since the second constraint is equivalent to $x_2 \le 6$, we can rewrite this problem as an equivalent linear programming problem with one functional constraint and two upper bound constraints:

Maximize $Z = 3x_1 + 5x_2$,

subject to

$$3x_1 + 2x_2 \le 18$$
$$0 \le x_1 \le 4$$

$0 \leq x_2 \leq 6.$

Let x_3 be the slack variable for the functional constraint and also define new variables,

$y_1 = 4 - x_1 \geq 0$ and $y_2 = 6 - x_2 \geq 0.$

Application of the Upper Bound Technique

With x_3 being the basic variable and x_1 and x_2 being nonbasic, the initial simplex tableau gives the first set of equations, labeled as iteration 0, shown below.

Iteration	Basic Variable	Eq	Coefficient of: Z	x_1	x_2	x_3	Right Side
0	Z	(0)	1	-3	-5	0	0
	x_3	(1)	0	3	2	1	18

Since there are negative coefficients in Eq. (0), this basic solution is not optimal. We choose x_2 as the entering basic variable (-5 < -3). From Eq. (1), $x_2 \leq 9$, and from the upper bound constraint, $x_2 \leq 6$. Thus, the smallest maximum feasible value of x_2 is 6. Because x_2 reaches its upper bound, replace x_2 by

$y_2 = 6 - x_2,$

so that $y_2 = 0$ becomes the new nonbasic variable and x_3 remains as a basic variable. This leads to the following simplex tableau.

Iteration	Basic Variable	Eq	Coefficient of: Z	x_1	y_2	x_3	Right Side
1	Z	(0)	1	-3	5	0	30
	x_3	(1)	0	3	-2	1	6

Since the coefficient of x_1 in Eq. (0) is negative, we choose x_1 as the entering basic variable. From Eq. (1), $x_1 \leq 2$, and from the upper bound constraint, $x_1 \leq 4$. Thus, the smallest maximum feasible value of x_1 is 2. Because x_1 does not reach its

upper bound, we still use x_1 as a basic variable, and then x_3 becomes the leaving basic variable in the usual way. This leads to the optimal simplex tableau shown below. The optimal solution is $x_1 = 2$, $y_2 = 0$ (so $x_2 = 6$) with $Z = 36$.

Iteration	**Basic Variable**	**Eq**	**Coefficient of:**				**Right Side**
			Z	**x_1**	**y_2**	**x_3**	
2	Z	(0)	1	3	1	0	36
	x_1	(1)	0	1	-2/3	1/3	2