Solved Examples for Chapter 7

Example for Section 7.2

Consider the following problem.

Maximize $Z = 2x_1 - 1x_2 + 1x_3$,

subject to

$$3x_1 + 1x_2 + 1x_3 \leq 60$$
$$1x_1 - 1x_2 + 2x_3 \leq 10$$
$$1x_1 + 1x_2 - 1x_3 \leq 20$$

and

$$x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0.$$

Let x_4, x_5 and x_6 denote the slack variables for the respective constraints. The final simplex tableau is

Basic Variable	Eq.	Coefficient of: Z	x_1	x_2	x_3	x_4	x_5	x_6	Right Side
Z	(0)	1	0	0	3/2	0	3/2	1/2	25
x_4	(1)	0	0	0	1	1	-1	-2	10
x_1	(2)	0	1	0	1/2	0	1/2	1/2	15
x_2	(3)	0	0	1	-3/2	0	-1/2	1/2	5

Now let us conduct sensitivity analysis by *independently* investigating each of the following six changes in the original model. For each change, we will use the sensitivity analysis procedure to revise this final tableau and (if needed) convert it to proper form from Gaussian elimination for identifying and evaluating the current basic solution. Then we will test this solution for feasibility and for optimality. If either test fails, we also will reoptimize to find a new optimal solution.

(a) Change the right-hand sides

$$\textbf{from} \quad \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 60 \\ 10 \\ 20 \end{bmatrix} \qquad \textbf{to} \qquad \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 70 \\ 20 \\ 10 \end{bmatrix}.$$

$\Delta b = \begin{bmatrix} 10 \\ 10 \\ -10 \end{bmatrix}$. With this change in b, the entries in the right-side column changes to the following values:

$$Z^* = y^* \bar{b} = \begin{bmatrix} 0 & 3/2 & 1/2 \end{bmatrix} \begin{bmatrix} 70 \\ 20 \\ 10 \end{bmatrix} = 35. \qquad S^* \bar{b} = \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1/2 & 1/2 \\ 0 & -1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 70 \\ 20 \\ 10 \end{bmatrix} = \begin{bmatrix} 30 \\ 15 \\ -5 \end{bmatrix}.$$

Therefore, the current (previously optimal) basic solution has become (x_1, x_2, x_3, x_4, x_5, x_6) = (15, -5, 0, 30, 0, 0), which fails the feasibility test. The dual simplex method (described in Sec. 8.1) now can be applied to the revised simplex tableau (the first one shown below) to find the new optimal solution (x_1, x_2, x_3, x_4, x_5, x_6) = (40/3, 0, 10/3, 80/3, 0, 0), as displayed in the second tableau below.

Basic Variable	**Eq**	**Coefficient of:**							**Right Side**
		Z	**x_1**	**x_2**	**x_3**	**x_4**	**x_5**	**x_6**	
Z	(0)	1	0	0	3/2	0	3/2	1/2	35
x_4	(1)	0	0	0	1	1	-1	-2	30
x_1	(2)	0	1	0	1/2	0	1/2	1/2	15
x_2	(3)	0	0	1	-3/2	0	-1/2	1/2	-5
Z	(0)	1	0	1	0	0	1	1	30
x_4	(1)	0	0	2/3	0	1	-4/3	-5/3	80/3
x_1	(2)	0	1	1/3	0	0	1/3	2/3	40/3
x_3	(3)	0	0	-2/3	1	0	- 1/3	-1/3	10/3

(b) Change the coefficients of x_1

from $\begin{bmatrix} c_1 \\ a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 1 \\ 1 \end{bmatrix}$ **to** $\begin{bmatrix} c_1 \\ a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 0 \end{bmatrix}$.

Since the only change is in the coefficients of x_1, we only need to recompute the column corresponding to the basic variable x_1 in the final tableau:

$$z_1 - \bar{c}_1 = y^* \bar{A}_1 - \bar{c}_1 = \begin{bmatrix} 0 & 3/2 & 1/2 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} - 1 = 2.$$

$$A_1^* = S^* \bar{A}_1 = \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1/2 & 1/2 \\ 0 & -1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.$$

The revised final tableau is

Basic Variable	**Eq**	**Coefficient of:**							**Right Side**
		Z	**x_1**	**x_2**	**x_3**	**x_4**	**x_5**	**x_6**	
Z	(0)	1	2	0	3/2	0	3/2	1/2	25
x_4	(1)	0	0	0	1	1	-1	-2	10
x_1	(2)	0	1	0	1/2	0	1/2	1/2	15
x_2	(3)	0	-1	1	-3/2	0	- 1/2	1/2	5

The new column for the basic variable x_1 is not in proper form from Gaussian elimination, so elementary row operations are required to restore proper form. After doing this, the proper form of the above revised tableau is

Basic Variable	**Eq**	**Coefficient of:**							**Right Side**
		Z	**x_1**	**x_2**	**x_3**	**x_4**	**x_5**	**x_6**	
Z	(0)	1	0	0	1/2	0	1/2	-1/2	-5
x_4	(1)	0	0	0	1	1	-1	-2	10
x_1	(2)	0	1	0	1/2	0	1/2	1/2	15
x_2	(3)	0	0	1	-1	0	0	1	20

Because of the negative coefficient for x_6 in row (0), the current (previously optimal) basic solution is feasible but not optimal. We can apply the simplex method to the above tableau in proper form and find the optimal solution $(x_1, x_2, x_3, x_4, x_5, x_6) = (5, 0, 0, 50, 0, 20)$, as displayed below.

Basic Variable	**Eq**	**Coefficient of:**							**Right Side**
		Z	x_1	x_2	x_3	x_4	x_5	x_6	
Z	(0)	1	0	1/2	0	0	1/2	0	5
x_4	(1)	0	0	2	-1	1	-1	0	50
x_1	(2)	0	1	-1/2	1	0	1/2	0	5
x_6	(3)	0	0	1	-1	0	0	1	20

(c) Change the coefficients of *x_3*

from $\begin{bmatrix} c_3 \\ a_{13} \\ a_{23} \\ a_{33} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \\ -1 \end{bmatrix}$ **to** $\begin{bmatrix} c_3 \\ a_{13} \\ a_{23} \\ a_{33} \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 1 \\ -2 \end{bmatrix}$.

With the change in the coefficients of x_3, we only need to recompute the column corresponding to the basic variable x_1 in the final tableau:

$$z_3 - \bar{c}_3 = y^* \bar{A}_3 - \bar{c}_3 = \begin{bmatrix} 0 & 3/2 & 1/2 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix} - 2 = -3/2.$$

$$A_3^* = S^* \bar{A}_3 = \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1/2 & 1/2 \\ 0 & -1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 6 \\ -1/2 \\ -3/2 \end{bmatrix}.$$

The revised final tableau is

Basic Variable	Eq	Coefficient of:							Right Side
		Z	x_1	x_2	X_3	x_4	x_5	x_6	
Z	(0)	1	0	0	-3/2	0	3/2	1/2	25
x_4	(1)	0	0	0	6	1	-1	-2	10
x_1	(2)	0	1	0	-1/2½	0	1/2	1/2	15
x_2	(3)	0	0	1	-3/2	0	- 1/2	1/2	5

The current (previously optimal) basic solution is feasible but not optimal. We can apply the simplex method to the above revised tableau and find the optimal solution $(x_1, x_2, x_3, x_4, x_5, x_6) = (95/6, 15/2, 5/3,0, 0, 0)$, as displayed below.

Basic Variable	Eq	Coefficient of:							Right Side
		Z	x_1	x_2	x_3	x_4	x_5	x_6	
Z	(0)	1	0	0	0	1/4	5/4	0	55/2
x_3	(1)	0	0	0	1	1/6	-1/6	-1/3	5/3
x_1	(2)	0	1	0	0	1/12	5/12	1/3	95/6
x_2	(3)	0	0	1	0	1/4	- 3/4	0	15/2

(d) Change the objective function to $Z = 3x_1 - 2x_2 + 3x_3$.

We only need to recompute the coefficients of x_1, x_2, and x_3 in Eq. (0):

$$z_1 - \bar{c}_1 = y^*A_1 - \bar{c}_1 = \begin{bmatrix} 0 & 3/2 & 1/2 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} - 3 = -1$$

$$z_3 - \bar{c}_2 = y^*A_2 - \bar{c}_2 = \begin{bmatrix} 0 & 3/2 & 1/2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} - (-2) = 1.$$

$$z_3 - \bar{c}_3 = y^*A_3 - \bar{c}_3 = \begin{bmatrix} 0 & 3/2 & 1/2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} - 3 = -1/2.$$

The revised tableau is

Basic Variable	Eq	Coefficient of:							Right Side
		Z	x_1	x_2	x_3	x_4	x_5	x_6	
Z	(0)	1	-1	1	- 1/2	0	3/2	1/2	25
x_4	(1)	0	0	0	1	1	-1	-2	10
x_1	(2)	0	1	0	1/2	0	1/2	1/2	15
x_2	(3)	0	0	1	-3/2	0	-1/2	1/2	5

As in part (b), we need to restore proper form from Gaussian elimination. The proper form of the above revised tableau is

Basic Variable	Eq	Coefficient of:							Right Side
		Z	x_1	x_2	x_3	x_4	x_5	x_6	
Z	(0)	1	0	0	3/2	0	5/2	1/2	35
x_4	(1)	0	0	0	1	1	-1	-2	10
x_1	(2)	0	1	0	1/2	0	1/2	1/2	15
x_2	(3)	0	0	1	-3/2	0	-1/2	1/2	5

The current basic solution is feasible and optimal.

(e) Introduce a new constraint $3x_1 - 2x_2 + x_3 \leq 30$. (Denote its slack variable by x_7).

Adding the new constraint (in augmented form) $3x_1 - 2x_2 + x_3 + x_7 = 30$ to the final tableau:

Basic Variable	Eq	Coefficient of: Z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	Right Side
Z	(0)	1	0	0	3/2	0	3/2	1/2	0	25
x_4	(1)	0	0	0	1	1	-1	-2	0	10
x_1	(2)	0	1	0	1/2	0	1/2	1/2	0	15
x_2	(3)	0	0	1	-3/2	0	-1/2	1/2	0	5
x_7	(4)	0	3	-2	1	0	0	0	1	30

We next need to restore proper form from Gaussian elimination into this new row. The proper form of the above tableau is

Basic Variable	Eq	Coefficient of: Z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	Right Side
Z	(0)	1	0	0	3/2	0	3/2	1/2	0	25
x_4	(1)	0	0	0	1	1	-1	-2	0	10
x_1	(2)	0	1	0	1/2	0	1/2	1/2	0	15
x_2	(3)	0	0	1	-3/2	0	-1/2	1/2	0	5
x_7	(4)	0	0	0	-7/2	0	-5/2	- 1/2	1	-5

The current basic solution is not feasible. We apply the dual simplex method (described in Sec. 8.1) to the above tableau and obtain the new optimal solution $(x_1, x_2, x_3, x_4, x_5, x_6, x_7) = (100/7, 50/7, 10/7, 60/7, 0, 0, 0)$.

(f) Introduce a new variable x_8 with coefficients

$$\begin{bmatrix} c_8 \\ a_{18} \\ a_{28} \\ a_{38} \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \\ 1 \\ 2 \end{bmatrix}$$

The column corresponding to the new variable x_8 in the final tableau is

$$z_8 - c_8 = y^*A_8 - c_8 = \begin{bmatrix} 0 & 3/2 & 1/2 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} - (-1) = 7/2,$$

$$A_8^* = S^*A_8 = \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1/2 & 1/2 \\ 0 & -1/2 & 1/2 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -7 \\ 3/2 \\ 1/2 \end{bmatrix}.$$

The final tableau with the new variable x_8 is

Basic Variable	**Eq**	**Coefficient of:**								**Right Side**
		Z	**x_1**	**x_2**	**x_3**	**x_4**	**x_5**	**x_6**	**x_8**	
Z	(0)	1	2	0	3/2	0	3/2	1/2	7/2	25
x_4	(1)	0	0	0	1	1	-1	-2	-7	10
x_1	(2)	0	1	0	1/2	0	1/2	1/2	3/2	15
x_2	(3)	0	0	1	-3/2	0	-1/2	1/2	1/2	5

The current basic solution is feasible and optimal.

Example for Section 7.3

Consider the following problem (previously analyzed in several Solved Examples for Chapter 4).

Maximize $\quad Z = 3x_1 + 2x_2$,

subject to

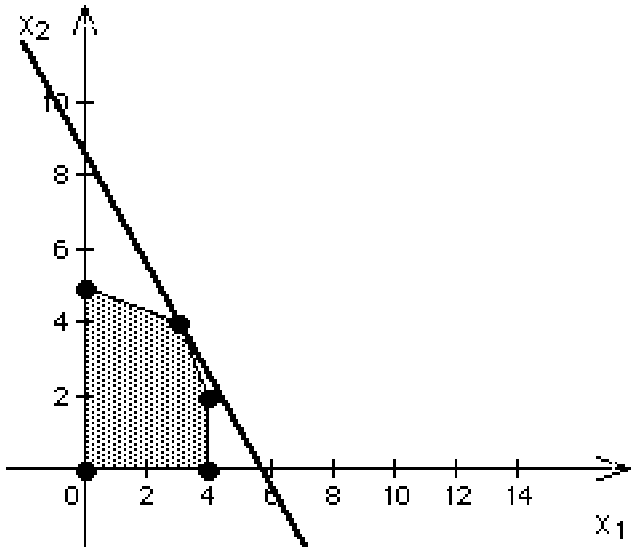
$$
\begin{aligned}
x_1 \qquad\;\; &\leq 4 \qquad \text{(resource 1)} \\
x_1 + 3x_2 &\leq 15 \qquad \text{(resource 2)} \\
2x_1 + \;\; x_2 &\leq 10 \qquad \text{(resource 3)}
\end{aligned}
$$

and

$$x_1 \geq 0, \quad x_2 \geq 0,$$

where Z measures the profit in dollars from the two activities and the right-hand sides are the number of units available of the respective resources.

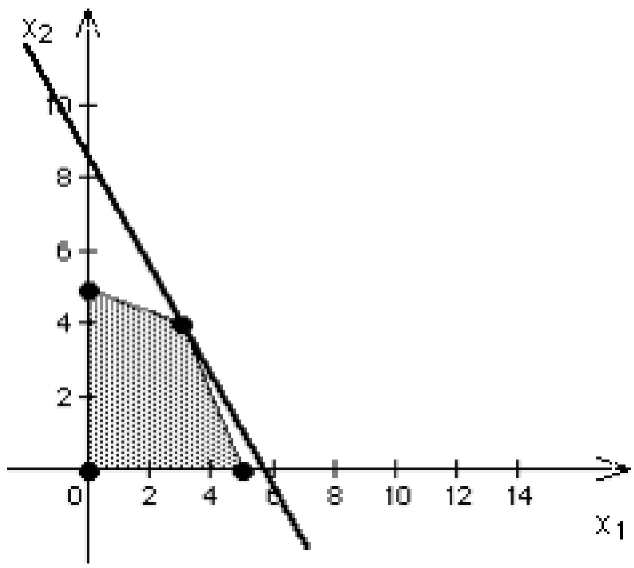
(*a*) Use the graphical method to solve this model.



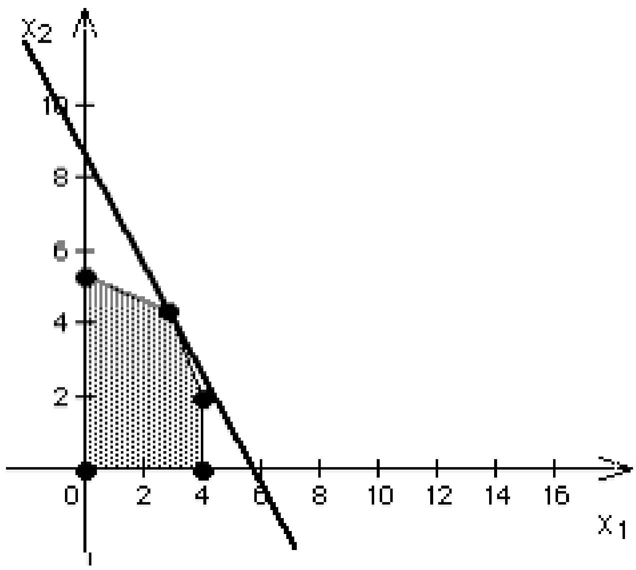
Optimal solution: $(x_1, x_2) = (3, 4)$ and Profit = \$17.

(*b*) Use graphical analysis to determine the shadow price for each of these resources by solving again after increasing the amount of the resource available by 1.

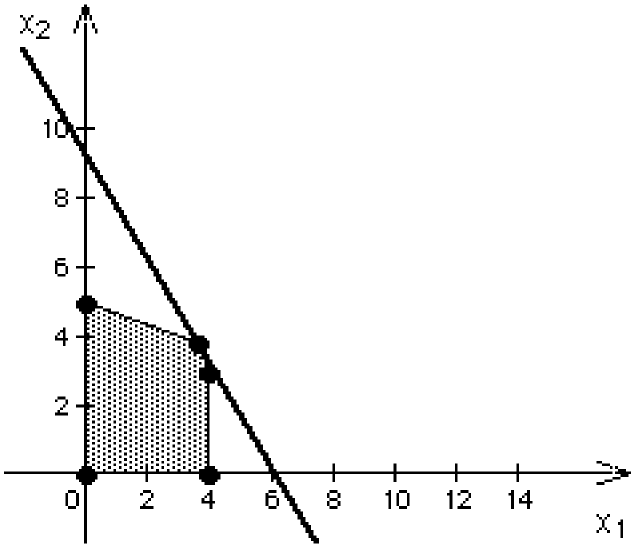
As depicted next, when the right-hand-side of the first constraint is increased to 5, the optimal solution remains the same. Hence, the shadow price for the first constraint is 0.



As depicted next, when the right-hand-side of the second constraint is increased to 16, the new optimal solution becomes $(x_1, x_2) = (2.8, 4.4)$ and Z =17.2. Hence, the shadow price for the second constraint is 17.2 – 17 = 0.2.



As depicted next, when the right-hand-side of the third constraint is increased to 11, the new optimal solution becomes $(x_1, x_2) = (3.6, 3.8)$ and Z = 18.4. Hence, the shadow price for the third constraint is 18.4 – 17 = 1.4.



(*c*) Use the spreadsheet model and Solver instead to do parts (*a*) and (*b*).

Original model:

	A	B	C	D	E	F
1		Activity 1	Activity 2			
2	Unit Profit	$3	$2			
3						
4		Resource Usage		Used		Available
5	Resource 1	1	0	3	≤	4
6	Resource 2	1	3	15	≤	15
7	Resource 3	2	1	10	≤	10
8						
9		Activity 1	Activity 2			Total Profit
10	Solution	3	4			$17.00

The shadow price for resource 1 is \$0.

	A	B	C	D	E	F
1		Activity 1	Activity 2			
2	Unit Profit	\$3	\$2			
3						
4		Resource Usage		Used		Available
5	Resource 1	1	0	3	≤	5
6	Resource 2	1	3	15	≤	15
7	Resource 3	2	1	10	≤	10
8						
9		Activity 1	Activity 2			Total Profit
10	Solution	3	4			\$17.00

The shadow price for resource 2 is \$0.20.

	A	B	C	D	E	F
1		Activity 1	Activity 2			
2	Unit Profit	\$3	\$2			
3						
4		Resource Usage		Used		Available
5	Resource 1	1	0	2.8	≤	4
6	Resource 2	1	3	16	≤	16
7	Resource 3	2	1	10	≤	10
8						
9		Activity 1	Activity 2			Total Profit
10	Solution	2.8	4.4			\$17.20

The shadow price for resource 3 is \$1.40.

	A	B	C	D	E	F
1		Activity 1	Activity 2			
2	Unit Profit	\$3	\$2			
3						
4		Resource Usage		Used		Available
5	Resource 1	1	0	3.6	≤	4
6	Resource 2	1	3	15	≤	15
7	Resource 3	2	1	11	≤	11
8						
9		Activity 1	Activity 2			Total Profit
10	Solution	3.6	3.8			\$18.40

(*d*) Use Solver's sensitivity report to obtain the shadow prices. Also use this report to find the range for the amount available of each resource over which the corresponding shadow price remains valid.

The shadow prices for the three resources are \$0, \$0.20, and \$1.40, respectively.
The allowable range for the right-hand side of the first resource is 3 to ∞.
The allowable range for the right-hand side of the second resource is 10 to 30.
The allowable range for the right-hand side of the third resource is 5 to 11.667.

Variable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
\$B\$10	Solution Activity 1	3	0	3	1	2.3333
\$C\$10	Solution Activity 2	4	0	2	7	0.5

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
\$D\$5	Resource 1 Used	3	0	4	1E+30	1
\$D\$6	Resource 2 Used	15	0.2	15	15	5
\$D\$7	Resource 3 Used	10	1.4	10	1.6667	5

(*e*) Describe why these shadow prices are useful when management has the flexibility to change the amounts of the resources being made available.

These shadow prices tell management that for each additional unit of the resource, profit will increase by \$0, or \$0.20, or \$1.40 for the three resources, respectively (for small changes). Management is then able to evaluate whether or not to change the amounts of resources being made available.