Solved Examples for Chapter 6

Example for Section 6.1

Construct the primal-dual table and the dual problem for the following linear programming model fitting our standard form.

Maximize $Z = 5x_1 + 4x_2 - x_3 + 3x_4$

subject to

$$3x_1 + 2x_2 - 3x_3 + x_4 \leq 24$$
$$3x_1 + 3x_2 + x_3 + 3x_4 \leq 36$$

and

$$x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0, \quad x_4 \geq 0.$$

Solution:

Laying out the parameters of this model in a table in the same order (except for placing the coefficients from the objective function in the bottom row) gives the following primal-dual table.

	x_1	x_2	x_3	x_4		
y_1	3	2	-3	1	$\leq$	24
y_2	3	3	1	3	$\leq$	36
	IV	IV	IV	IV		
	5	4	-1	3		

The numbers in the right-hand column of this table provide the coefficients in the objective function for the dual problem. The numbers in each functional constraint of the dual problem come from the other columns of this table. Therefore, the dual problem is

Minimize $W = 24y_1 + 36y_2$,

subject to

$$3y_1 + 3y_2 \geq 5$$
$$2y_1 + 3y_2 \geq 4$$

$$-3\,y_1 + 1\,y_2 \geq -1$$
$$1\,y_1 + 3\,y_2 \geq 3$$

and

$$y_1 \geq 0, \quad y_2 \geq 0.$$

Example for Section 6.5

Consider the following problem.

Maximize $Z = 2\,x_1 - 1\,x_2 + 1\,x_3$,

subject to

$$3\,x_1 + 1\,x_2 + 1\,x_3 \leq 60$$
$$1\,x_1 - 1\,x_2 + 2\,x_3 \leq 10$$
$$1\,x_1 + 1\,x_2 - 1\,x_3 \leq 20$$

and

$$x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0.$$

Let x_4, x_5, and x_6 denote the slack variables for the respective constraints. After applying the simplex method, the final simplex tableau is

Basic Variable	Eq.	Coefficient of: Z	x_1	x_2	x_3	x_4	x_5	x_6	Right Side
Z	(0)	1	0	0	3/2	0	3/2	1/2	25
x_4	(1)	0	0	0	1	1	-1	-2	10
x_1	(2)	0	1	0	1/2	0	1/2	1/2	15
x_2	(3)	0	0	1	-3/2	0	-1/2	1/2	5

The dual problem is

Minimize $W = 60\,y_1 + 10\,y_2 + 20\,y_3$,

subject to

$$3\,y_1 + 1\,y_2 + 1\,y_3 \geq 2$$
$$1\,y_1 - 1\,y_2 + 1\,y_3 \geq -1$$
$$1\,y_1 + 2\,y_2 - 1\,y_3 \geq 1$$

and

$$y_1 \geq 0, \quad y_2 \geq 0, \quad y_3 \geq 0.$$

From the final simplex tableau, we know that the primal optimal solution is $(x_1, x_2, x_3, x_4, x_5, x_6) = (15, 5, 0, 10, 0, 0)$ and the dual optimal solution is $(y_1, y_2, y_3) = (0, 3/2, 1/2)$.

Use duality theory to determine whether the current basic solution remains optimal after each of the following independent changes.

(a) Change the coefficients of x_3 from $\begin{bmatrix} c_3 \\ a_{13} \\ a_{23} \\ a_{33} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \\ -1 \end{bmatrix}$ **to** $\begin{bmatrix} c_3 \\ a_{13} \\ a_{23} \\ a_{33} \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 1 \\ -2 \end{bmatrix}$.

Since x_3 is a nonbasic variable, changing the coefficient of x_3 does not affect the feasibility of the primal optimal solution. For the current primal basic feasible solution to remain optimal, we need the corresponding dual basic solution to be feasible. Since the only change in the primal problem is in the coefficients of x_3, the only change in the dual problem is in its third constraint. In particular, the third constraint in the dual problem now has changed to

$$3\,y_1 + 1\,y_2 - 2\,y_3 \geq 2.$$

This constraint is violated at the dual basic solution $(y_1, y_2, y_3) = (0, 3/2, 1/2)$ since

$$3(0) + 1(3/2) - 2(1/2) = 1/2 < 2.$$

Thus, the current primal basic solution is not optimal.

(b) Introduce a new variable x_7 with coefficients

$$\begin{bmatrix} c_7 \\ a_{17} \\ a_{27} \\ a_{37} \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \\ 1 \\ 2 \end{bmatrix}.$$

If we let the new added primal variable $x_7 = 0$, it does not affect the feasibility of the primal basic solution that was optimal. The question is whether this solution is still optimal. The equivalent question is whether the corresponding dual basic solution, $(y_1, y_2, y_3) = (0, 3/2, 1/2)$, is still feasible. The only change in the dual problem is the addition of one new constraint. This new dual constraint is

$$-2\,y_1 + 1\,y_2 + 2\,y_3 \geq -1.$$

At $(y_1, y_2, y_3) = (0, 3/2, 1/2)$,

$$-2(0) + 1(3/2) + 2(1/2) = 5/2 > -1,$$

so the constraint is satisfied. Therefore, the current primal basic solution (with $x_7 = 0$) remains optimal.