

## Solved Examples for Chapter 5

### Example for Section 5.1

Consider the linear programming problem given in Table 6.1 as the dual problem for the Wyndor Glass Co. example. This problem is

Minimize  $W = 4 y_1 + 12 y_2 + 18 y_3$ ,  
subject to

$$\begin{aligned} y_1 + 3 y_3 &\geq 3 \\ 2 y_2 + 2 y_3 &\geq 5 \end{aligned}$$

and

$$y_1 \geq 0, \quad y_2 \geq 0, \quad y_3 \geq 0.$$

**(a) Identify the 10 sets of defining equations for this problem. For each one, solve (if a solution exists) for the corresponding corner-point solution, and classify it as a CPF solution or corner-point infeasible solution. For each corner-point solution, give the corresponding basic solution and its set of nonbasic variables.**

This problem has five constraints (two functional constraints and three nonnegativity constraints), where the constraint boundary equations for these constraints are obtained by replacing each  $\geq$  by  $=$ . Since there are three decision variables, each corner-point solution lies at the intersection of three of the constraint boundaries, so the corresponding three constraint boundary equations become the defining equations for that solution. There are 10 different sets of three constraint boundary equations (defining equations).

The basic solutions are obtained from the corner-point solutions by using the augmented form of the model. After introducing surplus variables  $y_4$  and  $y_5$ , the augmented form is

Minimize  $W = 4 y_1 + 12 y_2 + 18 y_3$ ,  
subject to

$$\begin{aligned} y_1 + 3 y_3 - y_4 &\geq 3 \\ 2 y_2 + 2 y_3 - y_5 &\geq 5 \end{aligned}$$

and

$$y_1 \geq 0, \quad y_2 \geq 0, \quad y_3 \geq 0, \quad y_4 \geq 0, \quad y_5 \geq 0.$$

Following are the 10 different ways of choosing three defining equations from the five constraint boundary equations. Solving these three defining equations gives the corner-point solution. The corresponding basic solution is obtained by then using the augmented form of the model to calculate the values of the surplus variables. The three variables that equal zero in the basic solution are the nonbasic variables.

(1) Defining Equations:  $y_1 = 0$ ,  $y_2 = 0$ ,  $y_3 = 0$

Corner-point solution: (0, 0, 0)

The corner-point solution (0, 0, 0) is infeasible.

The corresponding basic solution  $(y_1, y_2, y_3, y_4, y_5) = (0, 0, 0, -3, -5)$

Nonbasic variables:  $y_1, y_2, y_3$

(2) Defining Equations:  $y_1 = 0$ ,  $y_2 = 0$ ,  $y_1 + 3 y_3 = 3$

Corner-point solution: (0, 0, 1)

The corner-point solution (0, 0, 1) is infeasible.

The corresponding basic solution  $(y_1, y_2, y_3, y_4, y_5) = (0, 0, 1, 0, -3)$

Nonbasic variables:  $y_1, y_2, y_4$

(3) Defining Equations:  $y_1 = 0$ ,  $y_2 = 0$ ,  $2 y_2 + 2 y_3 = 5$

Corner-point solution: (0, 0, 2.5)

The corner-point solution (0, 0, 2.5) is a CPF.

The corresponding basic solution  $(y_1, y_2, y_3, y_4, y_5) = (0, 0, 2.5, 4.5, 0)$

Nonbasic variables:  $y_1, y_2, y_5$

(4) Defining Equations:  $y_1 = 0$ ,  $y_1 + 3 y_3 = 3$ ,  $y_3 = 0$

No solution exists.

(5) Defining Equations:  $y_1 = 0$ ,  $2 y_2 + 2 y_3 = 5$ ,  $y_3 = 0$

Corner-point solution: (0, 2.5, 0)

The corner-point solution (0, 2.5, 0) is infeasible.

The corresponding basic solution  $(y_1, y_2, y_3, y_4, y_5) = (0, 2.5, 0, -3, 0)$

Nonbasic variables:  $y_1, y_3, y_5$

(6) Defining Equations:  $y_2 = 0$ ,  $y_1 + 3 y_3 = 3$ ,  $y_3 = 0$

Corner-point solution: (3, 0, 0)

The corner-point solution (3, 0, 0) is infeasible.

The corresponding basic solution  $(y_1, y_2, y_3, y_4, y_5) = (3, 0, 0, 0, -5)$

Nonbasic variables:  $y_2, y_3, y_4$

(7) Defining Equations:  $y_2 = 0, \quad 2y_2 + 2y_3 = 5, \quad y_3 = 0$

No solution exists.

(8) Defining Equations:  $y_2 = 0, \quad y_1 + 3y_3 = 3, \quad 2y_2 + 2y_3 = 5$

Corner-point solution:  $(-4.5, 0, 2.5)$

The corner-point solution  $(-4.5, 0, 2.5)$  is infeasible.

The corresponding basic solution  $(y_1, y_2, y_3, y_4, y_5) = (-4.5, 0, 2.5, 0, 0)$

Nonbasic variables:  $y_2, y_4, y_5$

(9) Defining Equations:  $y_3 = 0, \quad y_1 + 3y_3 = 3, \quad 2y_2 + 2y_3 = 5$

Corner-point solution:  $(3, 2.5, 0)$

The corner-point solution  $(3, 2.5, 0)$  is a CPF.

The corresponding basic solution  $(y_1, y_2, y_3, y_4, y_5) = (3, 2.5, 0, 0, 0)$

Nonbasic variables:  $y_3, y_4, y_5$

(10) Defining Equations:  $y_1 = 0, \quad y_1 + 3y_3 = 3, \quad 2y_2 + 2y_3 = 5$

Corner-point solution:  $(0, 1.5, 1)$

The corner-point solution  $(0, 1.5, 1)$  is a CPF.

The corresponding basic solution  $(y_1, y_2, y_3, y_4, y_5) = (0, 1.5, 1, 0, 0)$

Nonbasic variables:  $y_1, y_4, y_5$

## Example for Sections 5.2 and 5.4

Consider the following model.

Maximize  $Z = 4x_1 + 3x_2 + 6x_3,$

subject to

$$3x_1 + x_2 + 3x_3 \leq 30$$

$$2x_1 + 2x_2 + 3x_3 \leq 40$$

and

$$x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0.$$

We introduce  $x_4$  and  $x_5$  as slack variables for the respective constraints. The augmented form of the model then is

$$\text{Maximize } Z = 4x_1 + 3x_2 + 6x_3,$$

subject to

$$3x_1 + x_2 + 3x_3 + x_4 = 30$$

$$2x_1 + 2x_2 + 3x_3 + x_5 = 40$$

and

$$x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0, \quad x_4 \geq 0, \quad x_5 \geq 0.$$

Using matrix notation, we have

$$\mathbf{c} = [4 \quad 3 \quad 6 \quad 0 \quad 0], \quad \mathbf{A} = \begin{bmatrix} 3 & 1 & 3 & 1 & 0 \\ 2 & 2 & 3 & 0 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 30 \\ 40 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}.$$

Now let us apply the revised simplex method step by step to solve this problem.

### Iteration 0:

Since  $x_4$  and  $x_5$  are the initial basic variables,

$$\mathbf{x}_B = \begin{bmatrix} x_4 \\ x_5 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{B}^{-1}, \text{ so } \begin{bmatrix} x_4 \\ x_5 \end{bmatrix} = \mathbf{B}^{-1}\mathbf{b} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 30 \\ 40 \end{bmatrix} = \begin{bmatrix} 30 \\ 40 \end{bmatrix}.$$

$$\mathbf{c}_B = [0, \quad 0].$$

### Iteration 1:

The coefficients in Eq. (0) are

$$\begin{aligned} \mathbf{c}_B \mathbf{B}^{-1} \mathbf{A} - \mathbf{c} &= [-4 \quad -3 \quad 6] \text{ for } x_1, x_2, x_3, \\ \mathbf{c}_B \mathbf{B}^{-1} &= [0 \quad 0] \text{ for } x_4, x_5. \end{aligned}$$

Since -6 is the most negative coefficient, we choose  $x_3$  as the entering basic variable. The coefficients of  $x_3$  in Eqs. (1) and (2) are

$$\mathbf{B}^{-1} \begin{bmatrix} 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}.$$

The right-hand side of these equations was identified in Iteration 0 as

$$\mathbf{B}^{-1} \mathbf{b} = \begin{bmatrix} 30 \\ 40 \end{bmatrix}, \text{ where this gives the value of } \mathbf{x}_B = \begin{bmatrix} x_4 \\ x_5 \end{bmatrix}.$$

Applying the minimum ratio test, since  $(30/3) < (40/3)$ , we choose  $x_4$  as the leaving basic variable. Thus,

$$\mathbf{x}_B = \begin{bmatrix} x_3 \\ x_5 \end{bmatrix}.$$

Since the coefficients of  $x_3$  in Eqs. (1) and (2) are  $\begin{bmatrix} 3 \\ 3 \end{bmatrix}$  and Eq. (1) yielded the

leaving basic variable,

$$\boldsymbol{\eta} = \begin{bmatrix} 1/3 \\ -3/3 \end{bmatrix} = \begin{bmatrix} 1/3 \\ -1 \end{bmatrix}, \quad \text{so } \mathbf{B}^{-1} \text{ becomes}$$

$$\mathbf{B}^{-1} = \begin{bmatrix} 1/3 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/3 & 0 \\ -1 & 1 \end{bmatrix}.$$

Therefore,

$$\begin{bmatrix} x_3 \\ x_5 \end{bmatrix} = \mathbf{B}^{-1}\mathbf{b} = \begin{bmatrix} 1/3 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 30 \\ 40 \end{bmatrix} = \begin{bmatrix} 10 \\ 10 \end{bmatrix}.$$

$$\mathbf{c}_B = [6, 0].$$

### Iteration 2:

The coefficients in Eq. (0) now are

$$\begin{aligned} \mathbf{c}_B \mathbf{B}^{-1} \mathbf{A} - \mathbf{c} &= [2 \quad -1 \quad 0] \quad \text{for } x_1, x_2, x_3, \\ \mathbf{c}_B \mathbf{B}^{-1} &= [2 \quad 0] \quad \text{for } x_4, x_5. \end{aligned}$$

Since only  $x_2$  has a negative coefficient, it becomes the entering basic variable. The coefficients of  $x_2$  in Eqs. (1) and (2) are

$$\mathbf{B}^{-1} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1/3 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 1 \end{bmatrix}.$$

The right-hand side of these equations was identified in Iteration 1 as

$$\mathbf{B}^{-1}\mathbf{b} = \begin{bmatrix} 10 \\ 10 \end{bmatrix}, \quad \text{where this gives the value of } \mathbf{x}_B = \begin{bmatrix} x_3 \\ x_5 \end{bmatrix}.$$

Applying the minimum ratio test, we have

$\frac{10}{1/3}$  for  $x_5$  and  $\frac{10}{-1}$  for  $x_3$ , but the negative denominator for  $x_3$  rules it out as a candidate, so we choose  $x_5$  as the leaving basic variable. Thus,

$$\mathbf{x}_B = \begin{bmatrix} x_3 \\ x_2 \end{bmatrix}.$$

Since the coefficients of  $x_2$  in Eqs. (1) and (2) are  $\begin{bmatrix} 1/3 \\ 1 \end{bmatrix}$  and Eq. (2) yielded the leaving basic variable,

$$\boldsymbol{\eta} = \begin{bmatrix} -(1/3)/1 \\ 1/1 \end{bmatrix} = \begin{bmatrix} -1/3 \\ 1 \end{bmatrix}, \quad \text{so } \mathbf{B}^{-1} \text{ becomes}$$

$$\mathbf{B}^{-1} = \begin{bmatrix} 1 & -1/3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1/3 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2/3 & -1/3 \\ -1 & 1 \end{bmatrix}.$$

Therefore,

$$\begin{bmatrix} x_3 \\ x_2 \end{bmatrix} = \mathbf{B}^{-1} \mathbf{b} = \begin{bmatrix} 2/3 & -1/3 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 30 \\ 40 \end{bmatrix} = \begin{bmatrix} 20/3 \\ 10 \end{bmatrix}.$$

$$\mathbf{c}_B = [6, \ 3].$$

### Optimality Test:

The coefficients in Eq. (0) now are

$$\begin{aligned} \mathbf{c}_B \mathbf{B}^{-1} \mathbf{A} - \mathbf{c} &= [6 \ 3] \begin{bmatrix} 2/3 & -1/3 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 & 3 \\ 2 & 2 & 3 \end{bmatrix} - [4 \ 3 \ 6] \\ &= [1 \ 0 \ 0] \quad \text{for } x_1, x_2, x_3, \\ \mathbf{c}_B \mathbf{B}^{-1} &= [1 \ 1] \quad \text{for } x_4, x_5. \end{aligned}$$

Since all these coefficients are nonnegative, the current solution is optimal. This solution is

$$\mathbf{x}_B = \begin{bmatrix} x_3 \\ x_2 \end{bmatrix} = \mathbf{B}^{-1} \mathbf{b} = \begin{bmatrix} 20/3 \\ 10 \end{bmatrix},$$

as found in Iteration 2. The other variables are nonbasic variables, so  $x_1 = 0$ ,  $x_4 = 0$ ,

and  $x_5 = 0$ .

$$Z = 4(0) + 3(10) + 6(20/3) = 70.$$

### Example for Section 5.3

Consider the following problem.

Maximize  $Z = x_1 - x_2 + 2x_3$ ,  
subject to

$$x_1 + x_2 + 3x_3 \leq 15$$

$$2x_1 - x_2 + x_3 \leq 2$$

$$-x_1 + x_2 + x_3 \leq 4$$

and

$$x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0.$$

Let  $x_4$ ,  $x_5$ , and  $x_6$  denote the slack variables for the respective constraints. After the simplex method is applied, a portion of the final simplex tableau is as follows:

Basic Variable	Eq	Coefficient of:							Right Side
		Z	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	
Z	(0)	1				0	$\frac{3}{2}$	$\frac{1}{2}$	
$x_4$	(1)	0				1	-1	-2	
$x_3$	(2)	0				0	$\frac{1}{2}$	$\frac{1}{2}$	
$x_2$	(3)	0				0	$-\frac{1}{2}$	$\frac{1}{2}$	

**(a) Use the fundamental insight presented in Sec. 5.3 to identify the missing numbers in the final simplex tableau. Show your calculations. Identify the defining equations of the CPF solution corresponding to the optimal BF solution in the final simplex tableau.**



From the coefficients for  $(x_4, x_5, x_6)$  in the final simplex tableau, we observe that

$$\mathbf{S}^* = \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1/2 & 1/2 \\ 0 & -1/2 & 1/2 \end{bmatrix} \quad \text{and} \quad \mathbf{y}^* = [0 \quad 3/2 \quad 1/2].$$

Thus, by the fundamental insight presented in Sec. 5.3, the constraint coefficients for  $(x_1, x_2, x_3)$  in the final tableau are

$$\mathbf{S}^* \mathbf{A} = \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1/2 & 1/2 \\ 0 & -1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 3 \\ 2 & -1 & 1 \\ -1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 0 & 1 \\ -3/2 & 1 & 0 \end{bmatrix}.$$

The coefficients in the objective function for  $(x_1, x_2, x_3)$  in the final tableau are

$$\mathbf{y} \mathbf{A} - \mathbf{c} = [0 \quad 3/2 \quad 1/2] \begin{bmatrix} 1 & 1 & 3 \\ 2 & -1 & 1 \\ -1 & 1 & 1 \end{bmatrix} - [1 \quad -1 \quad 2] = [3/2 \quad 0 \quad 0].$$

The final right-hand side is  $\mathbf{S}^* \mathbf{b} = \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1/2 & 1/2 \\ 0 & -1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 15 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix}, \quad \text{and}$

$$Z^* = \mathbf{y}^* \mathbf{b} = [0 \quad 3/2 \quad 1/2] \begin{bmatrix} 15 \\ 2 \\ 4 \end{bmatrix} = 5.$$

Therefore, the complete final simplex tableau is

Basic Variable	Eq	Coefficient of:							Right Side
		Z	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	
Z	(0)	1	3/2	0	0	0	3/2	1/2	5
$x_4$	(1)	0	1	0	0	1	-1	-2	5
$x_3$	(2)	0	1/2	0	1	0	1/2	1/2	3
$x_2$	(3)	0	-3/2	1	0	0	-1/2	1/2	1

**(b) Identify the defining equations for the CPF solution corresponding to the optimal BF solution in the final simplex tableau.**

Since the optimal BF solution is  $(x_1, x_2, x_3, x_4, x_5, x_6) = (0, 1, 3, 5, 0, 0)$ , the corresponding CPF solution is  $(x_1, x_2, x_3) = (0, 1, 3)$ . The nonbasic variables are  $x_1, x_5, x_6$ , and these are the indicating variables that indicate that the following constraints hold with equality.

$$\begin{array}{rcl} x_1 & & \geq 0 \\ 2x_1 - x_2 + x_3 & \leq & 2 \\ -x_1 + x_2 + x_3 & \leq & 4 \end{array}$$

Therefore, the defining equations for this CPF solution are

$$\begin{array}{l} x_1 = 0, \\ 2x_1 - x_2 + x_3 = 2, \\ -x_1 + x_2 + x_3 = 4. \end{array}$$