Solved Examples for Chapter 4

Example for Section 4.1

Consider the following linear programming model.

Maximize $Z = 3x_1 + 2x_2$,

subject to

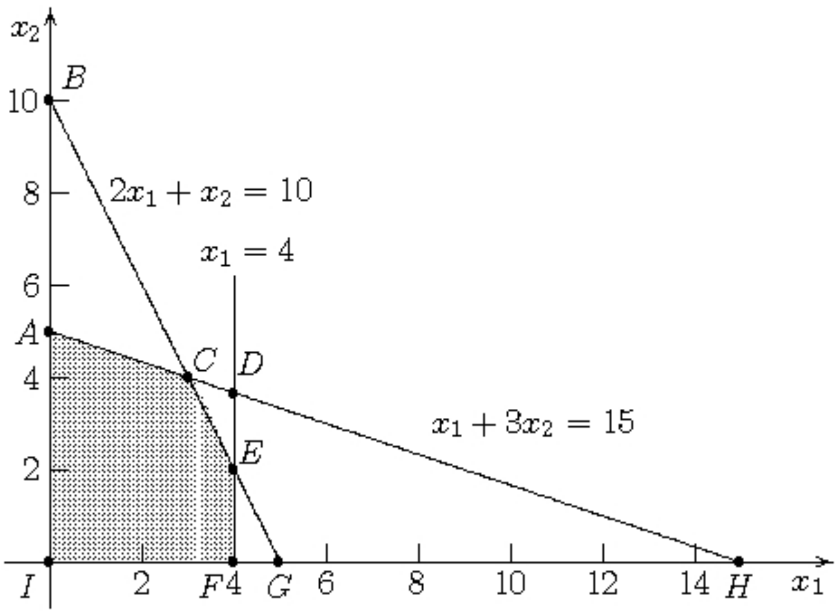
$$\begin{aligned} x_1 \quad\quad\quad &\leq 4 \\ x_1 + 3x_2 &\leq 15 \\ 2x_1 + \quad x_2 &\leq 10 \end{aligned}$$

and

$x_1 \geq 0, \quad x_2 \geq 0.$

(a) Use graphical analysis to identify all the *corner-point solutions* for this model. Label each as either feasible or infeasible.

The graph showing all the constraint boundary lines and the corner-point solutions at their intersections is shown below.



The exact value of (x_1, x_2) for each of these nine corner-point solutions (A, B, ..., I) can be identified by obtaining the simultaneous solution of the corresponding two constraint boundary equations. The results are summarized in the following table.

Corner-point solutions	(x_1, x_2)	Feasibility
A	(0, 5)	Feasible
B	(0,10)	Infeasible
C	(3, 4)	Feasible
D	(4, 11/3)	Infeasible
E	(4, 2)	Feasible
F	(4, 0)	Feasible
G	(5, 0)	Infeasible
H	(15, 0)	Infeasible
I	(0, 0)	Feasible

(b) Calculate the value of the objective function for each of the CPF solutions. Use this information to identify an optimal solution.

The objective value of each corner-point feasible solution is calculated in the following table:

Corner-point feasible solutions	(x_1, x_2)	Objective Value Z
A	(0, 5)	3*0+2*5 = 10
C	(3, 4)	3*3+2*4 = 17
E	(4, 2)	3*4+2*2 = 16
F	(4, 0)	3*4+2*0 = 12
I	(0, 0)	3*0+0*0 = 0

Since point C has the largest value of Z, $(x_1, x_2) = (3, 4)$ must be an optimal solution.

(c) Use the solution concepts of the simplex method given in Sec. 4.1 to identify which sequence of CPF solutions would be examined by the simplex method to reach an optimal solution.

CPF solution I:

By Solution Concept 3, we choose the origin, point I = (0, 0), to be the initial CPF solution. By Solution Concept 6, we know that I is not optimal since two adjacent CPF solutions, A = (0, 5) with Z = 10 and F = (4, 0) with Z = 12, have a larger value of Z (so moving toward either adjacent CPF solution gives a positive rate of improvement in Z). By Solution Concept 5, we choose F because the rate of improvement in Z of F (= 12/4 = 3) is greater than that of A (= 10/5=2).

CPF solution F:

The CPF solution F is not optimal because one adjacent CPF solution, E = (4, 2) with Z = 16, has a larger value of Z. We then move to CPF solution E.

CPF solution E:

The CPF solution E is not optimal because one adjacent CPF solution, C = (3,4) with Z = 17, has a larger value of Z. We then move to CPF solution C.

CPF solution C:

By Solution Concept 6, the CPF solution C is optimal since its adjacent CPF solutions, A and E, have smaller values of Z so moving toward either of these adjacent CPF solutions would give a negative rate of improvement in Z.

Therefore, the sequence of CPF solutions examined by the simplex method would be I -> F -> E -> C.

Example for Section 4.2

Reconsider the following linear programming model (previously analyzed in the preceding example).

Maximize $Z = 3x_1 + 2x_2$,

subject to

$$\begin{aligned} x_1 \quad\quad &\leq 4 \\ x_1 + 3x_2 &\leq 15 \\ 2x_1 + \quad x_2 &\leq 10 \end{aligned}$$

and

$x_1 \geq 0, \quad x_2 \geq 0.$

(a) Introduce slack variables in order to write the functional constraints in augmented form.

We introduce x_3, x_4, and x_5 as the slack variables for the respective constraints. The resulting augmented form of the model is

Maximize $Z = 3x_1 + 2x_2$,

subject to

$$\begin{aligned} x_1 \quad\quad + x_3 \quad\quad\quad\quad &= 4 \\ x_1 + 3x_2 \quad\quad + x_4 \quad\quad &= 15 \\ 2x_1 + \quad x_2 \quad\quad\quad\quad + x_5 &= 10 \end{aligned}$$

and

$x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0, \quad x_4 \geq 0, \quad x_5 \geq 0.$

(b) For each CPF solution, identify the corresponding BF solution by calculating the values of the slack variables. For each BF solution, use the values of the variables to identify the nonbasic variables and the basic variables.

CPF solution I = (0, 0):

Plug in $x_1 = x_2 = 0$ into the augmented form. The values of the slack variables are $x_3 = 4$, $x_4 = 15$, $x_5 = 10$.

The BF solution is $(x_1, x_2, x_3, x_4, x_5) = (0, 0, 4, 15, 10)$.

Since $x_1 = x_2 = 0$, we know that x_1 and x_2 are the two nonbasic variables.

Since $x_3 > 0$, $x_4 > 0$, $x_5 > 0$, we know that x_3, x_4, and x_5 are basic variables.

CPF solution A = (0, 5):

Plug in $x_1 = 0$ and $x_2 = 5$ into the augmented form. The values of the slack variables are $x_3 = 4$, $x_4 = 0$, $x_5 = 5$.

The BF solution is $(x_1, x_2, x_3, x_4, x_5) = (0, 5, 4, 0, 5)$.

Since $x_1 = x_4 = 0$, we know that x_1 and x_4 are the two nonbasic variables.

Since $x_2 > 0$, $x_3 > 0$, $x_5 > 0$, we know that x_2, x_3, and x_5 are basic variables.

CPF solution C = (3, 4):

Plug in $x_1 = 3$ and $x_2 = 4$ into the augmented form. The values of the slack variables are $x_3 = 1$, $x_4 = 0$, $x_5 = 0$.

The BF solution is $(x_1, x_2, x_3, x_4, x_5) = (3, 4, 1, 0, 0)$.

Since $x_4 = x_5 = 0$, we know that x_4 and x_5 are the two nonbasic variables.

Since $x_1 > 0$, $x_2 > 0$, $x_3 > 0$, we know that x_1, and x_2 and x_3 are basic variables.

CPF solution E = (4, 2):

Plug in $x_1 = 4$ and $x_2 = 2$ into the augmented form. The values of the slack variables are $x_3 = 0$, $x_4 = 5$, $x_5 = 0$.

The BF solution is $(x_1, x_2, x_3, x_4, x_5) = (4, 2, 0, 5, 0)$.

Since $x_3 = x_5 = 0$, we know that x_3 and x_5 are the two nonbasic variables.

Since $x_1 > 0$, $x_2 > 0$, $x_4 > 0$, we know that x_1, x_2, and x_4 are basic variables.

CPF solution F = (4, 0):

Plug in $x_1 = 4$ and $x_2 = 0$ into the augmented form. The values of the slack variables are $x_3 = 0$, $x_4 = 11$, $x_5 = 2$.

The BF solution is $(x_1, x_2, x_3, x_4, x_5) = (4, 0, 0, 11, 2)$.

Since $x_2 = x_3 = 0$, we know that x_2 and x_3 are the two nonbasic variables.

Since $x_1 > 0$, $x_4 > 0$, $x_5 > 0$, we know that x_1, x_4, and x_5 are basic variables.

Summary of results:

Label	CPF solution	BF solution	Nonbasic variables	Basic variables
I	(0, 0)	(0, 0, 4, 15, 10)	x_1, x_2	x_3, x_4, x_5
A	(0, 5)	(0, 5, 4, 0, 5)	x_1, x_4	x_2, x_3, x_5
C	(3, 4)	(3, 4, 1, 0, 0)	x_4, x_5	x_1, x_2, x_3
E	(4, 2)	(4, 2, 0, 5, 0)	x_3, x_5	x_1, x_2, x_4
F	(4, 0)	(4, 0, 0, 11, 2)	x_2, x_3	x_1, x_4, x_5

(c) For each BF solution, demonstrate (by plugging in the solution) that, after the nonbasic variables are set equal to zero, this BF solution also is the simultaneous solution of the system of equations obtained in part (a).

BF solution I = (0, 0, 4, 15, 10): Plugging this solution into the equations yields:

$$
\begin{array}{lllllll}
0 & & + \ 4 & & & = 4 \\
0 & + 3(0) & & + 15 & & = 15 \\
2(0) & + \ 0 & & & +10 & = 10,
\end{array}
$$

so the equations are satisfied.

BF solution A = (0, 5, 4, 0, 5): Plugging this solution into the equations yields

$$
\begin{array}{lllllll}
0 & & + \ 4 & & & = 4 \\
0 & + 3(5) & & + 0 & & = 15 \\
2(0) & + 5 & & & +5 & = 10,
\end{array}
$$

so the equations are satisfied.

BF Solution C = (3, 4, 1, 0, 0): Plugging this solution into the equations yields

$$
\begin{array}{lllllll}
3 & & + \ 1 & & & = 4 \\
3 & + 3(4) & & + 0 & & = 15 \\
2(3) & + \ 4 & & & + 0 & = 10,
\end{array}
$$

so the equations are satisfied.

BF solution E= (4, 2, 0, 5, 0): Plugging this solution into the equations yields

$$
\begin{array}{lllllll}
4 & & + 0 & & & = \ 4 \\
4 & + 3(2) & & + \ 5 & & = 15 \\
2(4) & + \ 2 & & & + 0 & = 10,
\end{array}
$$

so the equations are satisfied.

BF solution F = (4, 0, 0, 11, 2): Plugging this solution into the equations yields

$$\begin{aligned} 4 \qquad\qquad + 0 \qquad\qquad\qquad &= 4 \\ 4 + 3(0) \qquad + 11 \qquad\quad &= 15 \\ 2(4) + 0 \qquad\qquad\qquad + 2 &= 10, \end{aligned}$$

so the equations are satisfied.

Example for Section 4.3

Reconsider the following linear programming model (previously considered in the preceding two examples).

Maximize $Z = 3x_1 + 2x_2$,

subject to

$$\begin{aligned} x_1 \qquad &\leq 4 \\ x_1 + 3x_2 &\leq 15 \\ 2x_1 + x_2 &\leq 10 \end{aligned}$$

and

$$x_1 \geq 0, \quad x_2 \geq 0.$$

We introduce x_3, x_4, and x_5 as slack the variables for the respective constraints. The resulting augmented form of the model is

Maximize $Z = 3x_1 + 2x_2$,

subject to

$$\begin{aligned} x_1 \qquad + x_3 \qquad\qquad\qquad &= 4 \\ x_1 + 3x_2 \qquad + x_4 \qquad &= 15 \\ 2x_1 + x_2 \qquad\qquad\qquad + x_5 &= 10 \end{aligned}$$

and

$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0, x_5 \geq 0.$$

(a) Work through the simplex method (in algebraic form) to solve this model.

Initialization:

Let x_1 and x_2 be the nonbasic variables, so $x_1 = x_2 = 0$. Solving for x_3, x_4, and x_5 from the equations for the constraints:

$$
\begin{array}{llr}
(1) & x_1 \qquad\quad + x_3 & = 4 \\
(2) & x_1 + 3x_2 \qquad\quad + x_4 & = 15 \\
(3) & 2x_1 + \ x_2 \qquad\qquad\quad + x_5 & = 10
\end{array}
$$

we obtain the initial BF solution (0, 0, 4, 15, 10).

The objective function is $Z = 3x_1 + 2x_2$. The current BF solution is not optimal since we can improve Z by increasing x_1 or x_2.

Iteration 1:

$Z = 3x_1 + 2x_2$, so equation (0) is

(0) $Z - 3x_1 - 2x_2 = 0$.

If we increase x_1, the rate of improvement in $Z = 3$.

If we increase x_2, the rate of improvement in $Z = 2$.

Hence, we choose x_1 as the entering basic variable.

Next, we need to decide how far we can increase x_1. Since we need variables x_3, x_4, and x_5 to stay nonnegative, from equations (1), (2), and (3), we have

$$
\begin{array}{llllll}
(1) & x_3 = 4 - x_1 & \geq 0 & \Rightarrow & x_1 \leq 4. & \leftarrow \text{minimum} \\
(2) & x_4 = 15 - x_1 & \geq 0 & \Rightarrow & x_1 \leq 15. & \\
(3) & x_5 = 10 - 2x_1 & \geq 0 & \Rightarrow & x_1 \leq 5. &
\end{array}
$$

Thus, the entering basic variable x_1 can be increased to 4, at which point x_3 has decreased to 0. The variable x_3 becomes the new nonbasic variable. Proper form from Gaussian elimination is restored by adding 3 times equation (1) to equation (0), subtracting equation (1) from equation (2), and subtracting 2 times equation (1) from equation (3). This yields the following system of equations:

$$
\begin{array}{llcccccl}
(0) & Z & -2x_2 & + & 3x_3 & & & = 12 \\
(1) & x_1 & & + & x_3 & & & = 4 \\
(2) & & 3x_2 & - & x_3 & +x_4 & & = 11 \\
(3) & & x_2 & - & 2x_3 & & +x_5 & = 2.
\end{array}
$$

Thus, the new BF solution is (4, 0, 0, 11, 2) with Z = 12.

Iteration 2:

Using the new equation (0), the objective function becomes $Z = 2x_2 - 3x_3 + 12$. The current BF solution is nonoptimal since we can increase x_2 to improve Z with the rate of improvement in Z = 2. Hence, we choose x_2 as the entering basic variable.

Next, we need to decide how far we can increase x_2. Since we need the variables x_1, x_4 and x_5 to stay nonnegative, from equations (1), (2), and (3) in iteration 1, we have

$$
\begin{array}{llll}
(1) & x_1 = 4 & \geq 0 \Rightarrow & \text{no upper bound on } x_2 \\
(2) & x_4 = 11 - 3x_2 & \geq 0 \Rightarrow & x_2 \leq 11/3 \\
(3) & x_5 = 2 - x_2 & \geq 0 \Rightarrow & x_2 \leq 2. \leftarrow \text{minimum}
\end{array}
$$

Thus, x_2 can be increased to 2, at which point x_5 has decreased to 0, so x_5 becomes the leaving basic variable. Thus, x_5 becomes a nonbasic variable. After restoring proper form from Gaussian elimination, we obtain the following system of equations:

$$
\begin{array}{llcccccl}
(0) & Z & & - & x_3 & & +2x_5 & = 16 \\
(1) & x_1 & & + & x_3 & & & = 4 \\
(2) & & & & 5x_3 & +x_4 & -3x_5 & = 5 \\
(3) & & x_2 & - & 2x_3 & & +x_5 & = 2.
\end{array}
$$

Thus, the new BF solution is (4, 2, 0, 5, 0) with Z = 16.

Iteration 3:

Using the new equation (0), the objective function becomes $Z = x_3 - 2x_5 + 16$. The current BF solution is nonoptimal since we can increase x_3 to improve Z with the rate of improvement in Z = 1. Hence, we choose x_3 as the entering basic variable.

Next, we need to decide how far we can increase x_2. Since we need variables x_1, x_2, and x_4 to stay nonnegative, from equations (1), (2), and (3) in iteration 2, we have

(1)	$x_1 = 4 - x_3$	≥ 0	$\Rightarrow$	$x_3 \leq 4.$	
(2)	$x_4 = 5 - 5\,x_3$	≥ 0	$\Rightarrow$	$x_3 \leq 1.$	$\leftarrow$ minimum
(3)	$x_2 = 2 + 2\,x_3$	≥ 0	$\Rightarrow$ no upper bound on x_3.		

Thus, x_3 can be increased to 1, at which point x_4 has decreased to 0, so x_4 becomes the leaving basic variable. Thus, x_4 becomes a nonbasic variable. After restoring proper form from Gaussian elimination, we obtain the following system of equations:

$$
\begin{array}{llcccccl}
(0) & Z & & & + (1/5)\,x_4 & + & (7/5)\,x_5 & = 17 \\
(1) & & x_1 & & - (1/5)\,x_4 & + & (3/5)\,x_5 & = 3 \\
(2) & & & x_3 & + (1/5)\,x_4 & - & (3/5)\,x_5 & = 1 \\
(3) & & x_2 & & + (2/5)\,x_4 & - & (1/5)\,x_5 & = 4.
\end{array}
$$

Thus, the new BF solution is (3, 4, 1, 0, 0) with Z = 17. Since increasing either x_4 or x_5 will decrease Z, the current BF solution is optimal.

(b) Verify the optimal solution you obtained by using a software package based on the simplex method.

Using Solver (which employs the simplex method) to solve a spreadsheet formulation of this linear programming model finds the optimal solution as

$(x_1, x_2) = (3, 4)$ with $Z = 17$, as displayed next.

	A	B	C	D	E	F
1		X1	X2			
2	Unit Profit	3	2			
3						
4				Totals		Limit
5	Constraint 1	1	0	3	<=	4
6	Constraint 2	1	3	15	<=	15
7	Constraint 3	2	1	10	<=	10
8						
9						Total Profit
10	Solution	3	4			17

Solver Parameters

Set Objective Cell: TotalProfit

To: Max

By Changing Variable Cells:

Solution

Subject to the Constraints:

Totals <= Limit

Solver Options:

Make Variables Nonnegative

Solving Method: Simplex LP

Example for Section 4.4

Repeat the example for Section 4.3, using the tabular form of the simplex method this time.

The augmented form of the model is

Maximize $Z = 3x_1 + 2x_2$,

subject to

$$
\begin{aligned}
x_1 \quad\quad\; + x_3 \quad\quad\quad\quad\quad &= 4 \\
x_1 + 3x_2 \quad\quad + x_4 \quad\quad &= 15 \\
2x_1 + x_2 \quad\quad\quad\quad + x_5 &= 10
\end{aligned}
$$

and

$$x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0, \quad x_4 \geq 0, \quad x_5 \geq 0$$

Let x_1 and x_2 be the nonbasic variables and x_3, x_4, and x_5 be the nonbasic variables. The simplex tableau for this initial BF solution is

Basic Variable	Eq	Coefficient of:						Right Side	Ratio
		Z	x_1	x_2	x_3	x_4	x_5		
Z	(0)	1	-3	-2	0	0	0	0	
x_3	(1)	0	1	0	1	0	0	4	4 ← minimum
x_4	(2)	0	1	3	0	1	0	15	15
x_5	(3)	0	2	1	0	0	1	10	(10/2)=5

This BF solution is nonoptimal since the coefficients of x_1 and x_2 in Eq. (0) are negative. This means that if we increase either x_1 or x_2, we will increase the objective function value Z.

Iteration 1.

Since the most negative coefficient in Eq. (0) is –3 for x_1 (3 > 2), the nonbasic variable x_1 is to be changed to a basic variable. Performing the minimum ratio test on x_1, as shown in the last column of the above tableau, the leaving basic variable is x_3. After using elementary row operations to restore proper form from Gaussian elimination, the new simplex tableau with basic variables x_1, x_4, and x_5 becomes

Basic Variable	Eq	Coefficient of:						Right Side	Ratio
		Z	x_1	x_2	x_3	x_4	x_5		
Z	(0)	1	0	-2	3	0	0	12	
x_1	(1)	0	1	0	1	0	0	4	
x_4	(2)	0	0	3	-1	1	0	11	11/3
x_5	(3)	0	0	1	-2	0	1	2	2 ← minimum

Iteration 2.

Since the coefficient for x_2 in Eq. (0) is –3, we can improve Z by increasing x_2. The nonbasic variable x_2 is to be changed to a basic variable. Performing the minimum ratio test on x_2, as shown in the last column of the above tableau, the leaving basic variable is x_5. After restoring proper form from Gaussian elimination, the new simplex tableau with basic variables x_1, x_2, and x_4 becomes

Basic Variable	Eq	Coefficient of:						Right Side	Ratio
		Z	x_1	x_2	x_3	x_4	x_5		
Z	(0)	1	0	0	-1	0	2	16	
x_1	(1)	0	1	0	1	0	0	4	4
x_4	(2)	0	0	0	5	1	-3	5	1 ← minimum
x_2	(3)	0	0	1	-2	0	1	2	

Iteration 3.

Since the coefficient for x_3 in Eq. (0) is –1, we can improve Z by increasing x_3. The nonbasic variable x_3 is to be changed to a basic variable. Performing the minimum ratio test on x_3, as shown in the last column of the above tableau, the leaving basic variable is x_4. After restoring proper form from Gaussian elimination, the new simplex tableau with basic variables x_1, x_2, and x_3 becomes

Basic Variable	Eq	Coefficient of:						Right Side
		Z	x_1	x_2	x_3	x_4	x_5	
Z	(0)	1	0	0	0	1/5	7/5	17
x_1	(1)	0	1	0	0	-1/5	3/5	3
x_3	(2)	0	0	0	1	1/5	-3/5	1
x_2	(3)	0	0	1	0	2/5	-1/5	4

Since all the coefficients in Eq. (0) are nonnegative, the current BF is optimal. The optimal solution is (3, 4, 1, 0, 0) with Z = 17.

Example for Section 4.6

Consider the following problem.

Minimize $Z = 3x_1 + 2x_2 + x_3$,

subject to

$$x_1 + x_2 = 7$$
$$3x_1 + x_2 + x_3 \geq 10$$

and

$$x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0.$$

After introducing the surplus variable x_4, the above linear programming problem becomes

Minimize $Z = 3x_1 + 2x_2 + x_3$,

subject to

$$x_1 + x_2 = 7$$
$$3x_1 + x_2 + x_3 - x_4 = 10$$

and

$$x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0, \quad x_4 \geq 0.$$

(a) Using the Big M method, work through the simplex method step by step to solve the problem.

After introducing the artificial variables $\bar{x}_5$ and $\bar{x}_6$, the form of the problem becomes

$$\text{Minimize} \quad Z = 3x_1 + 2x_2 + x_3 + M\bar{x}_5 + M\bar{x}_6,$$

subject to

$$\begin{aligned} x_1 + x_2 \qquad\qquad\quad + \bar{x}_5 \qquad &= 7 \\ 3x_1 + x_2 + x_3 - x_4 \qquad + \bar{x}_6 &= 10 \end{aligned}$$

and

$$x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0, \quad x_4 \geq 0, \quad \bar{x}_5 \geq 0, \quad \bar{x}_6 \geq 0.$$

where M represents a huge positive number.

Converting from minimization to maximization, we have

$$\text{Maximize} \quad (-Z) = -3x_1 - 2x_2 - x_3 - M\bar{x}_5 - M\bar{x}_6$$

subject to

$$\begin{aligned} x_1 + x_2 \qquad\qquad\quad + \bar{x}_5 \qquad &= 7 \\ 3x_1 + x_2 + x_3 - x_4 \qquad + \bar{x}_6 &= 10 \end{aligned}$$

and

$$x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0, \quad x_4 \geq 0, \quad \bar{x}_5 \geq 0, \quad \bar{x}_6 \geq 0.$$

Let $\bar{x}_5$ and $\bar{x}_6$ be the basic variables. The corresponding simplex tableau is as follows.

Basic Variable	**Eq**	**Coefficient of:**							**Right Side**
		Z	x_1	x_2	x_3	x_4	$\bar{x}_5$	$\bar{x}_6$	
Z	(0)	-1	-4M+3	-2M+2	-M+1	M	0	0	-17M
$\bar{x}_5$	(1)	0	$\boxed{1}$	1	0	0	1	0	7
$\bar{x}_6$	(2)	0	3	1	1	-1	0	1	10

Iteration 1:

Since M is a huge positive number, the most negative coefficient in Eq. (0) is –4M+3 for x_1. Therefore, the nonbasic variable x_1 is to be changed to a basic variable.

Performing the minimum ratio test on x_1, the leaving basic variable is $\bar{x}_6$. After restoring proper form from Gaussian elimination, the new simplex tableau with basic variables $\bar{x}_5$ and x_1 becomes

Basic Variable	Eq	Coefficient of:							Right Side
		Z	x_1	x_2	x_3	x_4	$\bar{x}_5$	$\bar{x}_6$	
Z	(0)	-1	0	-(2/3)M+1	(1/3)M	-(1/3)M+1	0	(4/3)M-1	-(11/3)M-10
$\bar{x}_5$	(1)	0	0	2/3	-1/3	1/3	1	-1/3	11/3
x_1	(2)	0	1	1/3	1/3	-1/3	0	1/3	10/3

Iteration 2:

The most negative coefficient in Eq. (0) now is –(2/3)M+1 for x_2, so the nonbasic variable x_2 is to be changed to a basic variable. Performing the minimum ratio test on x_2, the leaving basic variable is $\bar{x}_5$. The new simplex tableau with basic variables x_2 and x_1 becomes

Basic Variable	Eq	Coefficient of:							Right Side
		Z	x_1	x_2	x_3	x_4	$\bar{x}_5$	$\bar{x}_6$	
Z	(0)	-1	0	0	0.5	0.5	M-1.5	M-0.5	-15.5
x_2	(1)	0	0	1	-0.5	0.5	1.5	-0.5	5.5
x_1	(2)	0	1	0	0.5	-0.5	-0.5	0.5	1.5

The current BF solution is optimal since all the coefficients in Eq.(0) are nonnegative. The resulting optimal solution is
$(x_1, x_2, x_3) = (1.5, 5.5, 0)$ with Z = 15.5.

(b) Using the two-phase method, work through the simplex method step by step to solve the problem.

We introduce the artificial variables $\bar{x}_5$ and $\bar{x}_6$.

The Phase 1 problem then is:

$$\text{Minimize} \quad Z = \bar{x}_5 + \bar{x}_6,$$

subject to

$$\begin{aligned} x_1 + x_2 \qquad\qquad\qquad + \bar{x}_5 \qquad\quad &= 7 \\ 3x_1 + x_2 + x_3 - x_4 \qquad\qquad + \bar{x}_6 &= 10 \end{aligned}$$

and

$$x_1 \ge 0, \quad x_2 \ge 0, \quad x_3 \ge 0, \quad x_4 \ge 0, \quad \bar{x}_5 \ge 0, \quad \bar{x}_6 \ge 0,$$

or equivalently,

$$\text{Maximize} \quad (-Z) = -\bar{x}_5 - \bar{x}_6,$$

subject to

$$\begin{aligned} x_1 + x_2 \qquad\qquad\qquad + \bar{x}_5 \qquad\quad &= 7 \\ 3x_1 + x_2 + x_3 - x_4 \qquad\qquad + \bar{x}_6 &= 10 \end{aligned}$$

and

$$x_1 \ge 0, \quad x_2 \ge 0, \quad x_3 \ge 0, \quad x_4 \ge 0, \quad \bar{x}_5 \ge 0, \quad \bar{x}_6 \ge 0.$$

Let $\bar{x}_5$ and $\bar{x}_6$ be the basic variables. The current system of equations is

$$\begin{aligned} &(0) \quad -Z \qquad\qquad\qquad\qquad\quad + \bar{x}_5 + \bar{x}_6 = 0 \\ &(1) \qquad x_1 + x_2 \qquad\qquad\quad + \bar{x}_5 \qquad\ = 7 \\ &(2) \qquad 3x_1 + x_2 + x_3 - x_4 \qquad + \bar{x}_6 = 10 \end{aligned}$$

To restore proper form from Gaussian elimination, we need to eliminate the basic variables, $\bar{x}_5$ and $\bar{x}_6$, from Eq. (0). This is done by subtracting both Eq. (1) and Eq. (2) from Eq. (0), which yields the following new Eq. (0).

$$(0) \quad -Z - 4x_1 - 2x_2 - x_3 + x_4 = -17.$$

Using the initial system of equations with this Eq. (0) to get started, the simplex method yields the following sequence of simplex tableaux for the Phase 1 problem.

Iteration	Basic Variable	Eq	Coefficient of:							Right Side
			Z	x_1	x_2	x_3	x_4	$\bar{x}_5$	$\bar{x}_6$	
(0)	Z	(0)	-1	-4	-2	-1	1	0	0	-17
	$\bar{x}_5$	(1)	0	1	1	0	0	1	0	7
	$\bar{x}_6$	(2)	0	3	1	1	-1	0	1	10
(1)	Z	(0)	-1	0	-2/3	1/3	-1/3	0	4/3	-11/3
	$\bar{x}_5$	(1)	0	0	2/3	-1/3	1/3	1	-1/3	11/3
	x_1	(2)	0	1	1/3	1/3	-1/3	0	1/3	10/3
(2)	Z	(0)	-1	0	0	0	0	1	1	0
	x_2	(1)	0	0	1	-0.5	0.5	1.5	-0.5	5.5
	x_1	(2)	0	1	0	0.5	-0.5	-0.5	0.5	1.5

Therefore, the optimal solution for the Phase 1 problem is
$(x_1, x_2, x_3, x_4, \bar{x}_5, \bar{x}_6) = (1.5, 5.5, 0, 0, 0, 0)$ with $Z = 0$.

Now using the original objective function, the Phase 2 problem is

Minimize $\quad Z = 3x_1 + 2x_2 + x_3,$

subject to

$$x_1 + x_2 = 7$$
$$3x_1 + x_2 + x_3 - x_4 = 10$$

and

$$x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0, \quad x_4 \geq 0,$$

or equivalently,

Maximize $\quad (-Z) = -3x_1 - 2x_2 - x_3,$

subject to

$$x_1 + x_2 = 7$$
$$3x_1 + x_2 + x_3 - x_4 = 10$$

and

$$x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0, \quad x_4 \geq 0.$$

Using the optimal solution for the Phase 1 problem (after eliminating the artificial variables, which are no longer needed) as the initial BF solution for the Phase 2 problem, we obtain the following simplex tableau.

Basic Variable	Eq	Coefficient of:					Right Side
		Z	x_1	x_2	x_3	x_4	
Z	(0)	-1	0	0	0.5	-0.5	-15.5
x_2	(1)	0	0	1	-0.5	0.5	5.5
x_1	(2)	0	1	0	0.5	-0.5	1.5

This tableau reveals that the current BF solution is also optimal. Hence, the optimal solution is $(x_1, x_2, x_3, x_4) = (1.5, 5.5, 0, 0)$ with Z = 15.5.

(c) Compare the sequence of BF solutions obtained in parts (a) and (b). Which of these solutions are feasible only for the artificial problem obtained by introducing artificial variables and which are actually feasible for the real problem?.

The sequence of BF solutions obtained in part (a) and (b) are the same. All these BF solutions except the last one are feasible only for the artificial problem obtained by introducing artificial variables. Only the final BF solution represents a feasible solution for the real problem.

(d) Use a software package based on the simplex method to solve the problem.

Using Solver (which employs the simplex method) to solve a spreadsheet formulation of the problem yields the following optimal solution:
$(x_1, x_2, x_3) = (1.5, 5.5, 0)$ with $Z = 15.5$, as displayed next..

	A	B	C	D	E	F	G
1		X1	X2	X3			
2	Unit Cost	3	2	1			
3							
4					Totals		Limit
5	Constraint 1	1	1	0	7	=	7
6	Constraint 2	3	1	1	10	>=	10
7							
8							Total Cost
9	Solution	1.5	5.5	0			15.5

Solver Parameters

Set Objective Cell: TotalCost

To: Min

By Changing Variable Cells:

Solution

Subject to the Constraints:

E5 = G5

E6 >= G6

Solver Options:

Make Variables Nonnegative

Solving Method: Simplex LP

Example for Section 4.7

Reconsider the linear programming model previously analyzed in the example for Sections 4.1, 4.2, 4.3, and 4.4. This model is again shown below, where the right-hand sides of the functional constraints now are interpreted as the amounts available of the respective resources.

Maximize $Z = 3x_1 + 2x_2$,

subject to

(1) $x_1 \leq 4$ (resource 1)

(2) $x_1 + 3x_2 \leq 15$ (resource 2)

(3) $2x_1 + x_2 \leq 10$ (resource 3)

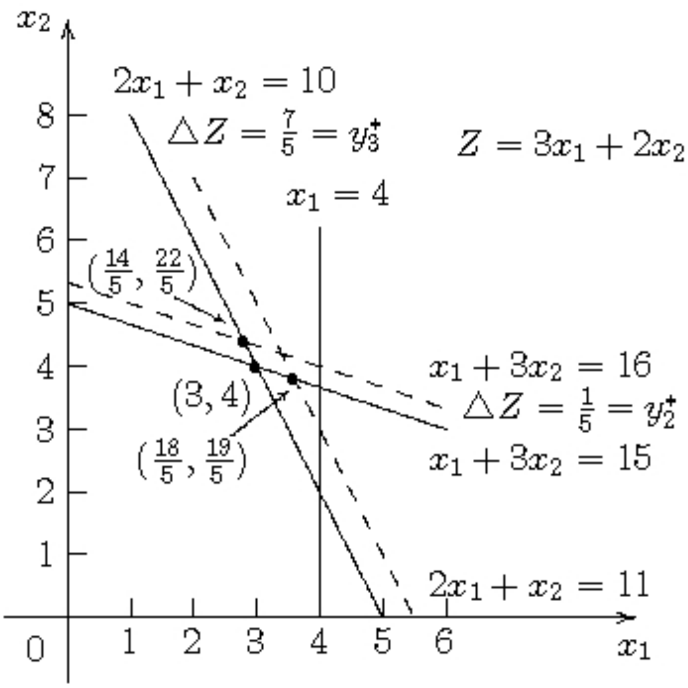
and

$x_1 \geq 0, \quad x_2 \geq 0.$

The optimal solution is $(x_1, x_2) = (3, 4)$ with $Z = 17$.

(a) Use graphical analysis as in Fig. 4.8 to determine the shadow prices for the respective resources.

The following figure summarizes the analysis.



From the figure, we can see the following.

Constraint (1) ($x_1 \leq 4$): Constraint (1) is not binding at the optimal solution (3, 4), since a small change in $b_1 = 4$ will not change the optimal value of Z. Hence, $y_1^* = 0$.

Constraint (2) ($x_1 + 3x_2 \leq 15$): Constraint (2) is binding at (3, 4). We increase b_2 from 15 to 16. The new optimal solution is (14/5, 22/5) with Z = 3*(14/5) + 2*(22/5) = 86/5.

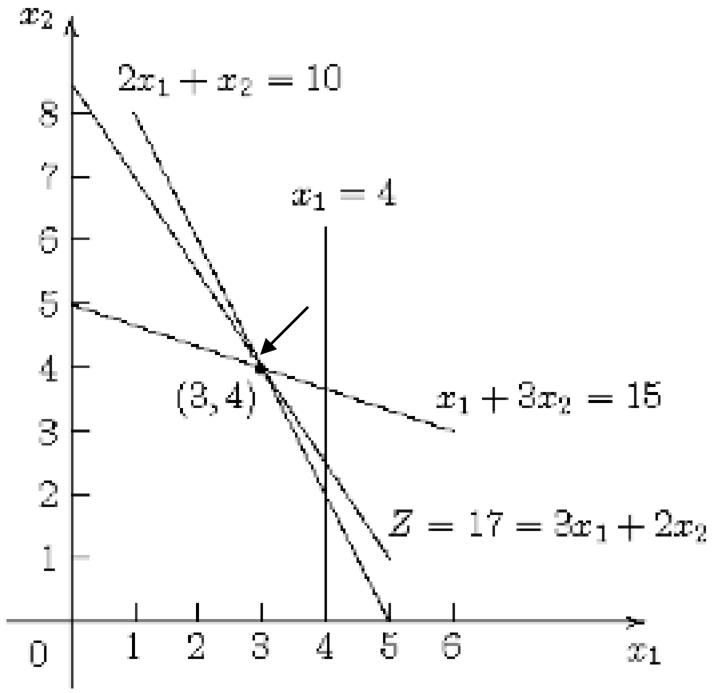
$$y_2^* = \Delta Z = 86/5 - 17 = 1/5.$$

Constraint (3) ($2x_1 + x_2 \leq 10$): Constraint (3) is binding at (3, 4). We increase b_3 from 10 to 11. The new optimal solution is (18/5, 19/5) with Z = 3*(18/5) + 2*(19/5) = 92/5.

$$y_3^* = \Delta Z = 92/5 - 17 = 1.4.$$

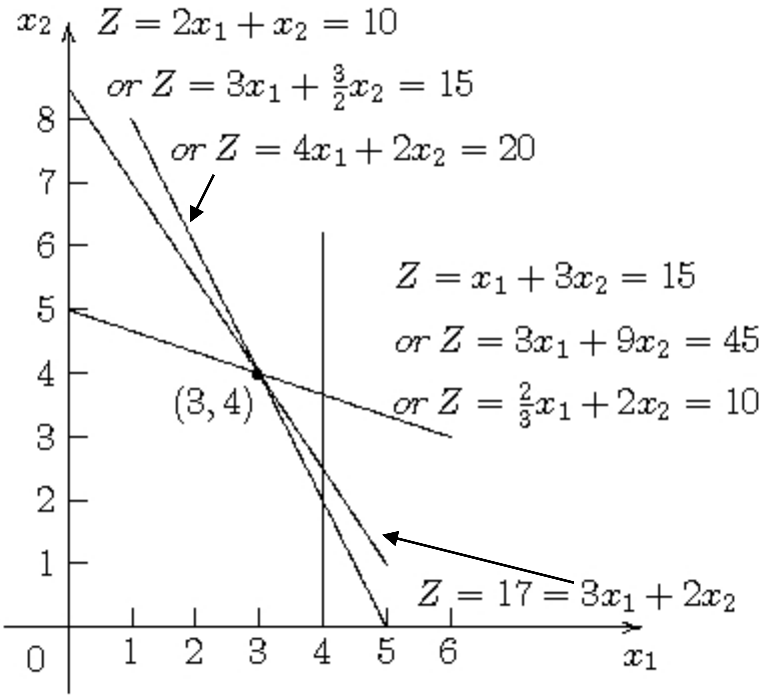
(b) Use graphical analysis to perform sensitivity analysis on this model. In particular, check each parameter of the model to determine whether it is a *sensitive* parameter (a parameter whose value cannot be changed without changing the optimal solution) by examining the graph that identifies the optimal solution.

From part (a), we know that b_1 is not a sensitive parameter, while b_2 and b_3 are sensitive parameters. Similarly, since constraint (1) is not binding at the optimal solution (3, 4), the coefficients $a_{11} = 1$ and $a_{12} = 0$ of constraint (1) are not sensitive. Since constraint (2) and (3) are binding at the optimal solution, the coefficients $a_{21} = 1$, $a_{22} = 3$, $a_{31} = 2$, and $a_{32} = 1$ are sensitive parameters. From the following figure, we can see that at the optimal solution, the objective function $Z = 3x_1 + 2x_2$ is not parallel to constraint (2) or constraint (3). Hence, the coefficients $c_1 = 3$ and $c_2 = 2$ are not sensitive parameters.



(c) Use graphical analysis as in Fig. 4.9 to determine the allowable range for each c_j value (coefficient of x_j in the objective function) over which the current optimal solution will remain optimal.

From the following graph, we can see that the current optimal solution will remain optimal for $2/3 \le c_1 \le 4$ (with c_2 fixed at 2) and $3/2 \le c_2 \le 9$ (with c_1 fixed at 3), since the objective function line will rotate around to coincide with one of the constraint boundary lines at each of the endpoints of these intervals.



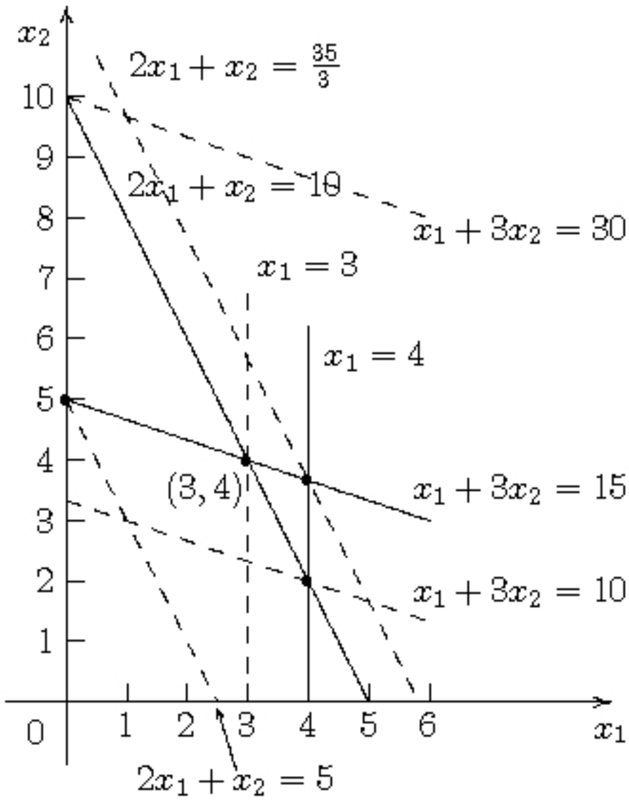
(d) Changing just one b_i value (the right-hand side of functional constraint *i*) will shift the corresponding constraint boundary. If the current optimal CPF solution lies on this constraint boundary, this CPF solution also will shift. Use graphical analysis to determine the allowable range for each b_i value over which this CPF solution will remain feasible.

From the following graph, we can see the following.

For Constraint (1) ($x_1 \leq 4$): The allowable range for b_1 is $3 \leq b_1 \leq \infty$ since (3, 4) remains feasible over this range.

For Constraint (2) ($x_1 + 3x_2 \leq 15$): The allowable range for b_2 is $10 \leq b_2 \leq 30$. For $b_2 < 10$, the intersection of $x_1 + 3x_2 = b_2$ and $2x_1 + x_2 = 10$ violates the $x_1 \leq 4$ constraint. For $b_2 > 30$, this intersection violates the $x_1 \geq 0$ constraint.

For Constraint (3) ($2x_1 + x_2 \leq 10$): The allowable range for b_3 is $5 \leq b_3 \leq 35/3$. For $b_3 < 5$, the intersection of $x_1 + 3x_2 = 15$ and $2x_1 + x_2 = b_3$ violates the $x_1 \geq 0$ constraint. For $b_3 > 35/3$, this intersection violates the $x_1 \leq 4$ constraint.



(e) Verify your answers in parts (a), (c), and (d) by using a computer package based on the simplex method to solve the problem and then to generate sensitivity analysis information.

Using Solver (which employs the simplex method), the sensitivity analysis report (which verifies these answers) is generated, as shown after the following spreadsheet.

	A	B	C	D	E	F
1		X1	X2			
2	Unit Profit	3	2			
3						
4				Totals		Limit
5	Constraint 1	1	0	3	<=	4
6	Constraint 2	1	3	15	<=	15
7	Constraint 3	2	1	10	<=	10
8						
9						Total Profit
10	Solution	3	4			17

Variable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
B10	Solution X1	3	0	3	1	2.33333
C10	Solution X2	4	0	2	7	0.5

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
D5	Constraint 1 Totals	3	0	4	1E+30	1
D6	Constraint 2 Totals	15	0.2	15	15	5
D7	Constraint 3 Totals	10	1.4	10	1.66667	5

Example for Section 4.9

Use the interior-point algorithm in your OR Courseware to solve the following model (previously analyzed in the examples for Sections 4.1, 4.2, 4.3, 4.4, and 4.7). Choose $\alpha = 0.5$ from the Option menu, use $(x_1, x_2) = (0.1, 0.4)$ as the initial trial solution, and run 15 iterations. Draw a graph of the feasible region, and then plot the trajectory of the trial solutions through this feasible region.

Maximize $\quad Z = 3x_1 + 2x_2,$

subject to

$$\begin{aligned} x_1 \qquad\quad &\le 4 \\ x_1 + 3x_2 &\le 15 \\ 2x_1 + x_2 &\le 10 \end{aligned}$$

and

$$x_1 \ge 0, \quad x_2 \ge 0.$$

We use the IOR tutorial with $\alpha = 0.5$, which generates the following output:

Solve Automatically by the Interior Point Algorithm: (Alpha = 0.5)

Iteration	x_1	x_2	Z
0	0.1	0.4	1.1
1	0.30854	2.61382	6.15325
2	0.35481	3.74006	8.54455
3	0.42446	4.28768	9.84874
4	0.60705	4.51223	10.8456
5	1.3213	4.41686	12.7976
6	2.19583	4.13808	12.7976
7	2.63337	3.99813	15.8964
8	2.85	3.93243	16.4149
9	2.95476	3.90669	16.6777
10	3.00139	3.90533	16.8148
11	3.01647	3.92112	16.8916
12	3.01519	3.94665	16.9389
13	3.00904	3.97043	16.968
14	3.0046	3.98506	16.9839
15	3.0023	3.99253	16.992

The trajectory of the trial solutions through the feasible region is shown in the following figure.