Solved Examples for Chapter 20

Example for Sections 20.3 and 20.4

You need to generate 10 random observations from the probability distribution

$$P\{X = n\} = \begin{cases} 0.10 & \text{if } n = 0, 1, 2, \ldots, 9 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Prepare to do this by generating 16 random integer numbers from the mixed congruential generator, $x_{n+1} \equiv (5x_n + 3)$ (modulo 16) and $x_0 = 1$.

Using the mixed congruential generator $x_{n+1} = (5x_n+3)$(modulo 16) and $x_0 = 1$, we generate the following 16 random numbers:

$X = \{1, 8, 11, 10, 5, 12, 15, 14, 9, 0, 3, 2, 13, 4, 7, 6\}$.

To illustrate,

$$x_1 = 5x_0 + 3 = 5(1) + 3 = 8,$$

$$\frac{5x_1 + 3}{16} = \frac{5(8) + 3}{16} = 2 + \frac{11}{16}, \qquad \text{so } x_2 = 11,$$

$$\frac{5x_2 + 3}{16} = \frac{5(11) + 3}{16} = 3 + \frac{10}{16}, \qquad \text{so } x_3 = 10,$$

etc.

(b) Use the single-digit random integer numbers from part (a) to generate the desired random observations.

Using only the single-digit random numbers from part (a), we obtain the following random observations:

$$X = \{1, 8, 5, 9, 0, 3, 2, 4, 7, 6\}.$$

(c) Note that once a particular value of X is generated in part (b), it can never be repeated because each of the 16 possible random integer numbers is generated exactly once in part (a). In which ways does this violate the desirable properties of random observations? What change would you make in what was done in parts (a) and (b) to alleviate this problem?

The random numbers generated above are not independent of each other. The change we would make is to use a vastly large modulo in part (a) and then use only the last digit of the resulting random numbers.

(d) Now convert the first 10 random integer numbers from part (a) to (approximate) uniform random numbers, and then apply the inverse transformation method to obtain the desired random observations.

As indicated in Sec. 20.3, we use the following formula to obtain the (approximated) uniform random number for each of the first 10 random integer numbers from part (a):

$$\text{Uniform random number} = \frac{\text{random integer number } + \ 1/2}{16}.$$

The inverse transformation method then leads to using the first digit after the decimal point of the uniform random number as the random observation from the probability distribution. The following table shows the resulting random observations.

Random Integer Number	Uniform Random Number	Random observation by inverse transformation method
1	0.0938	0
8	0.5313	5
11	0.7188	7
10	0.6563	6
5	0.3438	3
12	0.7813	7
15	0.9688	9
14	0.9063	9
9	0.5938	5
0	0.0313	0

(e) Does the procedure prescribed in part (d) actually give a probability of 0.10 of generating each of the 10 possible values of X each time? Explain. What change would you make in what was done in parts (a) and (d) to alleviate this problem?

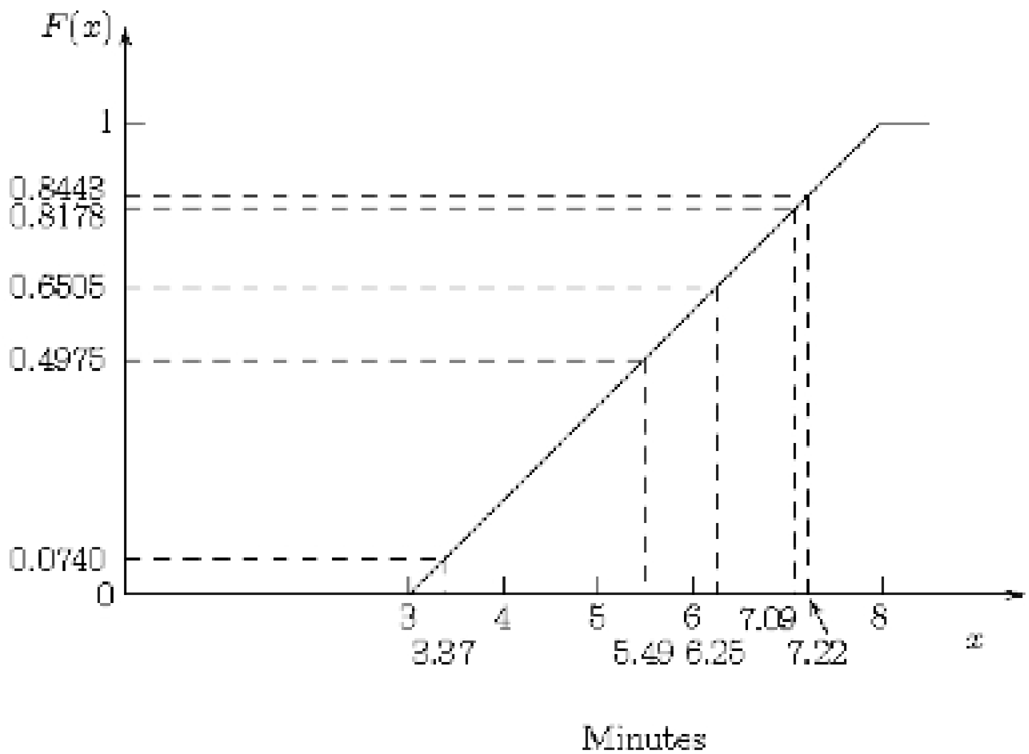
No. With a modulo of 16, the procedure in part (a) only generates random integer numbers between 0 and 15. The procedure in part (d) then converts these numbers into integers between 0 and 9. The result is that some "random" observations between 0 and 9 have two random integer numbers that will convert into that observation while other "random" observations have only one such random integer number. Therefore, we are not getting $P\{x = n\} = 0.1$ for $n = 0, 1, \cdots$. We can use a vastly large modulo and then apply part (d) to alleviate the problem.

Example for Section 20.4

Eddie's Bicycle Shop has a thriving business repairing bicycles. Trisha runs the reception area where customers check in their bicycles to be repaired and then later pick up their bicycles and pay their bills. She estimates that the time required to serve a customer on each visit has a uniform distribution between 3 minutes and 8 minutes.

Apply the inverse transformation method as indicated below to simulate the service times for five customers by using the following five uniform random numbers: 0.6505, 0.0740, 0.8443, 0.4975, 0.8178.

(a) Apply this method graphically.



Thus, the five simulated service times shown along the x-axis are 6.25, 3.37, 7.22, 5.49, and 7.09, in that order.

(b) Apply this method algebraically.

Given a random number r, the corresponding service time is calculated as follows:

$$F(x) = \frac{x-3}{5} = r \qquad \Rightarrow \qquad x = 5r + 3.$$

The service time for each random number is given in the following table.

Random number r	**Service Time** x
0.6505	6.25
0.0740	3.37
0.8443	7.22
0.4975	5.49
0.8178	7.09

(c) Calculate the average of the five service times and compare it to the mean of the service-time distribution.

The average of the five service times is

$$\frac{6.25+3.37+7.22+5.49+7.09}{5}=5.88,$$

which is higher than the mean of the service-time distribution, 5.5.

(d) Use Excel to generate 500 random observations and calculate the average. Compare this average to the mean of the service-time distribution.

Results will vary. As shown on the following spreadsheet, we used the ***Rand()*** function in Excel to generate uniform random numbers between 0 and 1. The 500-day simulation yielded an average service time of 5.487 days, which is very close to the true mean of 5.5 days.

	A	B	C	D	E
1					
2		**Day**	**Random Number**	**Service Time**	
3		1	0.4433	5.22	
4		2	0.8833	7.42	
5		3	0.2622	4.31	
6		4	0.0038	3.02	
7		5	0.3911	4.96	
8		6	0.3701	4.85	
9		7	0.2905	4.45	
10		8	0.9392	7.70	
11		9	0.8184	7.09	
12		10	0.6835	6.42	
492		490	0.5648	5.82	
493		491	0.0902	3.45	
494		492	0.4244	5.12	
495		493	0.0972	3.49	
496		494	0.9637	7.82	
497		495	0.0962	3.48	
498		496	0.2008	4.00	
499		497	0.0191	3.10	
500		498	0.5196	5.60	
501		499	0.2375	4.19	
502		500	0.0527	3.26	
503					
504			**Average Service Time**	5.49	
505					