Solved Examples for Chapter 15

Example for Section 15.2

For the game having the following payoff table, determine the optimal strategy for each player by successively eliminating dominated strategies. (Indicate the order in which you eliminated strategies.)

		Player 2		
	Strategy	**1**	**2**	**3**
	1	1	2	0
Player 1	2	2	-3	-2
	3	0	3	-1

Since $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} > \begin{bmatrix} 0 \\ -2 \\ -1 \end{bmatrix}$, strategy 1 of player 2 is dominated by strategy 3.
Hence, we eliminate strategy 1 from Player 2.

Since $\begin{bmatrix} -3 & -2 \end{bmatrix} < \begin{bmatrix} 2 & 0 \end{bmatrix}$ and $\begin{bmatrix} -3 & -2 \end{bmatrix} < \begin{bmatrix} 3 & -1 \end{bmatrix}$, strategy 2 of Player 1 is dominated by both strategy 1 and strategy 3. Hence, we eliminate strategy 2 from Player 1.

Since $\begin{bmatrix} 2 \\ 3 \end{bmatrix} > \begin{bmatrix} 0 \\ -1 \end{bmatrix}$, strategy 2 of player 2 is dominated by strategy 3.
Hence, we eliminate strategy 2 from Player 2.

Since 0 > -1, strategy 3 of Player 1 is dominated by strategy 1, so we eliminate strategy 3 from Player 1.

Therefore, Player 1 will choose strategy 1 and Player 2 will choose strategy 3, resulting in a payoff of 0 to both players.

Example for Section 15.4

For the game having the following payoff table, use the graphical procedure described in Sec. 15.4 to determine the value of the game and the optimal mixed strategy for each player according to the minimax criterion.

	Strategy	Player 2 1	 2	 3
	1	1	-1	3
Player 1	2	0	4	1
	3	3	-2	5
	4	-3	6	-2

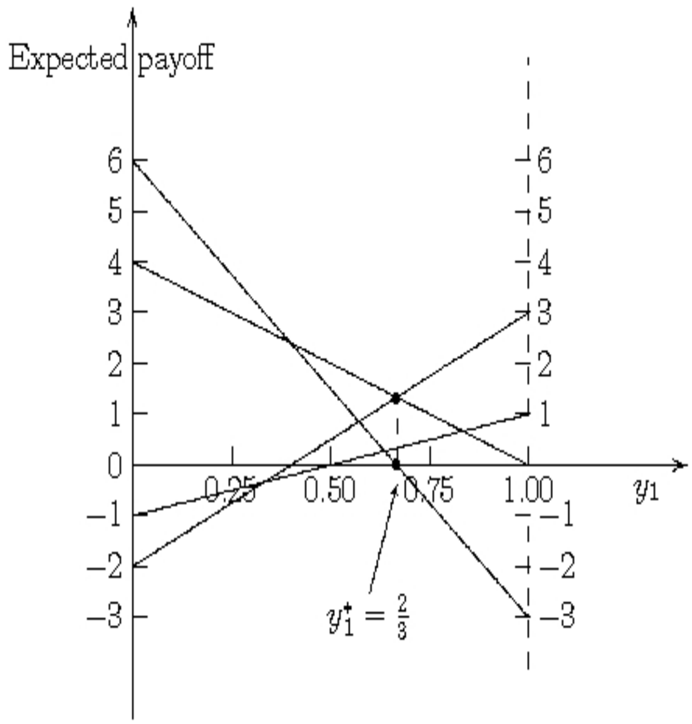
Since strategy 3 of Player 2 is dominated by strategy 1, we can eliminate strategy 3 from Player 2 and reduce the payoff table to

		Probability	Player 2 y_1	 $1-y_1$
	Probability	**Pure Strategy**	**1**	**2**
Player 1	x_1	**1**	1	-1
	x_2	**2**	0	4
	x_3	**3**	3	-2
	x_4	**4**	-3	6

For each of the pure strategies for Player 1, the expected payoff for Player 1 is

(x_1, x_2, x_3, x_4)	Expected payoff
(1, 0, 0, 0)	$y_1 - (1 - y_1) = 2y_1 - 1$
(0, 1, 0, 0)	$4(1 - y_1) = -4y_1 + 4$
(0, 0, 1, 0)	$3y_1 - 2(1 - y_1) = 5y_1 - 2$
(0, 0, 0, 1)	$-3y_1 + 6(1 - y_1) = -9y_1 + 6$

The following graph plots these expected payoffs for Player 1 over the range $0 \le y \le 1$ for each of the four pure strategies for Player 1.



When Player 2 chooses a value of y_1 and Player 1 chooses some mixed strategy (x_1, x_2, x_3, x_4), the expected payoff for Player 1 (and so the expected loss for Player 2) will be the corresponding weighted average of the points at y_1 on the four lines. Thus,

$$\text{Expected payoff for Player 1} = x_1(2y_1 - 1) + x_2(-4y_1 + 4) + x_3(5y_1 - 2) + x_4(-9y_1 + 6).$$

Player 1 wants to maximize this expected payoff. Given y_1, Player 1 can do this by choosing the pure strategy that corresponds to the top line for that y_1 in the graph. Since the expected payoff for Player 1 is the expected loss for Player 2, the minimax criterion says that Player 2 should minimize his maximum expected loss by selecting the value of y_1 where the top line reaches its lowest point. This lowest point is shown in the graph as the dot where the $(-4y_1 + 4)$ and $(5y_1 - 2)$ lines intersect.

To solve for the value of y_1 where these two lines intersect, we set

$$-4y_1 + 4 = 5y_1 - 2,$$

which yields $y_1 = 2/3$. Thus, the optimal mixed strategy for Player 2 is

$$(y_1^*, 1 - y_1^*) = \left(\frac{2}{3}, \frac{1}{3}\right).$$

The value of the game is the expected payoff for Player 1 at this point, so

$$\text{Value of the game} = -4\left(\frac{2}{3}\right) + 4 = \frac{4}{3}.$$

To find the corresponding optimal mixed strategy $(x_1^*, x_2^*, x_3^*, x_4^*)$ for Player 1, note that this strategy must enable Player 1 to achieve an expected payoff equal to the value of the game (4/3) when Player 2 uses his optimal strategy (so $y_1^* = 2/3$). This requires having zero weight on the lines in the graph that correspond to (1, 0, 0, 0) or (0, 0, 0, 1), so

$$x_1^* = 0 \qquad \text{and} \qquad x_4^* = 0.$$

Furthermore, if Player 2 were to choose any other strategy (so $y_1 \neq 2/3$), Player 1 must be able to achieve an expected payoff at least as large as the value of the game, so

$$x_2^*(-4y_1 + 4) + x_3^*(5y_1 - 2) \begin{cases} \geq \frac{4}{3} & \text{for } 0 \leq y_1 \leq 1 \\ = \frac{4}{3} & \text{for } y_1 = \frac{2}{3}. \end{cases}$$

Since x_2^* and x_3^* are numbers, the left-hand side gives the equation of a line over $0 \leq y_1 \leq 1$, which must be a horizontal line to satisfy the $\geq 4/3$ condition on both sides of $y_1 = 2/3$, so $\geq 4/3$ can be replaced by $= 4/3$ for $0 \leq y_1 \leq 1$. Choosing any two values of $y_1 \neq 2/3$, say $y_1 = 0$ and $y_1 = 1$, this means that $x_2^*(-4y_1 + 4) + x_3^*(5y_1 -2) = 4/3$ reduces to

$$4x_2^* \; -2x_3^* = \frac{4}{3} \qquad \text{for } y_1 = 0,$$

$$3x_3^* = \frac{4}{3} \qquad \text{for } y_1 = 1.$$

These two equations yield $x_2^* = 5/9$ and $x_3^* = 4/9$. Therefore, the optimal mixed strategy for Player 1 is

$$(x_1^*, x_2^*, x_3^*, x_4^*) = (0, \frac{5}{9}, \frac{4}{9}, 0).$$