Solved Examples for Chapter 12

Example for Section 12.1

Pawtucket University is planning to buy new copier machines for its library. Three members of its Operations Research Department are analyzing what to buy. They are considering two different models: Model A, a high-speed copier, and Model B, a lower-speed but less expensive copier. Model A can handle 20,000 copies a day, and costs $6,000. Model B can handle 10,000 copies a day, but costs only $4,000. They would like to have at least six copiers so that they can spread them throughout the library. They also would like to have at least one high-speed copier. Finally, the copiers need to be able to handle a capacity of at least 75,000 copies per day. The objective is to determine the mix of these two copiers that will handle all these requirements at minimum cost.

(a) Formulate an IP model for this problem.

Let A be the number of Model A copiers to buy.
Let B be the number of Model B copiers to buy.
The problem formulation is as follows:

Minimize $C = 6{,}000A + 4{,}000B$,
subject to

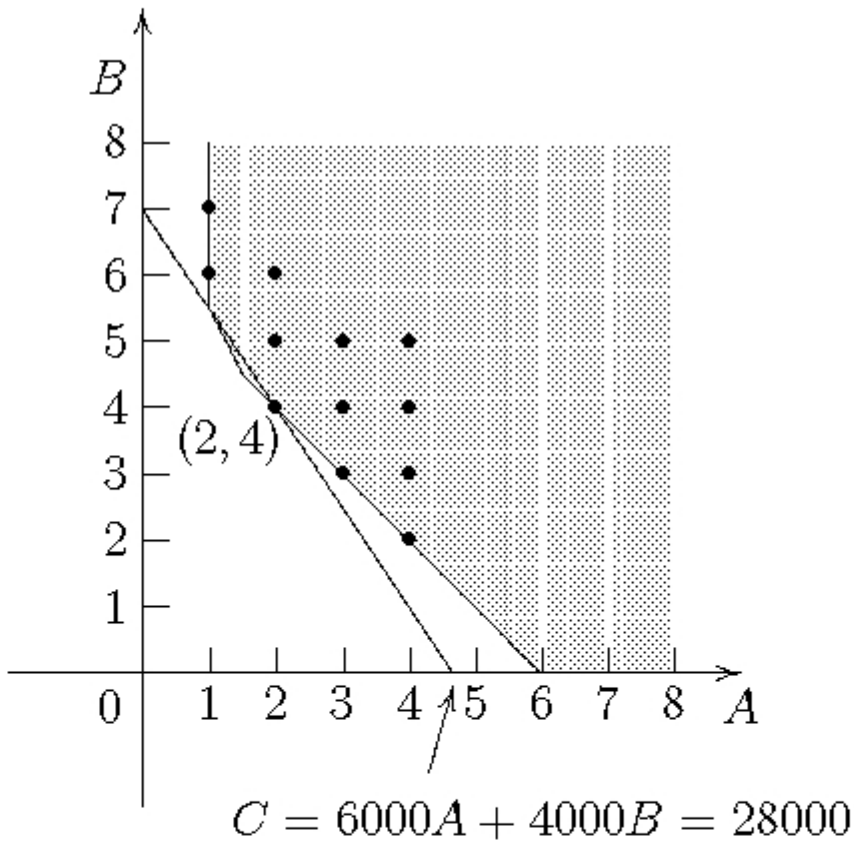
$$
\begin{aligned}
A + B &\geq 6 \\
A &\geq 1 \\
20{,}000A + 10{,}000B &\geq 75{,}000
\end{aligned}
$$

and

$$A \geq 0,\ B \geq 0,$$
$$A, B \text{ are integers.}$$

(b) Use a graphical approach to solve this model.

In the following figure, the dots indicate feasible points. As we can see, (A, B) = (2, 4) is the optimal solution with a minimum cost of $28,000.



(c) Use the computer to solve the model.

We use Excel to solve this problem, as shown in the following figure. Solver finds the optimal solution, $(A, B) = (2, 4)$ with a minimum cost of $\$28{,}000$.

	Model A	Model B			
Unit Cost	$6,000	$4,000			
					Required
			Totals		Amount
Total Need	1	1	6	>=	6
High-Speed Need	1	0	2	>=	1
Capacity	20,000	10,000	80000	>=	75,000
					Total Cost
Number to Purchase	2	4			$28,000

Solver Parameters

Set Objective Cell: TotalCost
To: Min
By Changing Variable Cells:
NumberToPurchase
Subject to the Constraints:
Totals >= RequiredAmount
NumberToPurchase = integer
Solver Options:
Make Variables Nonnegative
Solving Method: Simplex LP

Example for Section 12.4

A U.S. professor will be spending a short sabbatical leave at the University of Iceland. She wishes to bring all needed items with her on the airplane. After collecting the professional items she must have, she finds that airline regulations on space and weight for checked luggage will severely limit the clothes that she can take. (She plans to carry on a warm coat and then purchase a warm Icelandic sweater upon arriving in Iceland.) Clothes under consideration for checked luggage include 3 skirts, 3 slacks, 4 tops, and 3 dresses. The professor wants to maximize the number of outfits she will have in Iceland (including the special dress she will wear on the airplane). Each dress constitutes an outfit. Other outfits consist of a combination of a top and either a skirt or slacks. However, certain combinations are not fashionable and so will not qualify as an outfit.

In the following table, the combinations that will make an outfit are marked with an x.

		Top			
	1	**2**	**3**	**4**	**Icelandic Sweater**
1	x	x			x
Skirt 2	x			x	
3		x	x	x	x
1	x		x		
Slacks 2	x	x		x	x
3			x	x	x

The weight (in grams) and volume (in cubic centimeters) of each item are shown in the following table:

		Weight	**Volume**
Skirt	1	600	5,000
	2	450	3,500
	3	700	3,000
Slacks	1	600	3,500
	2	550	6,000
	3	500	4,000
Top	1	350	4,000
	2	300	3,500
	3	300	3,000
	4	450	5,000
Dress	1	600	6,000
	2	700	5,000
	3	800	4,000
Total allowed		4,000	32,000

Formulate a BIP model to choose which items of clothing to take. (Hint: After using binary decision variables to represent the individual items, you should introduce auxiliary binary variables to represent outfits involving combinations of items. Then use constraints and the objective function to ensure that these auxiliary variables have the correct values, given the values of the decision variables.)

We first define decision variables as follows.

$$\text{Let } s_i = \begin{cases} 1 & \text{if skirt i is taken,} \\ 0 & \text{if skirt i is not taken,} \quad \text{for } i = 1, 2, 3. \end{cases}$$

$$\text{Let } p_i = \begin{cases} 1 & \text{if slack i is taken,} \\ 0 & \text{if slack i is not taken,} \quad \text{for } i = 1, 2, 3. \end{cases}$$

$$\text{Let } t_i = \begin{cases} 1 & \text{if top i is taken,} \\ 0 & \text{if top i is not taken,} \quad \text{for } i = 1, 2, 3, 4. \end{cases}$$

Let $t_5 = 1$ indicate the use of the Icelandic sweater ("top #5").

$$\text{Let } d_i = \begin{cases} 1 & \text{if dress i is taken,} \\ 0 & \text{if dress i is not taken,} \quad \text{for } i = 1, 2, 3. \end{cases}$$

$$\text{Let } x_{ij} = \begin{cases} 1 & \text{if both skirt i and top j are taken,} \\ 0 & \text{otherwise,} \qquad \text{for relevant combinations of i and j.} \end{cases}$$

$$\text{Let } y_{ij} = \begin{cases} 1 & \text{if both slack i and top j are taken,} \\ 0 & \text{otherwise,} \qquad \text{for relevant combinations of i and j.} \end{cases}$$

The formulation of this problem is

$$\text{Maximize } Z = x_{11} + x_{12} + x_{15} + x_{21} + x_{24} + x_{32} + x_{33} + x_{34} + x_{35} + y_{11} + y_{13} + y_{21} + y_{22} + y_{24} + y_{25} + y_{33} + y_{34} + y_{35} + d_1 + d_2 + d_3,$$

subject to

$$600\, s_1 + 450\, s_2 + 700\, s_3 + 600\, p_1 + 550\, p_2 + 500\, p_3 + 350\, t_1 + 300\, t_2 + 300\, t_3 + 450\, t_4 + 600\, d_1 + 700\, d_2 + 800\, d_3 \le 4{,}000$$

$$5{,}000\, s_1 + 3{,}500\, s_2 + 3{,}000\, s_3 + 3{,}500\, p_1 + 6{,}000\, p_2 + 4{,}000\, p_3 + 4{,}000\, t_1 + 3{,}500\, t_2 + 3{,}000\, t_3 + 5{,}000\, t_4 + 6{,}000\, d_1 + 5{,}000\, d_2 + 4{,}000\, d_3 \le 32{,}000$$

$$x_{ij} \le 1/2\,(s_i + t_j) \quad \text{for } i = 1, 2, 3,\ j = 1, 2, 3, 4, 5$$

$$y_{ij} \le 1/2\,(p_i + t_j) \quad \text{for } i = 1, 2, 3,\ j = 1, 2, 3, 4, 5$$

s_i, p_i, d_i binary for $i = 1, 2, 3$

t_i binary for $i = 1, 2, 3, 4$

x_{ij}, y_{ij} binary for relevant combinations of i and j.

Example for Section 12.7

Consider the following IP problem.

Maximize $Z = 220x_1 + 80x_2$,
subject to

$$5x_1 + 2x_2 \leq 16$$
$$2x_1 - x_2 \leq 4$$
$$-x_1 + 2x_2 \leq 4$$

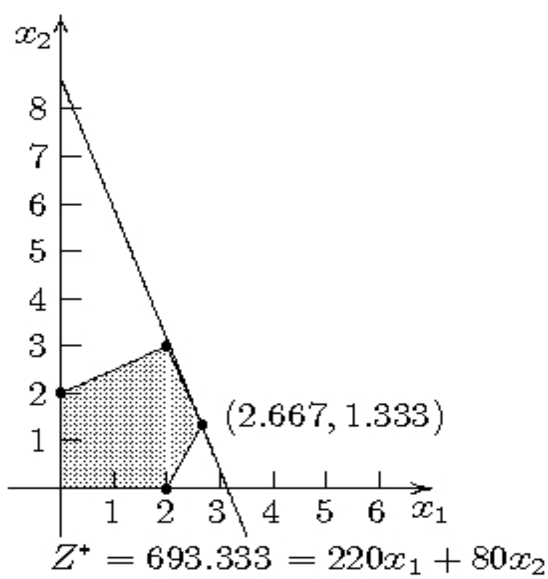
and

$$x_1 \geq 0, \quad x_2 \geq 0,$$
x_1, x_2 are integers.

(a) Use the MIP branch-and-bound algorithm presented in Sec. 12.7 to solve this problem by hand. For each subproblem, solve its LP relaxation graphically.

Initialization:

Relaxing the integer constraints, the graph below reveals that the optimal solution of the LP relaxation of the whole problem is $(x_1, x_2) = (2.667, 1.333)$ with an objective function value of Z = 693.333.



This LP-relaxation of the whole problem possesses feasible solutions and its optimal solution has noninteger values for x_1 and x_2, so the whole problem is not fathomed and we are ready to move on to the first full iteration.

Iteration 1:

The only remaining (unfathomed) subproblem at this point is the whole problem, so we use it for branching and bounding. In the above optimal solution for its LP-relaxation, both integer-restricted variables (x_1 and x_2) are noninteger, so we select the first one (x_1) to be the branching variable. Since $x_1^* = 2.667$ in this optimal solution, we will create two new subproblems below by adding the respective constraints,

$$x_1 \leq [x_1^*] \qquad \text{and} \qquad x_1 \geq [x_1^*] + 1,$$

where $[x_1^*]$ is the greatest integer $\leq x_1^*$, so $[x_1^*] = 2$.

Subproblem 1:

The original problem plus the additional constraint,

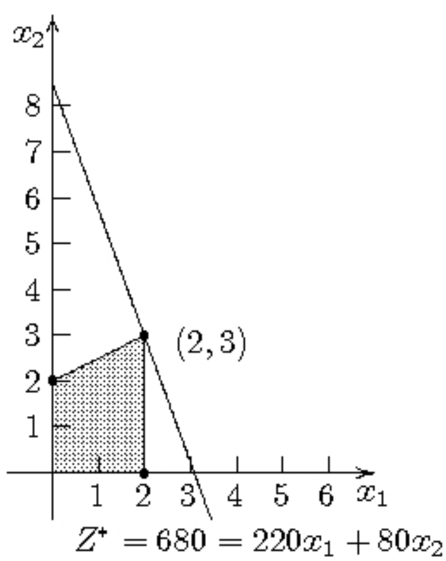
$$x_1 \leq 2.$$

Subproblem 2:

The original problem plus the additional constraint,

$$x_1 \geq 3.$$

For subproblem 1, the following graph shows that the optimal solution for its LP-relaxation is $(x_1, x_2) = (2, 3)$ with $Z = 680$.



Since the solution $(x_1, x_2) = (2, 3)$ is integer-valued, subproblem 1 is fathomed by fathoming test 3 and this solution becomes the first incumbent.

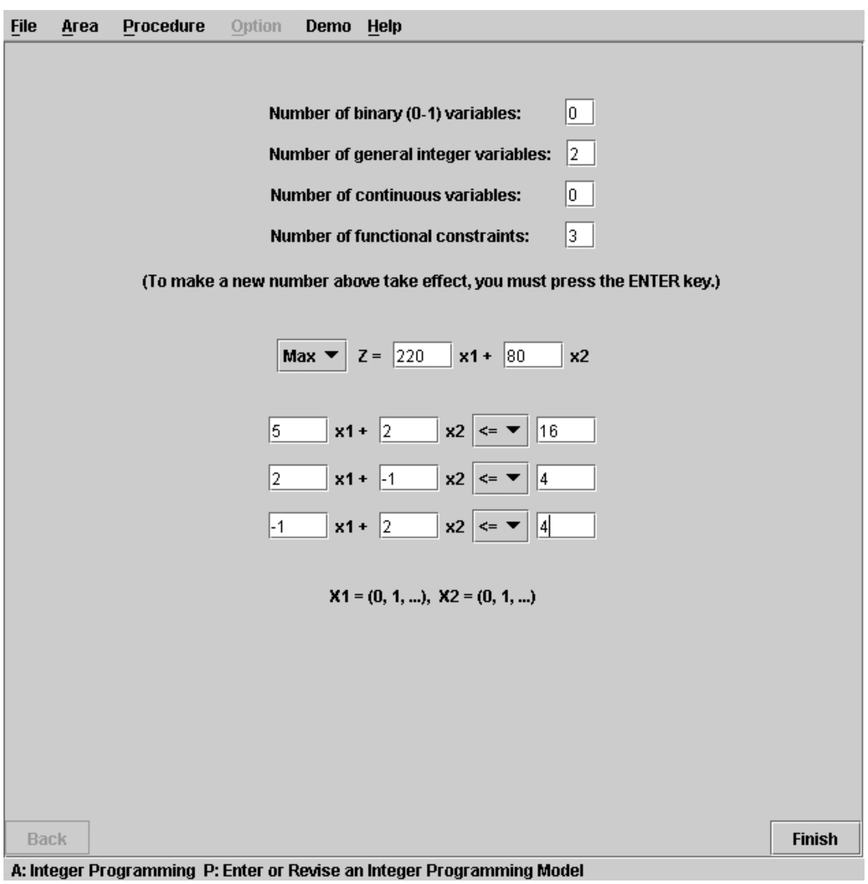
Incumbent = (2, 3) with $Z^* = 680$.

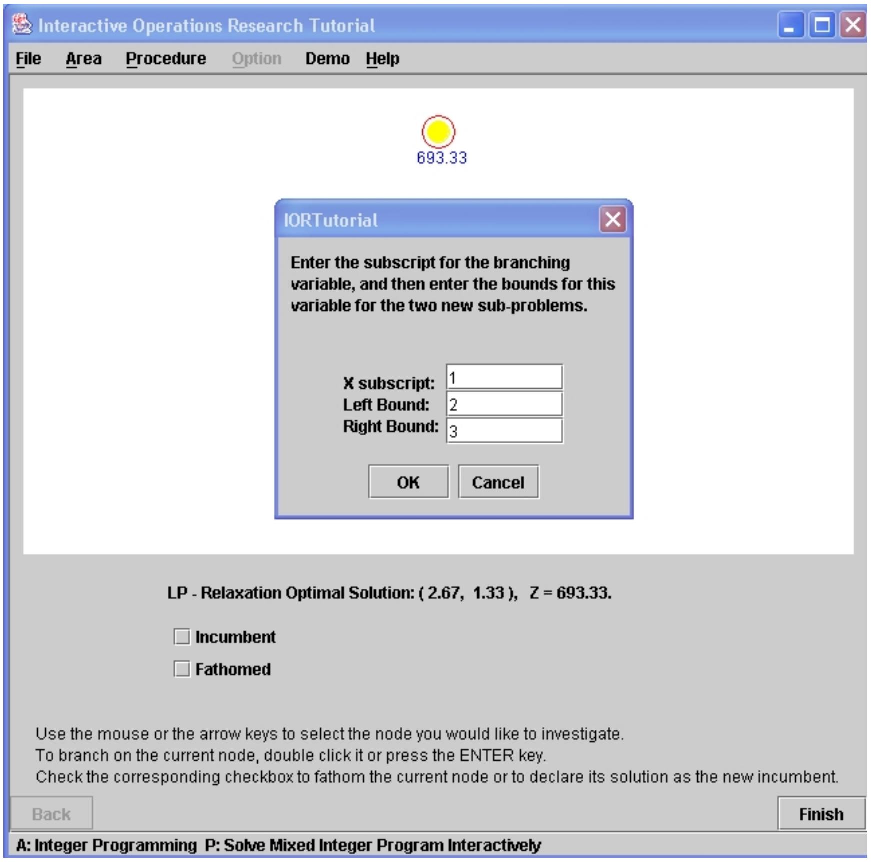
Now consider subproblem 2. Referring back to the graph of the LP-relaxation for the whole problem, it can be seen that the new constraint $x_1 \geq 3$ results in having no feasible solutions. Therefore, subproblem 2 is fathomed by fathoming test 2.

At this point, there are no remaining (unfathomed) subproblems, so the optimality test indicates that the current incumbent is optimal for the original whole problem, so no additional iterations are needed.

$$(x_1^*, x_2^*) = (2, 3) \qquad \text{with } Z^* = 680.$$

(b) Now use the interactive procedure for this algorithm in your IOR Tutorial to solve the problem.





Interactive Operations Research Tutorial
File
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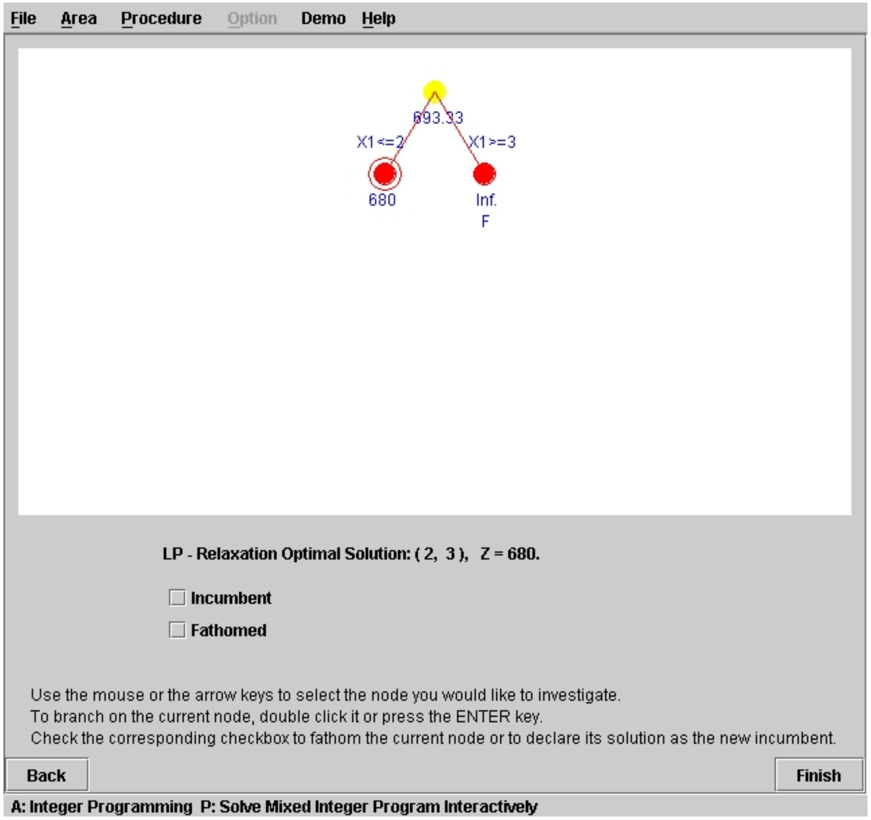
Procedure
Option
Demo
Help
693.33
IORTutorial
Enter the subscript for the branching variable, and then enter the bounds for this variable for the two new sub-problems.
X subscript: 1
Left Bound: 2
Right Bound: 3
OK
Cancel
LP - Relaxation Optimal Solution: (2.67, 1.33), Z = 693.33.
Incumbent
Fathomed
Use the mouse or the arrow keys to select the node you would like to investigate.
To branch on the current node, double click it or press the ENTER key.
Check the corresponding checkbox to fathom the current node or to declare its solution as the new incumbent.
Back
Finish
A: Integer Programming P: Solve Mixed Integer Program Interactively


File Area Procedure Option Demo Help
693.33
X1<=2
X1>=3
680
Inf.
F
LP - Relaxation Optimal Solution: (2, 3), Z = 680.
Incumbent
Fathomed
Use the mouse or the arrow keys to select the node you would like to investigate.
To branch on the current node, double click it or press the ENTER key.
Check the corresponding checkbox to fathom the current node or to declare its solution as the new incumbent.
Back
Finish
A: Integer Programming P: Solve Mixed Integer Program Interactively

(c) Check your answer by using an automatic procedure to solve the problem.

As shown in the following spreadsheet, Solver finds the optimal solution, $(x_1, x_2) = (2, 3)$ with $Z = 680$, which is identical to the solution found in part (b).

	X1	X2			
Unit Profit	220	80			
			Totals		Limit
Constraint 1	5	2	16	<=	16
Constraint 2	2	-1	1	<=	4
Constraint 3	-1	2	4	<=	4
					Total Profit
Solution	2	3			680

Solver Parameters
Set Objective Cell: TotalProfit
To: Max
By Changing Variable Cells:
Solution
Subject to the Constraints:
Totals <= Limit
Solution = integer
Solver Options:
Make Variables Nonnegative
Solving Method: Simplex LP