Solved Examples for Chapter 10

Example for Section 10.3

Sarah has just graduated from high school. As a graduation present, her parents have given her a car fund of $21,000 to help purchase and maintain a certain three-year-old used car for college. Since operating and maintenance costs go up rapidly as the car ages, Sarah's parents tell her that she will be welcome to trade in her car on another three-year-old car one or more times during the next three summers if she determines that this would minimize her total net cost. They also inform her that they will give her a *new* car in four years as a college graduation present, so she should definitely plan to trade in her car then. (These are pretty nice parents!)

The table gives the relevant data for *each* time Sarah purchases a three-year-old car. For example, if she trades in her car after two years, the next car will be in ownership year 1 during her junior year, etc.

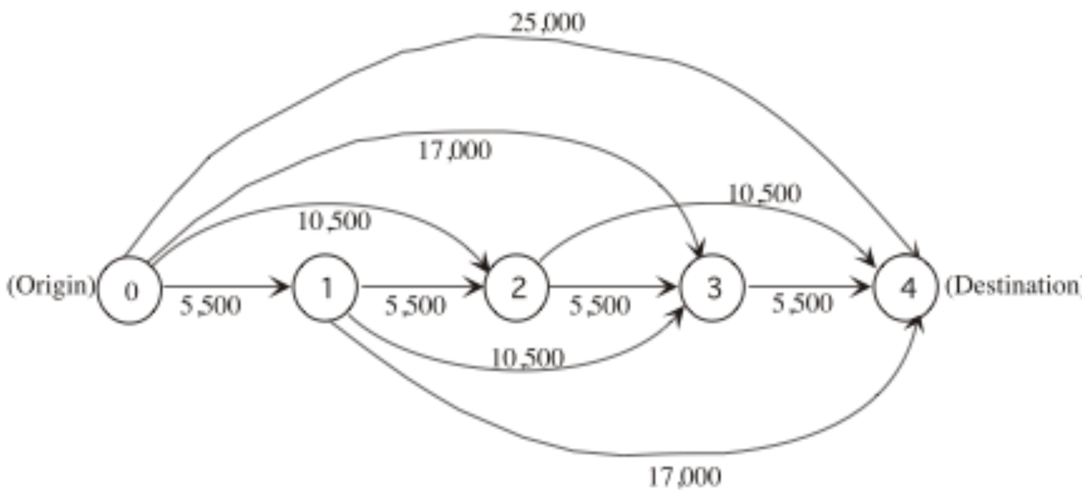
Sarah's Data Each Time She Purchases a Three-Year Old Car

Purchase Price	Operating and Maintenance Costs for Ownership Year				Trade-in Value at End of Ownership Year			
	1	2	3	4	1	2	3	4
$12,000	$2,000	$3,000	$4,500	$6,500	$8,500	$6,500	$4,500	$3,000

When should Sarah trade in her car (if at all) during the next three summers to minimize her total net cost of purchasing, operating, and maintaining the cars over her four years of college?

(a) Formulate this problem as a shortest-path problem.

The following figure shows the network formulation of this problem as a shortest path problem. Nodes 1, 2, 3, and 4 are the end of Sarah's year 1, 2, 3, and 4 of college, respectively. Node 0 is now, before starting college. Each arc from one node to a second node corresponds to the activity of purchasing a car at the time indicated by the first of these two nodes and then trading it in at the time indicated by the second node. Sarah begins by purchasing a car now, and she ends by trading in a car at the end of year 4, so node 0 is the *origin* and node 4 is the *destination*.



The number of arcs on the path chosen from the origin to the destination indicates how many times Sarah will purchase and trade in a car. For example, consider the path

$$0 \rightarrow 1 \rightarrow 3 \rightarrow 4$$

This corresponds to purchasing a car now, then trading it in at the end of year 1 to purchase a second car, then trading in the second car at the end of year 3 to purchase a third car, and then trading in this third car at the end of year 4.

Since Sarah wants to minimize her total net cost from now (node 0) to the end of year 4 (node 4), each arc length needs to measure the net cost of that arc's cycle of purchasing, maintaining, and trading in a car. Therefore,

$$\begin{aligned}\text{Arc length} &= \text{purchase price} \\ &\quad + \text{operating and maintenance costs} \\ &\quad - \text{trade-in value.}\end{aligned}$$

For example, consider the arc from node 1 to node 3. This arc corresponds to purchasing a car at the end of year 1, operating and maintaining it during ownership years 1 and 2, and then trading it in at the end of ownership year 2. Consequently,

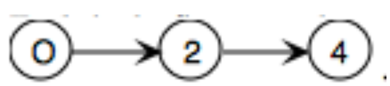
$$\begin{aligned}\text{Length of arc from } 1 \text{ to } 3 &= 12{,}000 + 2{,}000 + 3{,}000 - 6{,}500 \\ &= 10{,}500 \qquad \text{(in dollars).}\end{aligned}$$

The arc lengths calculated in this way are shown next to the arcs in the figure. Adding up the lengths of the arcs on any path from node 0 to node 4 then gives the total net cost for that particular plan for trading in cars over the next four years. Therefore, finding the *shortest path* from the origin to the destination identifies the plan that will minimize Sarah's total net cost.

(b) Use the algorithm described in Sec. 10.3 to solve this shortest-path problem.

n	Solved nodes connected to unsolved nodes	Its closest connected unsolved node	Total cost involved	nth nearest node	Its minimum cost	Its last connection
1	0	1	5,500	1	5,500	$0 \to 1$
2	0	2	10,500	2	10,500	$0 \to 2$
	1	2	5,500+5,500 = 11,000			
3	0	3	17,000			
	1	3	5,500+10,500 = 16,000	3	16,000	$1 \to 3$
	2	3	10,500+5,500 = 16,000	3	16,000	$2 \to 3$
4	0	4	25,000			
	1	4	5,500+17,000 = 22,500			
	2	4	10,500+10,500 = 21,000	4	21,000	$2 \to 4$
	3	4	16,000+5,500 = 21,500			

Thus, the shortest path turns out to be



Trade in the first car at the end of Year 2.
Trade in the second car at the end of Year 4.

The length of this path is 10,500 + 10,500 = 21,000, so Sarah's total net cost is $21,000. Recall that this is exactly the amount in Sarah's car fund provided by her parents. (These are *really* nice parents!)

(c) Formulate and solve a spreadsheet model for this problem.

The following figure shows a spreadsheet model for this problem. After applying Solver, the values of 1 in the changing cells OnRoute (D12:D21) identify the shortest (least expensive) path for scheduling trade-ins.

	B	C	D	E	F	G	H	I	J
3		Operating &	Trade-in Value	Purchase					
4		Maint. Cost	at End of Year	Price					
5	Year 1	$2,000	$8,500	$12,000					
6	Year 2	$3,000	$6,500						
7	Year 3	$4,500	$4,500						
8	Year 4	$6,500	$3,000						
9									
10									
11	**From**	**To**	**On Route**	**Cost**		**Nodes**	**Net Flow**		**Supply/Demand**
12	Year 0	Year 1	0	$5,500		Year 0	1	=	1
13	Year 0	Year 2	1	$10,500		Year 1	0	=	0
14	Year 0	Year 3	0	$17,000		Year 2	0	=	0
15	Year 0	Year 4	0	$25,000		Year 3	0	=	0
16	Year 1	Year 2	0	$5,500		Year 4	-1	=	-1
17	Year 1	Year 3	0	$10,500					
18	Year 1	Year 4	0	$17,000					
19	Year 2	Year 3	0	$5,500					
20	Year 2	Year 4	1	$10,500					
21	Year 3	Year 4	0	$5,500					
22									
23		**Total Cost**	$21,000						

Solver Parameters

Set Objective Cell: TotalCost

To: Min

By Changing Variable Cells:

OnRoute

Subject to the Constraints:

NetFlow = SupplyDemand

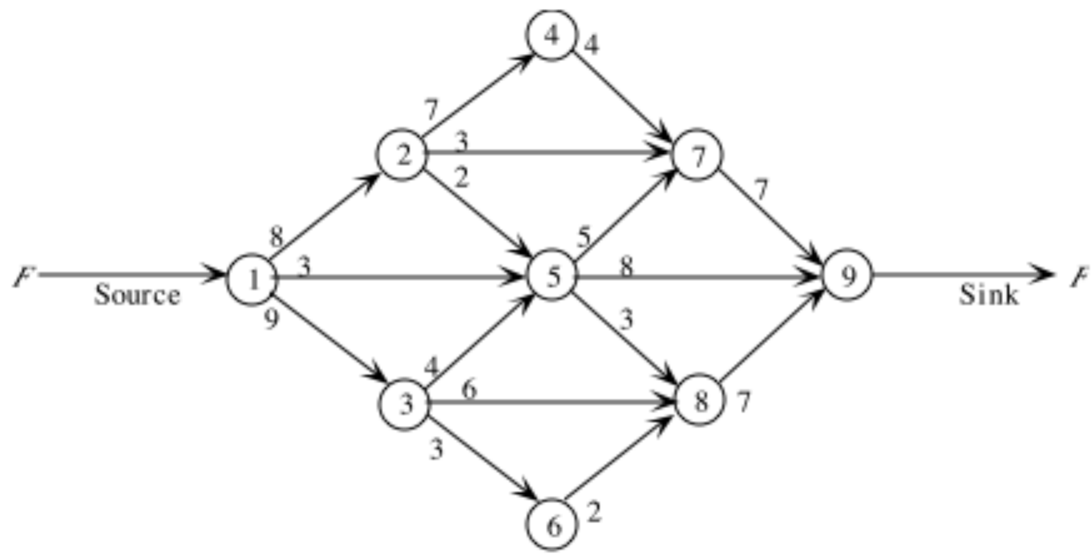
Solver Options:

Make Variables Nonnegative

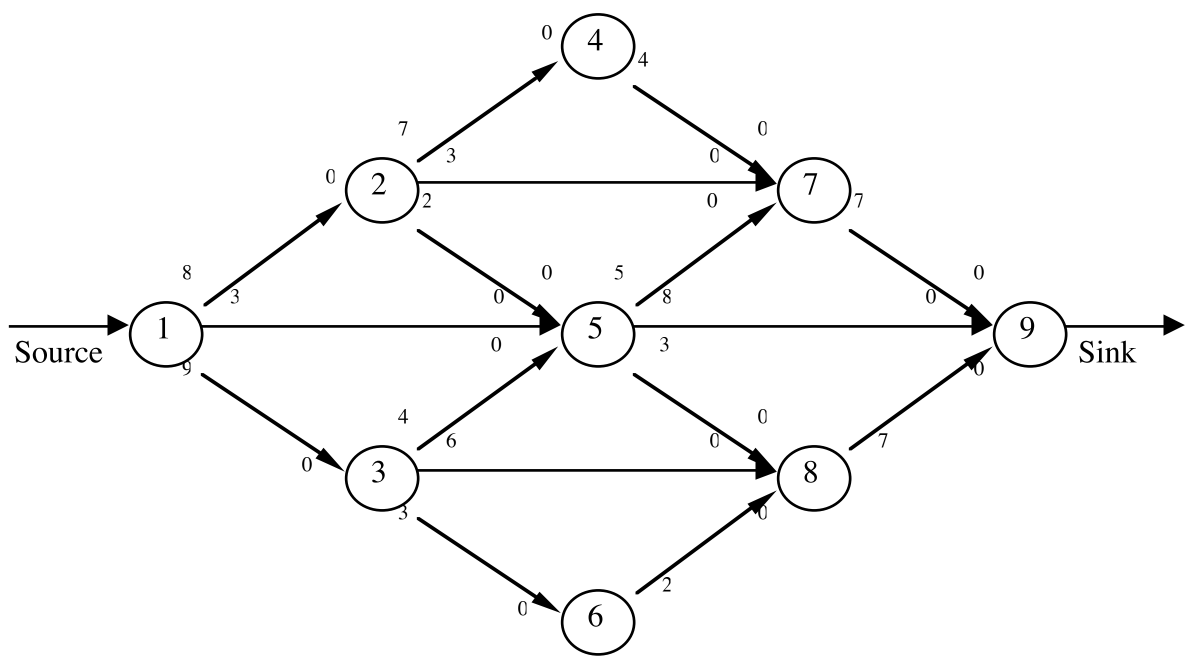
Solving Method: Simplex LP

Example for Section 10.5

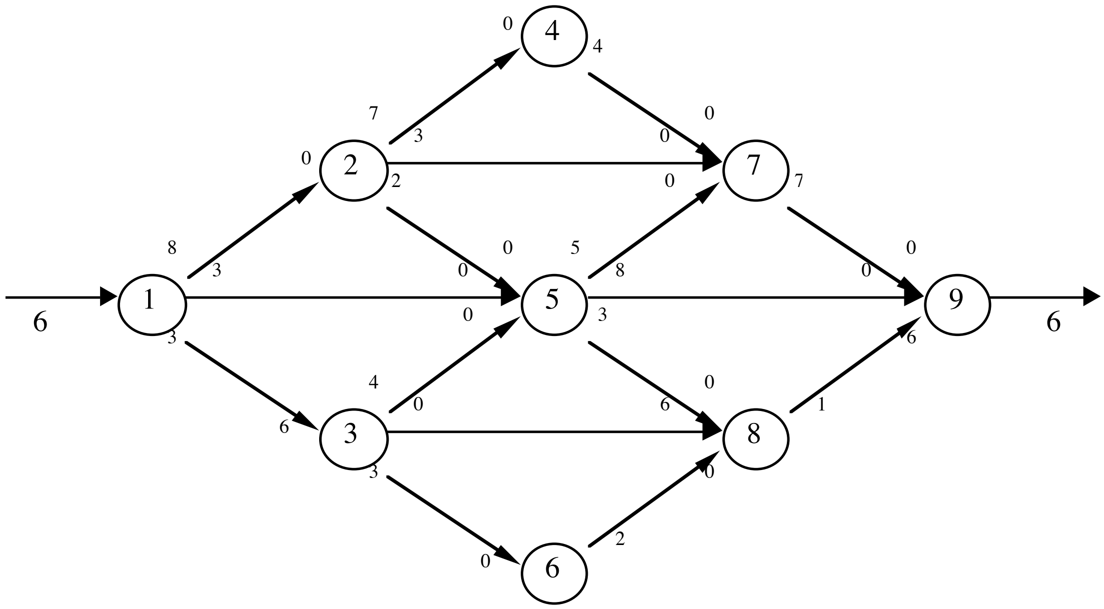
For the network shown below, use the augmenting path algorithm described in Sec. 10.5 to find the flow pattern giving the maximum flow from the source to the sink, given that the arc capacity from node i to node j is the number nearest node i along the arc between these nodes.



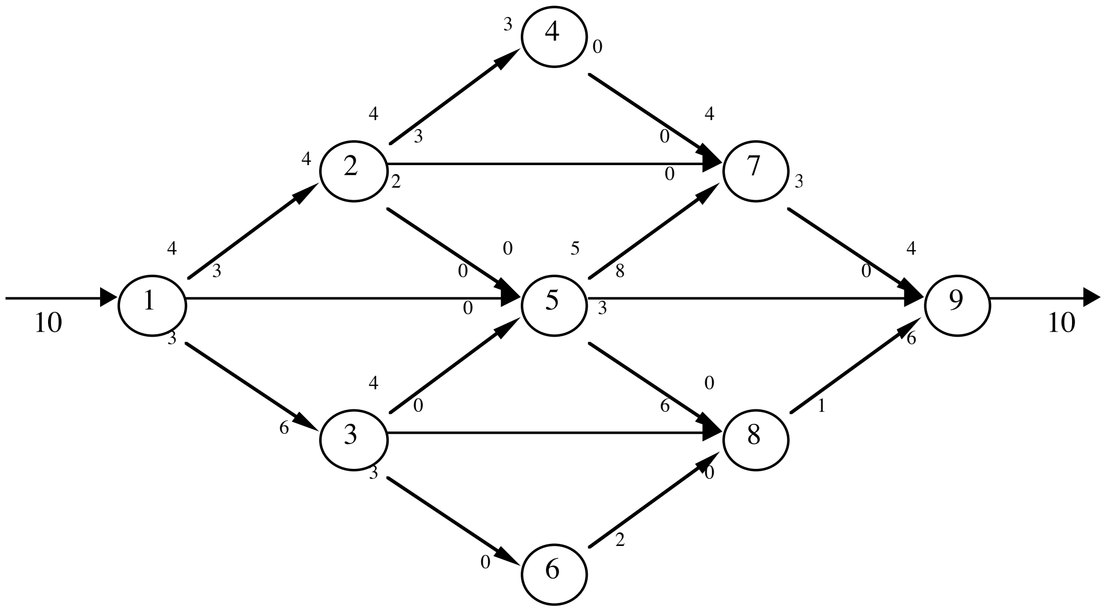
Iteration 0: The initial residual network is



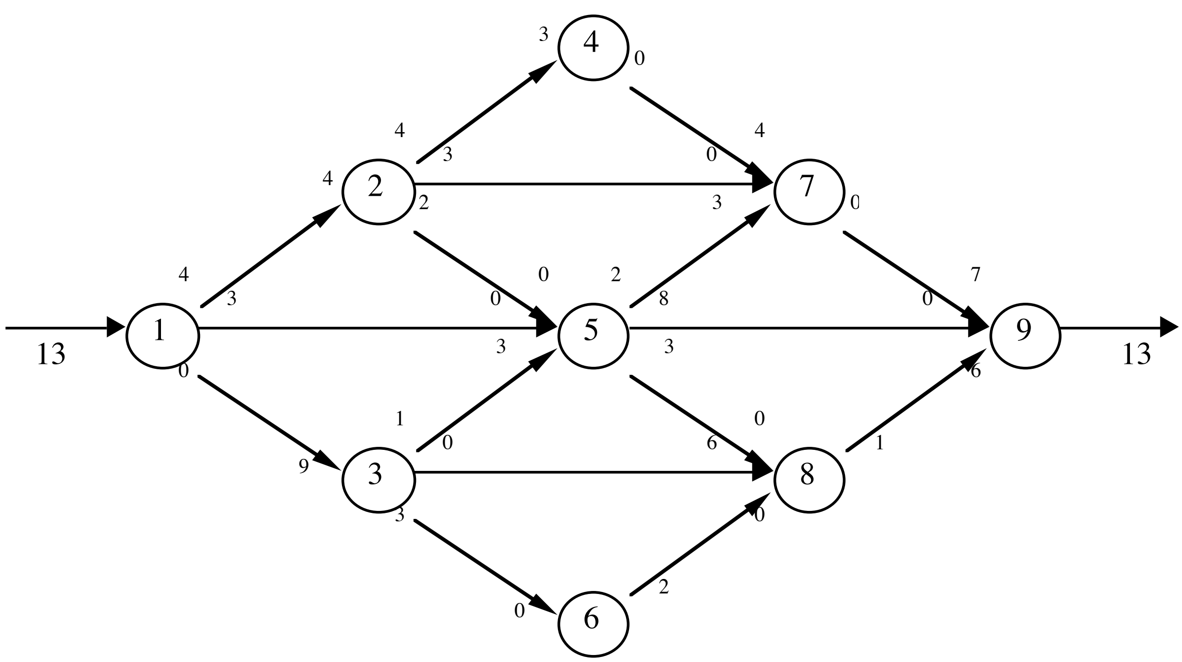
Iteration 1: One of the several augmenting paths is 1→ 3 → 8 → 9, which has a residual capacity of min{9, 6, 7} = 6. Any of the augmenting paths could be chosen, but suppose we select this one. By assigning a flow of 6 to this path, the resulting residual network is



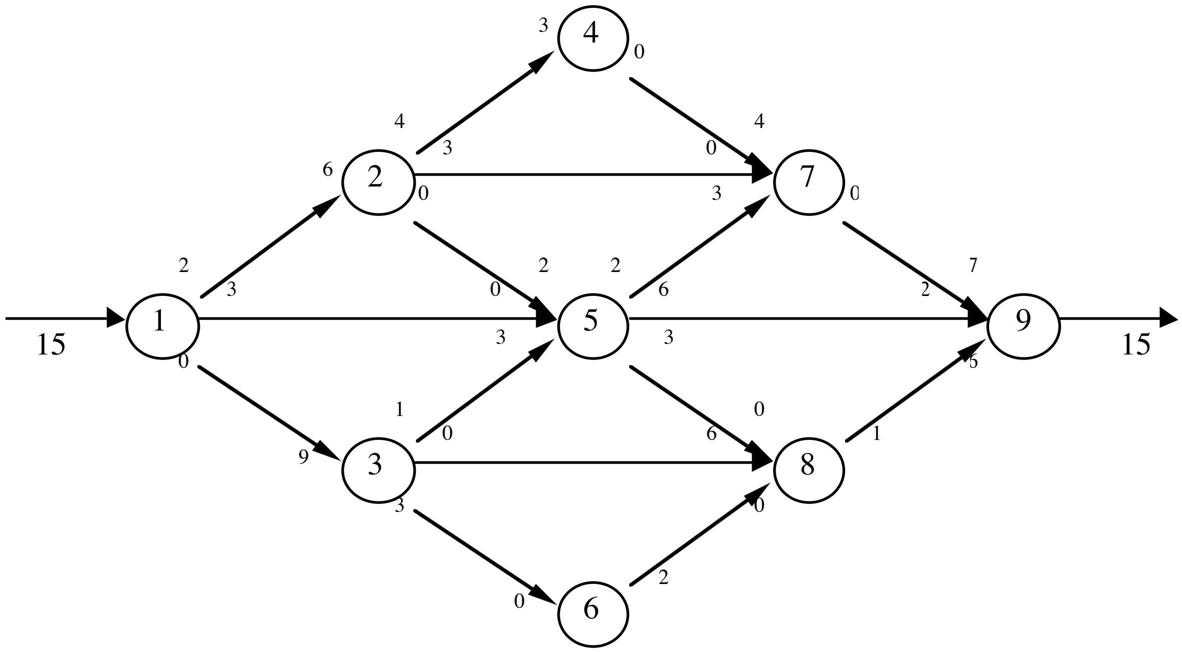
Iteration 2: Assign a flow of 4 to the augmenting path 1→ 2 → 4 → 7 → 9. The resulting residual network is



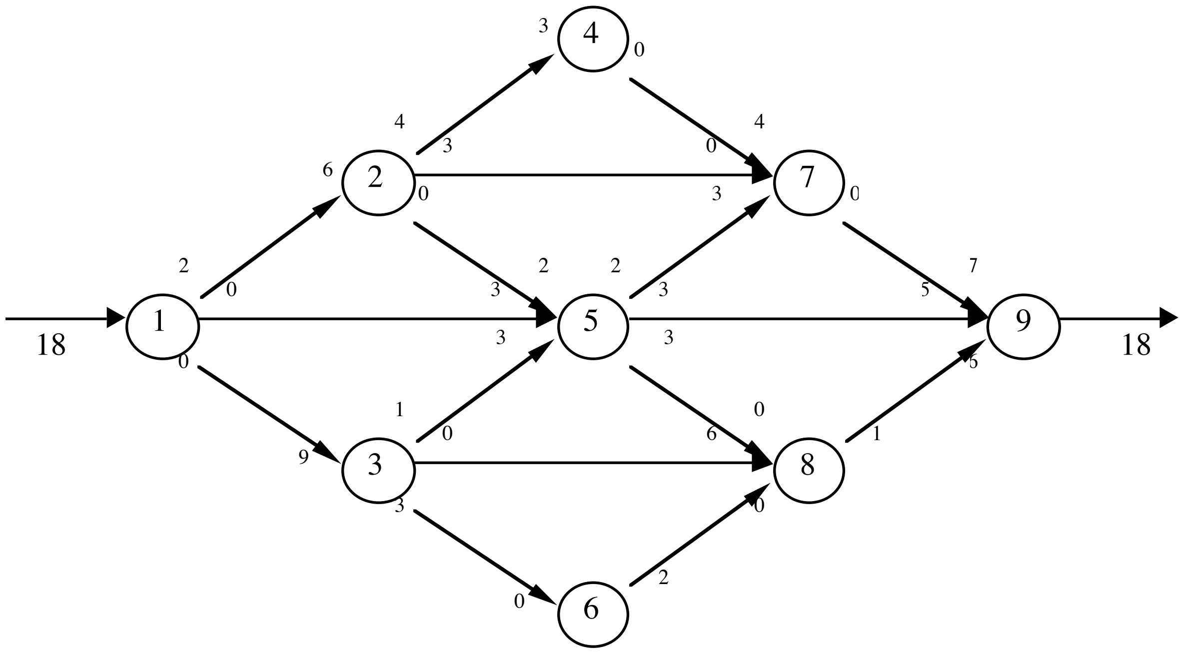
Iteration 3: Assign a flow of 3 to the augmenting path 1→ 3 → 5 → 7 → 9. The resulting residual network is



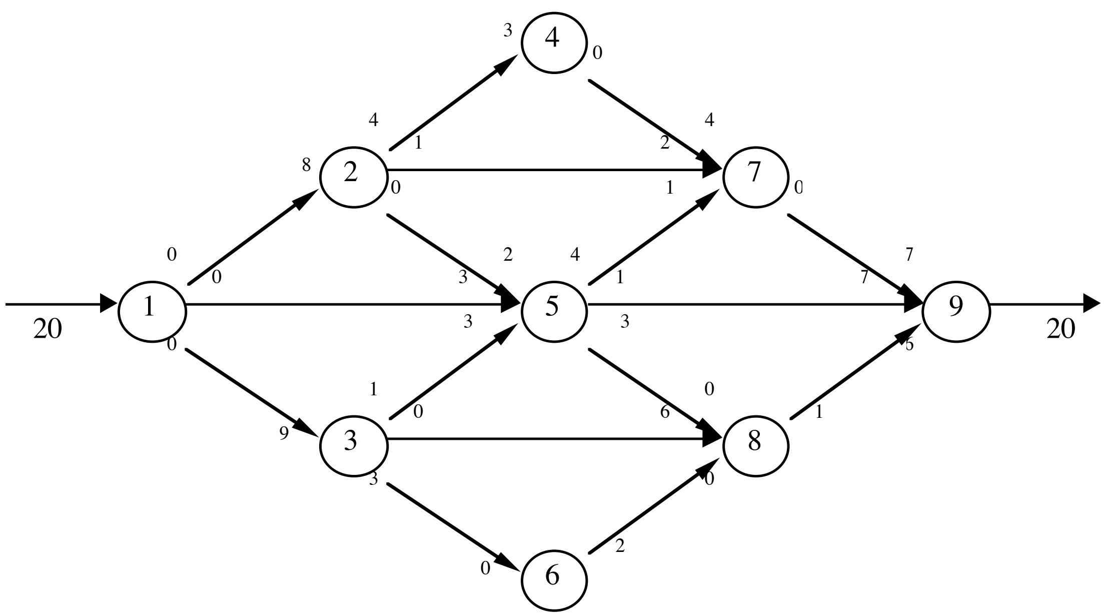
Iteration 4: Assign a flow of 2 to the augmenting path 1→ 2 → 5 → 9. The resulting residual network is



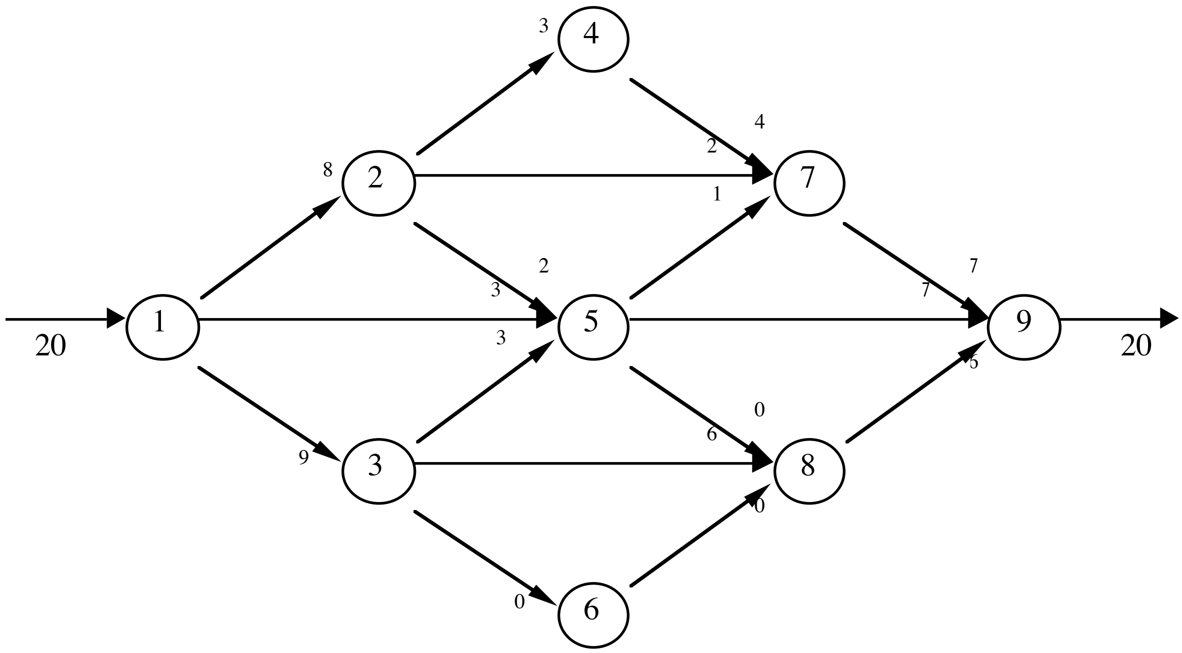
Iteration 5: Assign a flow of 3 to the augmenting path 1→ 5 → 9. The resulting residual network is



Iteration 6: Assign a flow of 2 to the augmenting path $1 \rightarrow 2 \rightarrow 7 \rightarrow 5 \rightarrow 9$. (Although flow between nodes 5 and 7 can only go in the direction from node 5 to node 7, this assignment of a flow of 2 to $7 \rightarrow 5$ is, in reality, simply reducing the previously assigned flow from node 5 to node 7 by 2 units.) The resulting residual network is



There are no more augmenting paths, so the current flow (given by the number at the end of the respective arcs) in the following network is optimal. The maximum flow is 20.



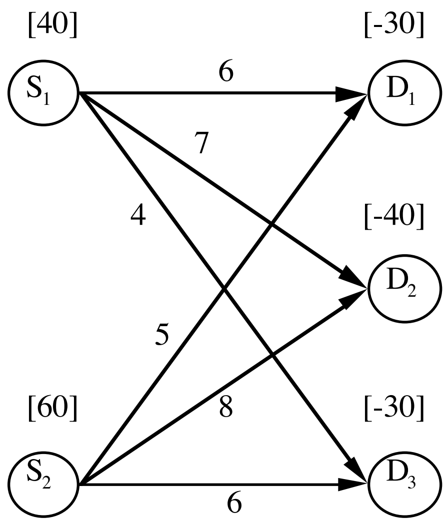
Example for Sections 10.6 and 10.7

Consider the transportation problem having the following parameter table:

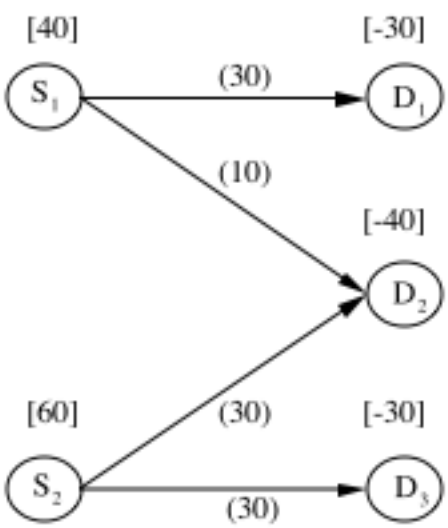
		Destination		
	1	2	3	**Supply**
Source 1	6	7	4	40
2	5	8	6	60
Demand	30	40	30	

Formulate the network representation of this problem as a minimum cost flow problem. Use the northwest corner rule to obtain an initial BF solution. Then use the network simplex method to solve the problem.

The network formulation of this problem is shown in the following figure, where the number next to each node is the net flow generated there and the number next to each arc is the cost per unit flow through that arc.



Using the northwest corner rule, we obtain the following initial BF solution, where the number in parentheses next to each arc is the flow through that arc.



Now we apply the network simplex method to the initial BF solution shown above.

Iteration 1:

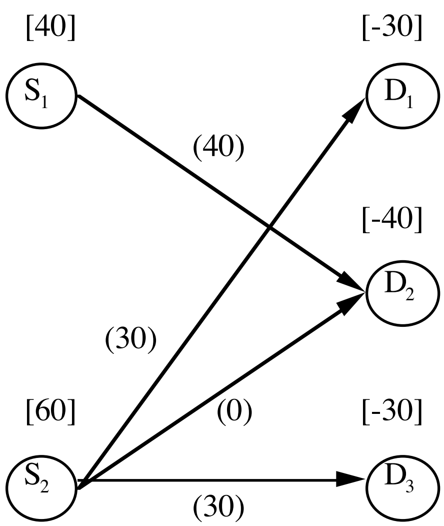
Increase $x_{S1\text{-}D3}$: if $\Delta x_{S1\text{-}D3} = 1$, the cycle created is $S_1 \rightarrow D_3 \rightarrow S_2 \rightarrow D_2 \rightarrow S_1$ and the incremental cost around this cycle is $\Delta Z = 4 - 6 + 8 - 7 = -1$.

Increase $x_{S2\text{-}D1}$: if $\Delta x_{S2\text{-}D1} = 1$, the cycle created is $S_2 \rightarrow D_1 \rightarrow S_1 \rightarrow D_2 \rightarrow S_2$ and the incremental cost around this cycle is $\Delta Z = 5 - 6 + 7 - 8 = -2$.

Hence, we choose to increase $x_{S2\text{-}D1}$ since it decreases the total cost Z at the fastest rate. Since $x_{S1\text{-}D1}$ and $x_{S2\text{-}D2}$ reach their lower bound simultaneously when we increase $x_{S2\text{-}D1}$, we

can choose either of them as the leaving basic variable. Suppose we choose $x_{S1\text{-}D1}$ as the leaving basic variable.

The resulting BF spanning tree is



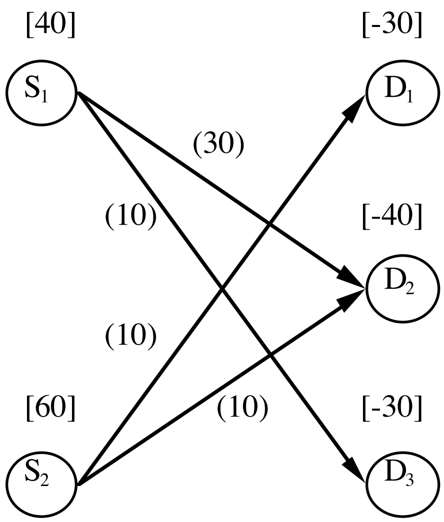
Iteration 2:

Increase $x_{S1\text{-}D3}$: if $\Delta x_{S1\text{-}D3} = 1$, the cycle created is $S_1 \to D_3 \to S_2 \to D_2 \to S_1$ and the incremental cost around this cycle is $\Delta Z = 4 - 6 + 8 - 7 = -1$.

Increase $x_{S1\text{-}D1}$: if $\Delta x_{S1\text{-}D1} = 1$, the cycle created is $S_1 \to D_1 \to S_2 \to D_2 \to S_1$ and the incremental cost around this cycle is $\Delta Z = 6 - 5 + 8 - 7 = 2$.

Hence, we choose to increase $x_{S1\text{-}D3}$ since it is the only option that decreases the total cost Z. Since $x_{S2\text{-}D3}$ reaches its lower bound first, we choose $x_{S2\text{-}D3}$ as the leaving basic variable.

The resulting BF spanning tree is



Optimality Test:

Increase $x_{S1\text{-}D1}$: If $\Delta x_{S1\text{-}D1} = 1$, the cycle created is $S_1 \rightarrow D_1 \rightarrow S_2 \rightarrow D_2 \rightarrow S_1$ and the incremental cost around this cycle is $\Delta Z = 6 - 5 + 8 - 7 = 2$.

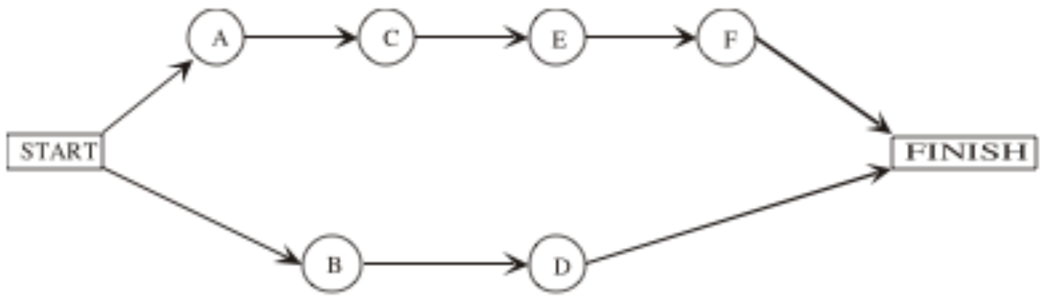
Increase $x_{S2\text{-}D3}$: ΔIf $x_{S2\text{-}D3} = 1$, the cycle created is $S_2 \rightarrow D_3 \rightarrow S_1 \rightarrow D_2 \rightarrow S_2$ and the incremental cost around this cycle is $\Delta Z = 6 - 4 + 7 - 8 = 1$.

Thus, increasing either of the nonbasic variables by making it an entering basic variable would only increase the total cost ($\Delta Z > 0$), so an improvement in Z cannot be achieved by introducing flow through either of the nonbasic arcs. Therefore, the BF solution shown above is optimal.

Example for Section 10.8

Sharon Lowe, Vice President for Marketing for the Electronic Toys Company, is about to begin a project to design an advertising campaign for a new line of toys. She wants the project completed within 47 days in time to launch the advertising campaign at the beginning of the Christmas season.

Sharon has identified the six activities (labeled A, B, ..., F) needed to execute this project. Considering the order in which these activities need to occur, she also has constructed the following project network.



To meet the deadline of 47 days, Sharon has decided to crash the project, using the CPM method of time-cost trade-offs to determine how to do this in the most economical way. She has gathered the data needed to apply this method, as given below.

Activity	Time (days)		Cost		Maximum Reduction in Time	Crash Cost per day saved
	Normal	Crash	Normal	Crash		
A	12	9	$210,000	$270,000	3	$20,000
B	23	18	$410,000	$460,000	5	$10,000
C	15	12	$290,000	$320,000	3	$10,000
D	27	21	$440,000	$500,000	6	$10,000
E	18	14	$350,000	$410,000	4	$15,000
F	6	4	$160,000	$210,000	2	$25,000

(*a*) Consider the lower path through the project network. Use marginal cost analysis to determine the most economical way of reducing the length of this path to 47 days.

The lower path is B-D with a path length of 50 days.
From the time-cost trade-off data, both activities B and D have a crash cost per day saved of $10,000, and both can be reduced by more than 3 days. Therefore, using marginal cost analysis, we find that the most economical way of reducing the length of this path to 47 days is to shorten either activity (it doesn't matter which one) by 3 days with an additional total cost of $30,000.

Activity to crash	Crash Cost	Length of Path B-D
		50
B or D	$10,000	49
B or D	$10,000	48
B or D	$10,000	47

(*b*) Repeat part (*a*) for the upper path through the project network. What is the total crashing cost for the optimal way of decreasing the estimated project duration to 47 days?

The upper path is A-C-E-F with a path length of 51 days.
Marginal cost analysis is performed in the table below. Of the activities on the path, activity C has the smallest crash cost per day saved ($10,000) and activity E is next ($15,000). Activity C can only be reduced by 3 days, so activity E also will need to be crashed somewhat. Therefore, we find that the most economical way of reducing the length of this path to 47 days is to shorten activity C by 3 days and activity E by 1 day with an additional total cost of $45,000.

Activity to crash	Crash Cost	Length of Path A-C-E-F
		51
C	$10,000	50
C	$10,000	49
C	$10,000	48
E	$15,000	47

Combining this result with the result from part (a), the total crashing cost for the optimal way of meeting the deadline of 47 days is $30,000 + $45,000 = $75,000.

(c) Formulate a linear programming model for the problem of determining the most economical way of meeting the deadline of 47 days.

The natural decision variables are

x_j = reduction in the duration of activity j due to crashing this activity, for j = A, B, ... , F.

Each of these variables has both a nonnegativity constraint and a maximum reduction constraint, where the upper bound for this latter constraint is given by the corresponding number in the next-to-last column (labeled Maximum Reduction in Time) of the table of data given at the end of the problem statement. Using the last column (labeled Crash Cost per Day Saved) of this same table, the objective function to be minimized is

$$Z = 20{,}000x_A + 10{,}000x_B + 10{,}000x_C + 10{,}000x_D + 15{,}000x_E + 25{,}000x_F.$$

Some additional variables also are needed in the formulation. In particular, let

y_{FINISH} = project duration,
y_j = start time of activity j (for j = C, D, E, F), given the values of $x_A, x_B, \ldots, x_F$.

(No such variable is needed for activities A and B, since these activities that simultaneously start the project are automatically assigned a start time of 0.) These variables need to satisfy the constraints,

$y_{FINISH} \leq 47$,
$y_j \geq y_i + t_i - x_i$,

where activity i is the immediate predecessor of activity j and t_i is the normal time of activity i (as given by the second column of the table of data).

Therefore, the complete linear programming model is

Minimize $Z = 20{,}000X_A + 10{,}000x_B + 10{,}000x_C + 10{,}000x_D + 15{,}000x_E + 25{,}000x_F$,

subject to the following constraints:

1. Maximum reduction constraints:

$x_A \leq 3$, $x_B \leq 5$, $x_C \leq 3$, $x_D \leq 6$, $x_E \leq 4$, $x_F \leq 2$.

2. Nonnegativity constraints:

$x_A \geq 0$, $x_B \geq 0$, $x_C \geq 0$, $x_D \geq 0$, $x_E \geq 0$, $x_F \geq 0$,
$y_C \geq 0$, $y_D \geq 0$, $y_E \geq 0$, $y_F \geq 0$, $y_{FINISH} \geq 0$.

3. Start time constraints:

$y_C \geq 0 + 12 - x_A$, $y_D \geq 0 + 23 - x_B$,
$y_E \geq y_C + 15 - x_C$, $y_F \geq y_E + 18 - x_E$.

4. Project duration constraint:

$y_{FINISH} \leq 47$.

(*d*) Use Excel to solve the problem.

The following spreadsheet shows how Excel finds an optimal solution: shorten activity B by 3 days, shorten activity C by 4 days. The total cost (sum of the normal cost and the crash cost) is $1,935,000.

	A	B	C	D	E	F	G	H	I	J
1						Maximum	Crash Cost			
2						Time	per day	Start	Time	Finish
3	Activity	Normal	Crash	Normal	Crash	Reduction	Saved	Time	Reduction	Time
4	A	12	9	$210,000	$270,000	3	$20,000	0	0	12
5	B	23	18	$410,000	$460,000	5	$10,000	0	3	20
6	C	15	12	$290,000	$320,000	3	$10,000	12	4	23
7	D	27	21	$440,000	$500,000	6	$10,000	20	0	47
8	E	18	14	$350,000	$410,000	4	$15,000	23	0	41
9	F	6	4	$160,000	$210,000	2	$25,000	41	0	47
10										
11							Finish Time	47	<=	47
12										
13									Total Cost	$1,930,000

Solver Parameters

Set Objective Cell: TotalCost

To: Min

By Changing Variable Cells:

StartTime, TimeReduction, FinishTime

Subject to the Constraints:

H6 >= J4

H8 >= J6

H9 >= J8

H7 >= J5

H11 >= J7

H11 >= J9

H11 <= J11

Solver Options:

Make Variables Nonnegative

Solving Method: Simplex LP