APPENDIX 6

Simultaneous Linear Equations

Consider the system of simultaneous linear equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2, \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m. \end{aligned}$$

It is commonly assumed that this system has a solution, and a unique solution, if and only if $m = n$. However, this assumption is an oversimplification. It raises the questions: Under what conditions will these equations have a simultaneous solution? Given that they do, when will there be only one such solution? If there is a unique solution, how can it be identified in a systematic way? These questions are the ones we explore in this appendix. The discussion of the first two questions assumes that you are familiar with the basic information about matrices in Appendix 4.

The preceding system of equations can also be written in matrix form as

$$\mathbf{Ax} = \mathbf{b},$$

where

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.$$

The first two questions can be answered immediately in terms of the properties of these matrices. First, the system of equations possesses at least one solution if and only if the *rank* of **A** equals the *rank* of [**A, b**]. (Notice that equality is guaranteed if the *rank* of **A** equals *m*.) This result follows immediately from the definitions of *rank* and *linear independence* given in Appendix 4, because if the *rank* of [**A, b**] exceeds the *rank* of **A** by 1 (the only other possibility), then **b** is *linearly independent* of the column vectors of **A** (that is, **b** cannot equal any linear combination **Ax** of these vectors).

Second, given that these ranks are equal, there are then two possibilities. If the *rank* of **A** is *n* (its maximum possible value), then the system of equations will possess exactly *one solution*. [This result follows from Theorem A4.1, the definition of a *basis,* and part (*b*) of Theorem A4.3.] If the *rank* of **A** is *less* than *n,* then there will exist an *infinite number of solutions.* (This result follows from the fact that for any *basis* of the column vectors of **A,** the $\mathbf{x}_j$ corresponding to column vectors not in this basis can be assigned any value, and there will still exist a solution for the other variables as before.)

Finally, it should be noted that if **A** and [**A, b**] have a *common rank r* such that $r < m$, then $(m - r)$ of the equations must be linear combinations of the other ones, so that these $(m - r)$ *redundant* equations can be deleted without affecting the solution(s). It then follows from the preceding results that this system of equations (with or without the redundant equations) possesses at least one solution, where the number of solutions is *one* if $r = n$ or *infinite* if $r < n$.

Now consider how to find a solution to the system of equations. Assume for the moment that $m = n$ and **A** is nonsingular, so that a unique solution exists. This solution can be obtained by the **Gauss-Jordan method of elimination** (commonly called **Gaussian elimination**), which proceeds as follows. To begin, eliminate the first variable from all but one (say, the first) of the equations by adding an appropriate multiple (positive or negative) of this equation to each of the others. (For convenience, this one equation would be divided by the coefficient of this variable, so that the final value of this coefficient is 1.) Next, proceed in the same way to eliminate

the second variable from all equations except one new one (say, the second). Then repeat this procedure for the third variable, the fourth variable, and so on, until each of the n variables remains in only one of the equations and each of the n equations contains exactly one of these variables. The desired solution can then be read from the equations directly.

To illustrate the *Gauss-Jordan method of elimination,* we consider the following system of linear equations:

$$\begin{array}{llcl}
(1) & x_1 - x_2 + 4x_3 & = & 10 \\
(2) & -x_1 + 3x_2 & = & 10 \\
(3) & 2x_2 + 5x_3 & = & 22.
\end{array}$$

The method begins by eliminating x_1 from all but the first equation. This first step is executed simply by adding Eq. (1) to Eq. (2), which yields

$$\begin{array}{llcl}
(1) & x_1 - x_2 + 4x_3 & = & 10 \\
(2) & 2x_2 + 4x_3 & = & 20 \\
(3) & 2x_2 + 5x_3 & = & 22.
\end{array}$$

The next step is to eliminate x_2 from all but the second equation. Begin this step by dividing Eq. (2) by 2, so that x_2 will have a coefficient of $+1$, as follows:

$$\begin{array}{llcl}
(1) & x_1 - x_2 + 4x_3 & = & 10 \\
(2) & x_2 + 2x_3 & = & 10 \\
(3) & 2x_2 + 5x_3 & = & 22.
\end{array}$$

Then add Eq. (2) to Eq. (1), and subtract two times Eq. (2) from Eq. (3), which yields

$$\begin{array}{llcl}
(1) & x_1 + 6x_3 & = & 20 \\
(2) & x_2 + 2x_3 & = & 10 \\
(3) & x_3 & = & 2.
\end{array}$$

The final step is to eliminate x_3 from all but the third equation. This step requires subtracting six times Eq. (3) from Eq. (1) and subtracting two times Eq. (3) from Eq. (2), which yields

$$\begin{array}{llcl}
(1) & x_1 & = & 8 \\
(2) & x_2 & = & 6 \\
(3) & x_3 & = & 2.
\end{array}$$

Thus, the desired solution is $(x_1, x_2, x_3) = (8, 6, 2)$, and the procedure is completed.

Now consider briefly what happens if the Gauss-Jordan method of elimination is applied when $m \neq n$ and/or $\mathbf{A}$ is singular. As we discussed earlier, there are three possible cases to consider. First, if the rank of $[\mathbf{A}, \mathbf{b}]$ exceeds the rank of $\mathbf{A}$ by 1, then *no solution* to the system of equations will exist. In this case, the Gauss-Jordan method obtains an equation where the left-hand side has vanished (i.e., all the coefficients of the variables are zero), whereas the right-hand side is nonzero. This signpost indicates that no solution exists, so there is no reason to proceed further.

The second case is where both of these ranks are equal to n, so that a *unique solution* exists. This case implies that $m \geq n$. If $m = n$, then the previous assumptions must hold and no difficulty arises. Therefore, suppose that $m > n$, so that there are $(m - n)$ redundant equations. In this case, all these redundant equations are eliminated (i.e., both the left-hand and right-hand sides would become zero) during the process of executing the Gauss-Jordan method, so the unique solution is identified just as it was before.

The final case is where both the ranks are equal to r, where $r < n$, so that the system of equations possesses an *infinite number of solutions*. In this case, at the completion of the Gauss-Jordan method, each of r variables remains in only one of the equations, and each of the r equations (any additional equations have vanished) contains exactly one of these variables. However, each of the other $(n - r)$ variables either vanishes or remains in some of the equations. Therefore, any solution obtained by assigning arbitrary values to the $(n - r)$ variables, and then identifying the respective values of the r variables from the single final equation in which each one appears, is a solution to the system of simultaneous equations. Equivalently, the transfer of these $(n - r)$ variables to the right-hand side of the equations (either before or after the method is executed) identifies the solutions for the r variables as a function of these extra variables.