Solutions Manual

for

Heat and Mass Transfer: Fundamentals & Applications

5th Edition

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McGraw-Hill, 2015

Chapter 5

NUMERICAL METHODS IN HEAT CONDUCTION

(Last updated August 1, 2015)

Why Numerical Methods?

5-1C Analytical solutions provide insight to the problems, and allows us to observe the degree of dependence of solutions on certain parameters. They also enable us to obtain quick solution, and to verify numerical codes. Therefore, analytical solutions are not likely to disappear from engineering curricula.

5-2C Analytical solution methods are limited to *highly simplified problems* in *simple geometries*. The geometry must be such that its entire surface can be described mathematically in a coordinate system by setting the variables equal to constants. Also, heat transfer problems can not be solved analytically if the *thermal conditions* are not sufficiently simple. For example, the consideration of the variation of thermal conductivity with temperature, the variation of the heat transfer coefficient over the surface, or the radiation heat transfer on the surfaces can make it impossible to obtain an analytical solution. Therefore, analytical solutions are limited to problems that are simple or can be simplified with reasonable approximations.

5-3C In practice, we are most likely to use a software package to solve heat transfer problems even when analytical solutions are available since we can do parametric studies very easily and present the results graphically by the press of a button. Besides, once a person is used to solving problems numerically, it is very difficult to go back to solving differential equations by hand.

5-4C The *energy balance method* is based on *subdividing* the medium into a sufficient number of volume elements, and then applying an *energy balance* on each element. The formal *finite difference method* is based on replacing derivatives by their finite difference approximations. For a specified nodal network, these two methods will result in the same set of equations.

5-5C The *analytical solutions* are based on (1) driving the governing differential equation by performing an energy balance on a differential volume element, (2) expressing the boundary conditions in the proper mathematical form, and (3) solving the differential equation and applying the boundary conditions to determine the integration constants. The *numerical solution* methods are based on replacing the *differential equations* by *algebraic equations*. In the case of the popular *finite difference* method, this is done by replacing the *derivatives* by *differences*. The analytical methods are simple and they provide solution functions applicable to the entire medium, but they are limited to simple problems in simple geometries. The numerical methods are usually more involved and the solutions are obtained at a number of points, but they are applicable to any geometry subjected to any kind of thermal conditions.

5-6C The experiments will most likely prove engineer B right since an approximate solution of a more realistic model is more accurate than the exact solution of a crude model of an actual problem.

Finite Difference Formulation of Differential Equations

5-7C A point at which the finite difference formulation of a problem is obtained is called a *node*, and all the nodes for a problem constitute the *nodal network*. The region about a node whose properties are represented by the property values at the nodal point is called the *volume element*. The distance between two consecutive nodes is called the *nodal spacing*, and a differential equation whose derivatives are replaced by differences is called a *difference equation*.

5-8 The finite difference formulation of steady two-dimensional heat conduction in a medium with heat generation and constant thermal conductivity is given by

$$\frac{T_{m-1,n} - 2T_{m,n} + T_{m+1,n}}{\Delta x^2} + \frac{T_{m,n-1} - 2T_{m,n} + T_{m,n+1}}{\Delta y^2} + \frac{\dot{e}_{m,n}}{k} = 0$$

in rectangular coordinates. This relation can be modified for the three-dimensional case by simply adding another index j to the temperature in the z direction, and another difference term for the z direction as

$$\frac{T_{m-1,n,j} - 2T_{m,n,j} + T_{m+1,n,j}}{\Delta x^2} + \frac{T_{m,n-1,j} - 2T_{m,n,j} + T_{m,n+1,j}}{\Delta y^2} + \frac{T_{m,n,j-1} - 2T_{m,n,j} + T_{m,n,j+1}}{\Delta z^2} + \frac{\dot{e}_{m,n,j}}{k} = 0$$

5-9 Finite difference formulation for an interior node, boundary node subject to convection and constant heat flux in case of variable thermal conductivity is to be determined.

Analysis The one dimensional steady state heat conduction equation with variable thermal conductivity is expressed as

$$\frac{d}{dx}\left(k\frac{dT}{dx}\right)+\dot{e}=0$$

Using Eq. (5-6), the first derivative of the temperature at the midpoints surrounding the node with variable thermal conductivity can be expressed for x as

$$k(T)\frac{dT}{dx}\bigg|_{m-\frac{1}{2}}=k_o\left[1+\beta\frac{(T_{m-1}+T_m)}{2}\right]\frac{(T_{m-1}-T_m)}{\Delta x}\ \text{ and }\ k(T)\frac{dT}{dx}\bigg|_{m+\frac{1}{2}}=k_o\left[1+\beta\frac{(T_{m+1}+T_m)}{2}\right]\frac{(T_{m+1}-T_m)}{\Delta x}$$

Using the definition of second derivative as the derivative of the first derivative we get

$$\frac{d}{dx}\left(k\frac{dT}{dx}\right)+\dot{e}=\frac{k_o\left[1+\beta\frac{(T_{m+1}+T_m)}{2}\right]\frac{(T_{m+1}-T_m)}{\Delta x}-k_o\left[1+\beta\frac{(T_{m-1}+T_m)}{2}\right]\frac{(T_{m-1}-T_m)}{\Delta x}}{\Delta x}+\dot{e}_m=0$$

Simplifying above equation yields,

$$(T_{m+1}-T_m)\left[1+\frac{\beta}{2}(T_{m+1}+T_m)\right]+(T_{m-1}-T_m)\left[1+\frac{\beta}{2}(T_{m-1}+T_m)\right]+\dot{e}_m\frac{\Delta x^2}{k_o}=0$$

$$\therefore(T_{m-1}-2T_m+T_{m+1})+\frac{\beta}{2}\left[T_{m-1}^2-2T_m^2+T_{m+1}^2\right]+\dot{e}_m\frac{\Delta x^2}{k_o}=0$$

For left boundary node exposed to constant heat flux apply energy balance to the half volume around the boundary node with all the heat transfer entering the volume element.

Replacing k by k(T) in Eq. 5-22 we get,

$$k_o\left[1+\beta\frac{(T_{m+1}+T_m)}{2}\right]\frac{(T_{m+1}-T_m)}{\Delta x}+\dot{q}+\dot{e}_m\frac{\Delta x}{2}=0\quad\therefore(T_{m+1}-T_m)+\frac{\beta}{2}\left(T_{m+1}^2-T_m^2\right)+\frac{\dot{q}\Delta x}{k_o}+\frac{\dot{e}_m\Delta x^2}{2k_o}=0$$

For right boundary node exposed to convection environment, apply energy balance to the half volume around the boundary node with all the heat transfer entering the volume element.

$$k_o\left[1+\beta\frac{(T_{m-1}+T_m)}{2}\right]\frac{(T_{m-1}-T_m)}{\Delta x}+h(T_\infty-T_m)+\frac{\dot{e}_m\Delta x}{2}=0$$

$$\therefore(T_{m-1}-T_m)+\frac{\beta}{2}\left(T_{m-1}^2-T_m^2\right)+\frac{h\Delta x}{k_o}(T_\infty-T_m)+\frac{\dot{e}_m\Delta x^2}{2k_o}=0$$

5-10 For a three dimensional steady state heat transfer without internal heat generation finite difference formulations are to be determined.

Analysis The three dimensional heat conduction equation for steady state conditions with variable thermal conductivity is expressed as

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)=0$$

Using Eq. (5-6), the first derivative of the temperature at the midpoints surrounding the node can be expressed for x, y and z directions as

x-direction: $$k\frac{dT}{dx}\bigg|_{m-\frac{1}{2},n,j}\cong k_{m-\frac{1}{2},n,j}\frac{T_{m,n,j}-T_{m-1,n,j}}{\Delta x}\quad \textit{and}\quad k\frac{dT}{dx}\bigg|_{m+\frac{1}{2},n,j}\cong k_{m+\frac{1}{2},n,j}\frac{T_{m+1,n,j}-T_{m,n,j}}{\Delta x}$$

y-direction: $$k\frac{dT}{dy}\bigg|_{n-\frac{1}{2},m,j}\cong k_{n-\frac{1}{2},m,j}\frac{T_{n,m,j}-T_{n-1,m,j}}{\Delta y}\quad \textit{and}\quad k\frac{dT}{dy}\bigg|_{n+\frac{1}{2},m,j}\cong k_{n+\frac{1}{2},m,j}\frac{T_{n+1,m,j}-T_{n,m,j}}{\Delta y}$$

z-direction: $$k\frac{dT}{dz}\bigg|_{j-\frac{1}{2},m,n}\cong k_{j-\frac{1}{2},m,n}\frac{T_{j,m,n}-T_{j-1,m,n}}{\Delta z}\quad \textit{and}\quad k\frac{dT}{dz}\bigg|_{j+\frac{1}{2},m,n}\cong k_{j+\frac{1}{2},m,n}\frac{T_{j+1,m,n}-T_{j,m,n}}{\Delta z}$$

Using the definition of second derivative as the derivative of the first derivative we get

$$\frac{d}{dx}\left(k\frac{dT}{dx}\right)=\frac{k\frac{dT}{dx}\Big|_{m+\frac{1}{2},n,j}-k\frac{dT}{dx}\Big|_{m-\frac{1}{2},n,j}}{\Delta x}=k_{m+\frac{1}{2},n,j}\frac{T_{m+1,n,j}-T_{m,n,j}}{\Delta x^2}-k_{m-\frac{1}{2},n,j}\frac{T_{m,n,j}-T_{m-1,n,j}}{\Delta x^2}$$

Similar for y and z directions we can show that

$$\frac{d}{dy}\left(k\frac{dT}{dy}\right)=k_{n+\frac{1}{2},m,j}\frac{T_{n+1,m,j}-T_{n,m,j}}{\Delta y^2}-k_{n-\frac{1}{2},m,j}\frac{T_{n,m,j}-T_{n-1,m,j}}{\Delta y^2}$$

$$\frac{d}{dz}\left(k\frac{dT}{dz}\right)=k_{j+\frac{1}{2},m,n}\frac{T_{j+1,m,n}-T_{j,m,n}}{\Delta z^2}-k_{j-\frac{1}{2},m,n}\frac{T_{j,m,n}-T_{j-1,m,n}}{\Delta z^2}$$

Putting these equations of the second derivatives in the three dimensional heat conduction equation with variable thermal conductivity we get the required finite difference formulation.

$$k_{m+\frac{1}{2},n,j}\frac{T_{m+1,n,j}-T_{m,n,j}}{\Delta x^2}-k_{m-\frac{1}{2},n,j}\frac{T_{m,n,j}-T_{m-1,n,j}}{\Delta x^2}+k_{n+\frac{1}{2},m,j}\frac{T_{n+1,m,j}-T_{n,m,j}}{\Delta y^2}-k_{n-\frac{1}{2},m,j}\frac{T_{n,m,j}-T_{n-1,m,j}}{\Delta y^2}+$$
$$k_{j+\frac{1}{2},m,n}\frac{T_{j+1,m,n}-T_{j,m,n}}{\Delta z^2}-k_{j-\frac{1}{2},m,n}\frac{T_{j,m,n}-T_{j-1,m,n}}{\Delta z^2}=0$$

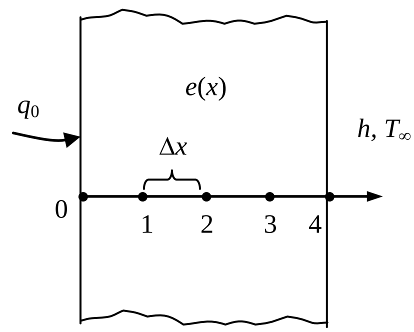
5-11 A plane wall with variable heat generation and constant thermal conductivity is subjected to uniform heat flux $\dot{q}_0$ at the left (node 0) and convection at the right boundary (node 4). Using the finite difference form of the 1st derivative, the finite difference formulation of the boundary nodes is to be determined.

Assumptions **1** Heat transfer through the wall is steady since there is no indication of change with time. **2** Heat transfer is one-dimensional since the plate is large relative to its thickness. **3** Thermal conductivity is constant and there is nonuniform heat generation in the medium. **4** Radiation heat transfer is negligible.

Analysis The boundary conditions at the left and right boundaries can be expressed analytically as

at $x = 0$: $$-k\frac{dT(0)}{dx} = q_0$$

at $x = L$: $$-k\frac{dT(L)}{dx} = h[T(L) - T_\infty]$$



Replacing derivatives by differences using values at the closest nodes, the finite difference form of the 1st derivative of temperature at the boundaries (nodes 0 and 4) can be expressed as

$$\left.\frac{dT}{dx}\right|_{\text{left, m}=0} \cong \frac{T_1 - T_0}{\Delta x} \quad \text{and} \quad \left.\frac{dT}{dx}\right|_{\text{right,m}=4} \cong \frac{T_4 - T_3}{\Delta x}$$

Substituting, the finite difference formulation of the boundary nodes become

at $x = 0$: $$-k\frac{T_1 - T_0}{\Delta x} = q_0$$

at $x = L$: $$-k\frac{T_4 - T_3}{\Delta x} = h[T_4 - T_\infty]$$

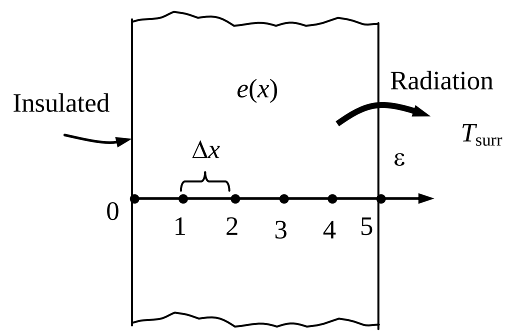
5-12 A plane wall with variable heat generation and constant thermal conductivity is subjected to insulation at the left (node 0) and radiation at the right boundary (node 5). Using the finite difference form of the 1st derivative, the finite difference formulation of the boundary nodes is to be determined.

Assumptions **1** Heat transfer through the wall is steady since there is no indication of change with time. **2** Heat transfer is one-dimensional since the plate is large relative to its thickness. **3** Thermal conductivity is constant and there is nonuniform heat generation in the medium. **4** Convection heat transfer is negligible.

Analysis The boundary conditions at the left and right boundaries can be expressed analytically as

At $x = 0$: $$-k\frac{dT(0)}{dx} = 0 \quad \text{or} \quad \frac{dT(0)}{dx} = 0$$

At $x = L$: $$-k\frac{dT(L)}{dx} = \varepsilon\sigma[T^4(L) - T_{surr}^4]$$



Replacing derivatives by differences using values at the closest nodes, the finite difference form of the 1st derivative of temperature at the boundaries (nodes 0 and 5) can be expressed as

$$\left.\frac{dT}{dx}\right|_{\text{left, m}=0} \cong \frac{T_1 - T_0}{\Delta x} \quad \text{and} \quad \left.\frac{dT}{dx}\right|_{\text{right, m}=5} \cong \frac{T_5 - T_4}{\Delta x}$$

Substituting, the finite difference formulation of the boundary nodes become

At $x = 0$: $$-k\frac{T_1 - T_0}{\Delta x} = 0 \quad \text{or} \quad T_1 = T_0$$

At $x = L$: $$-k\frac{T_5 - T_4}{\Delta x} = \varepsilon\sigma[T_5^4 - T_{surr}^4]$$

One-Dimensional Steady Heat Conduction

5-13C The finite difference form of a heat conduction problem by the *energy balance method* is obtained by *subdividing* the medium into a sufficient number of volume elements, and then applying an *energy balance* on each element. This is done by first *selecting* the nodal points (or nodes) at which the temperatures are to be determined, and then *forming elements* (or control volumes) over the nodes by drawing lines through the midpoints between the nodes. The properties *at the node* such as the temperature and the rate of heat generation represent the *average* properties of the element. The temperature is assumed to vary *linearly* between the nodes, especially when expressing heat conduction between the elements using Fourier's law.

5-14C The basic steps involved in the iterative Gauss-Seidel method are: (1) Writing the equations explicitly for each unknown (the unknown on the left-hand side and all other terms on the right-hand side of the equation), (2) making a reasonable initial guess for each unknown, (3) calculating new values for each unknown, *always using the most recent values*, and (4) repeating the process until desired convergence is achieved.

5-15C In a medium in which the finite difference formulation of a general interior node is given in its simplest form as

$$\frac{T_{m-1} - 2T_m + T_{m+1}}{\Delta x^2} + \frac{\dot{e}_m}{k} = 0$$

(*a*) heat transfer in this medium is **steady**, (*b*) it is **one-dimensional**, (*c*) there **is** heat generation, (*d*) the nodal spacing is **constant**, and (*e*) the thermal conductivity is **constant**.

5-16C In the finite difference formulation of a problem, an insulated boundary is best handled by replacing the insulation by a mirror, and treating the node on the boundary as an *interior* node. Also, a thermal symmetry line and an insulated boundary are treated the same way in the finite difference formulation.

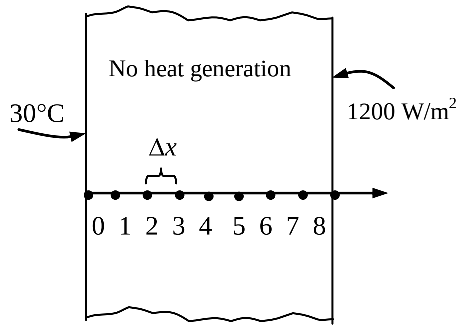
5-17C A node on an insulated boundary can be treated as an interior node in the finite difference formulation of a plane wall by replacing the insulation on the boundary by a *mirror*, and considering the reflection of the medium as its extension. This way the node next to the boundary node appears on both sides of the boundary node because of symmetry, converting it into an interior node.

5-18C In the energy balance formulation of the finite difference method, it is recommended that all heat transfer at the boundaries of the volume element be assumed to be *into* the volume element even for steady heat conduction. This is a valid recommendation even though it seems to violate the conservation of energy principle since the assumed direction of heat conduction at the surfaces of the volume elements has no effect on the formulation, and some heat conduction terms turn out to be negative.

5-19 A plane wall with no heat generation is subjected to specified temperature at the left (node 0) and heat flux at the right boundary (node 8). The finite difference formulation of the boundary nodes and the finite difference formulation for the rate of heat transfer at the left boundary are to be determined.

Assumptions **1** Heat transfer through the wall is given to be steady, and the thermal conductivity to be constant. **2** Heat transfer is one-dimensional since the plate is large relative to its thickness. **3** There is no heat generation in the medium.

Analysis Using the energy balance approach and taking the direction of all heat transfers to be towards the node under consideration, the finite difference formulations become



Left boundary node: $T_0 = 30$

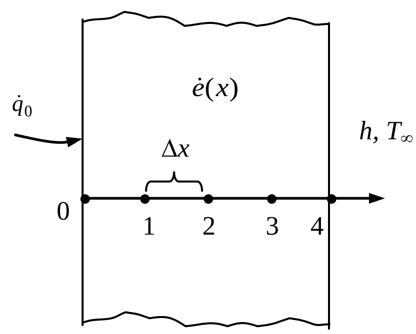
Right boundary node: $kA\dfrac{T_7 - T_8}{\Delta x} + \dot{q}_0 A = 0$ or $k\dfrac{T_7 - T_8}{\Delta x} + 1200 = 0$

Heat transfer at left surface:

$$\dot{Q}_{\text{left surface}} + kA\frac{T_1 - T_0}{\Delta x} = 0$$

5-20 A plane wall with variable heat generation and constant thermal conductivity is subjected to uniform heat flux $\dot{q}_0$ at the left (node 0) and convection at the right boundary (node 4). The finite difference formulation of the boundary nodes is to be determined.

Assumptions **1** Heat transfer through the wall is given to be steady, and the thermal conductivity to be constant. **2** Heat transfer is one-dimensional since the plate is large relative to its thickness. **3** Radiation heat transfer is negligible.



Analysis Using the energy balance approach and taking the direction of all heat transfers to be towards the node under consideration, the finite difference formulations become

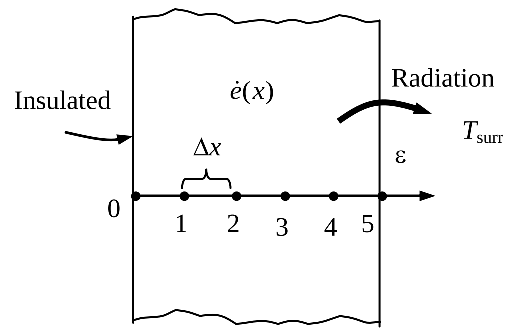
Left boundary node: $\dot{q}_0 A + kA\dfrac{T_1 - T_0}{\Delta x} + \dot{e}_0 (A\Delta x/2) = 0$

Right boundary node: $kA\dfrac{T_3 - T_4}{\Delta x} + hA(T_\infty - T_4) + \dot{e}_4 (A\Delta x/2) = 0$

5-21 A plane wall with variable heat generation and constant thermal conductivity is subjected to insulation at the left (node 0) and radiation at the right boundary (node 5). The finite difference formulation of the boundary nodes is to be determined.

Assumptions **1** Heat transfer through the wall is given to be steady and one-dimensional, and the thermal conductivity to be constant. **2** Convection heat transfer is negligible.

Analysis Using the energy balance approach and taking the direction of all heat transfers to be towards the node under consideration, the finite difference formulations become



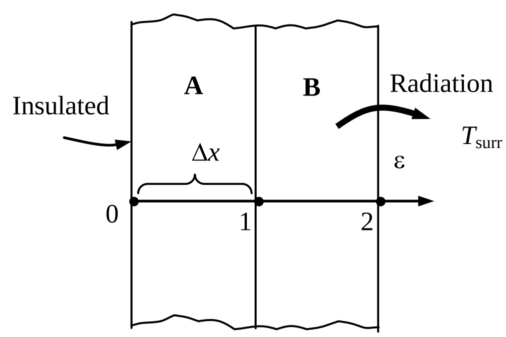
Left boundary node: $$kA\frac{T_1 - T_0}{\Delta x} + \dot{e}_0(A\Delta x/2) = 0$$

Right boundary node: $$\varepsilon\sigma A(T_{surr}^4 - T_5^4) + kA\frac{T_4 - T_5}{\Delta x} + \dot{e}_5(A\Delta x/2) = 0$$

5-22 A composite plane wall consists of two layers A and B in perfect contact at the interface where node 1 is. The wall is insulated at the left (node 0) and subjected to radiation at the right boundary (node 2). The complete finite difference formulation of this problem is to be obtained.

Assumptions **1** Heat transfer through the wall is given to be steady and one-dimensional, and the thermal conductivity to be constant. **2** Convection heat transfer is negligible. **3** There is no heat generation.

Analysis Using the energy balance approach and taking the direction of all heat transfers to be towards the node under consideration, the finite difference formulations become

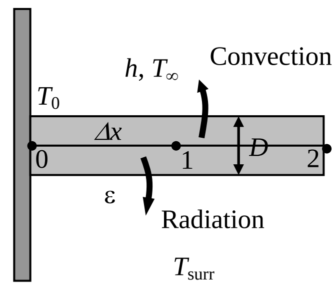


Node 0 (at left boundary): $$k_A A\frac{T_1 - T_0}{\Delta x} = 0 \rightarrow T_1 = T_0$$

Node 1 (at the interface): $$k_A A\frac{T_0 - T_1}{\Delta x} + k_B A\frac{T_2 - T_1}{\Delta x} = 0$$

Node 2 (at right boundary): $$\varepsilon\sigma A(T_{surr}^4 - T_2^4) + k_B A\frac{T_1 - T_2}{\Delta x} = 0$$

5-23 A pin fin with negligible heat transfer from its tip is considered. The complete finite difference formulation for the determination of nodal temperatures is to be obtained.



Assumptions **1** Heat transfer through the pin fin is given to be steady and one-dimensional, and the thermal conductivity to be constant. **2** Convection heat transfer coefficient is constant and uniform. **3** Heat loss from the fin tip is given to be negligible.

Analysis The nodal network consists of 3 nodes, and the base temperature T_0 at node 0 is specified. Therefore, there are two unknowns T_1 and T_2, and we need two equations to determine them. Using the energy balance approach and taking the direction of all heat transfers to be towards the node under consideration, the finite difference formulations become

Node 1 (at midpoint):

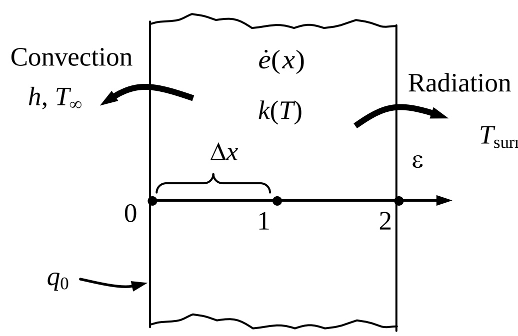
$$kA\frac{T_0 - T_1}{\Delta x} + kA\frac{T_2 - T_1}{\Delta x} + h(p\Delta x)(T_\infty - T_1) + \varepsilon\sigma(p\Delta x)\left[(T_{\text{surr}} + 273)^4 - (T_1 + 273)^4\right] = 0$$

Node 2 (at fin tip):

$$kA\frac{T_1 - T_2}{\Delta x} + h(p\Delta x/2)(T_\infty - T_2) + \varepsilon\sigma(p\Delta x/2)\left[(T_{\text{surr}} + 273)^4 - (T_2 + 273)^4\right] = 0$$

where $A = \pi D^2/4$ is the cross-sectional area and $p = \pi D$ is the perimeter of the fin.

5-24 A plane wall with variable heat generation and variable thermal conductivity is subjected to specified heat flux $\dot{q}_0$ and convection at the left boundary (node 0) and radiation at the right boundary (node 5). The complete finite difference formulation of this problem is to be obtained.



Assumptions **1** Heat transfer through the wall is given to be steady and one-dimensional, and the thermal conductivity and heat generation to be variable. **2** Convection heat transfer at the right surface is negligible.

Analysis Using the energy balance approach and taking the direction of all heat transfers to be towards the node under consideration, the finite difference formulations become

Node 0 (at left boundary):

$$\dot{q}_0 A + hA(T_\infty - T_0) + k_0 A\frac{T_1 - T_0}{\Delta x} + \dot{e}_0(A\Delta x/2) = 0$$

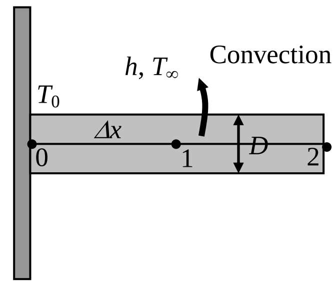
Node 1 (at the mid plane): $$k_1 A\frac{T_0 - T_1}{\Delta x} + k_1 A\frac{T_2 - T_1}{\Delta x} + \dot{e}_1(A\Delta x) = 0$$

Node 2 (at right boundary): $$\varepsilon\sigma A(T_{\text{surr}}^4 - T_2^4) + k_2 A\frac{T_1 - T_2}{\Delta x} + \dot{e}_2(A\Delta x/2) = 0$$

5-25 A pin fin with negligible heat transfer from its tip is considered. The complete finite difference formulation for the determination of nodal temperatures is to be obtained.

Assumptions **1** Heat transfer through the pin fin is given to be steady and one-dimensional, and the thermal conductivity to be constant. **2** Convection heat transfer coefficient is constant and uniform. **3** Radiation heat transfer is negligible. **4** Heat loss from the fin tip is given to be negligible.

Analysis The nodal network consists of 3 nodes, and the base temperature T_0 at node 0 is specified. Therefore, there are two unknowns T_1 and T_2, and we need two equations to determine them. Using the energy balance approach and taking the direction of all heat transfers to be towards the node under consideration, the finite difference formulations become



Node 1 (at midpoint): $$kA\frac{T_0-T_1}{\Delta x}+kA\frac{T_2-T_1}{\Delta x}+hp\Delta x(T_\infty-T_1)=0$$

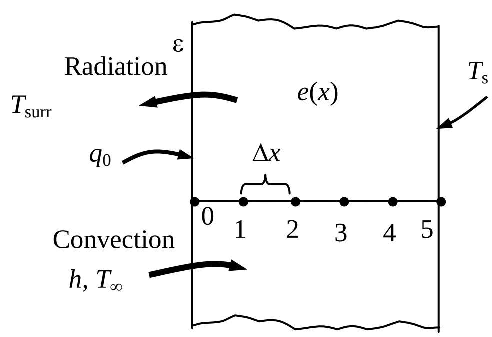
Node 2 (at fin tip): $$kA\frac{T_1-T_2}{\Delta x}+h(p\Delta x/2)(T_\infty-T_2)=0$$

where $A=\pi D^2/4$ is the cross-sectional area and $p=\pi D$ is the perimeter of the fin.

5-26 A plane wall with variable heat generation and constant thermal conductivity is subjected to combined convection, radiation, and heat flux at the left (node 0) and specified temperature at the right boundary (node 5). The finite difference formulation of the left boundary node (node 0) and the finite difference formulation for the rate of heat transfer at the right boundary (node 5) are to be determined.

Assumptions **1** Heat transfer through the wall is given to be steady and one-dimensional. **2** The thermal conductivity is given to be constant.

Analysis Using the energy balance approach and taking the direction of all heat transfers to be towards the node under consideration, the finite difference formulations become



Left boundary node (all temperatures are in K):

$$\varepsilon\sigma A(T_{\text{surr}}^4-T_0^4)+hA(T_\infty-T_0)+kA\frac{T_1-T_0}{\Delta x}+\dot{q}_0A+\dot{e}_0(A\Delta x/2)=0$$

Heat transfer at right surface:

$$\dot{Q}_{\text{right surface}}+kA\frac{T_4-T_5}{\Delta x}+\dot{e}_5(A\Delta x/2)=0$$

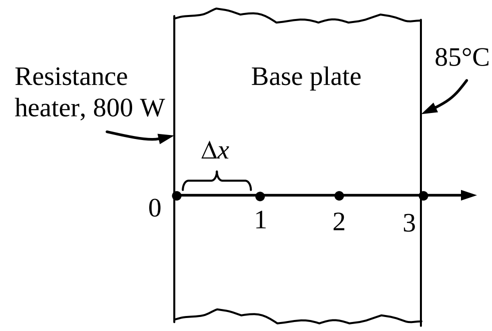
5-27 A plate is subjected to specified heat flux on one side and specified temperature on the other. The finite difference formulation of this problem is to be obtained, and the unknown surface temperature under steady conditions is to be determined.

Assumptions **1** Heat transfer through the base plate is given to be steady. **2** Heat transfer is one-dimensional since the plate is large relative to its thickness. **3** There is no heat generation in the plate. **4** Radiation heat transfer is negligible. **5** The entire heat generated by the resistance heaters is transferred through the plate.

Properties The thermal conductivity is given to be k = 20 W/m·°C.

Analysis The nodal spacing is given to be Δx=0.2 cm. Then the number of nodes M becomes

$$M = \frac{L}{\Delta x} + 1 = \frac{0.6\text{ cm}}{0.2\text{ cm}} + 1 = 4$$



The right surface temperature is given to be T_3 =85°C. This problem involves 3 unknown nodal temperatures, and thus we need to have 3 equations to determine them uniquely. Nodes 1 and 2 are interior nodes, and thus for them we can use the general finite difference relation expressed as

$$\frac{T_{m-1} - 2T_m + T_{m+1}}{\Delta x^2} + \frac{\dot{e}_m}{k} = 0 \rightarrow T_{m-1} - 2T_m + T_{m+1} = 0 \quad (\text{since } \dot{e} = 0), \quad \text{for } m = 1 \text{ and } 2$$

The finite difference equation for node 0 on the left surface subjected to uniform heat flux is obtained by applying an energy balance on the half volume element about node 0 and taking the direction of all heat transfers to be towards the node under consideration:

$$\text{Node 0 (left surface - heat flux)}: \quad \dot{q}_0 + k\frac{T_1 - T_0}{\Delta x} = 0$$

$$\text{Node 1 (interior)}: \quad T_0 - 2T_1 + T_2 = 0$$

$$\text{Node 2 (interior)}: \quad T_1 - 2T_2 + T_3 = 0$$

where

$$\Delta x = 0.2\text{ cm},\ k = 20\text{ W/m}\cdot{}^\circ\text{C},\ T_3 = 85^\circ\text{C, and } \dot{q}_0 = \dot{Q}_0 / A = (800\text{W})/(0.0160\text{ m}^2) = 50{,}000\text{ W/m}^2.$$

The system of 3 equations with 3 unknown temperatures constitute the finite difference formulation of the problem.

(b) The nodal temperatures under steady conditions are determined by solving the 3 equations above simultaneously with an equation solver to be

T_0 = **100°C**, T_1 =95°C, and T_2 =90°C

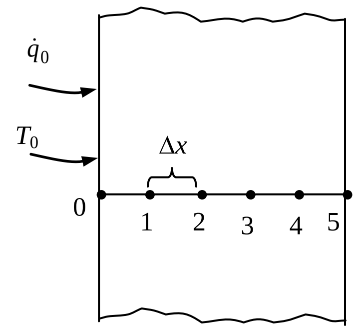
Discussion This problem can be solved analytically by solving the differential equation as described in Chap. 2, and the analytical (exact) solution can be used to check the accuracy of the numerical solution above.

5-28 A plane wall is subjected to specified heat flux and specified temperature on one side, and no conditions on the other. The finite difference formulation of this problem is to be obtained, and the temperature of the other side under steady conditions is to be determined.

Assumptions **1** Heat transfer through the plate is given to be steady and one-dimensional. **2** There is no heat generation in the plate.

Properties The thermal conductivity is given to be k = 2.5 W/m·°C.

Analysis The nodal spacing is given to be Δx=0.06 m.



Then the number of nodes M becomes

$$M = \frac{L}{\Delta x} + 1 = \frac{0.3\,\text{m}}{0.06\,\text{m}} + 1 = 6$$

Nodes 1, 2, 3, and 4 are interior nodes, and thus for them we can use the general finite difference relation expressed as

$$\frac{T_{m-1} - 2T_m + T_{m+1}}{\Delta x^2} + \frac{\dot{e}_m}{k} = 0 \rightarrow T_{m+1} - 2T_m + T_{m-1} = 0 \quad (\text{since } \dot{e} = 0), \quad \text{for } m = 1, 2, 3, \text{ and } 4$$

The finite difference equation for node 0 on the left surface is obtained by applying an energy balance on the half volume element about node 0 and taking the direction of all heat transfers to be towards the node under consideration,

$$\dot{q}_0 + k\frac{T_1 - T_0}{\Delta x} = 0 \longrightarrow 350\ \text{W/m}^2 + (2.5\ \text{W/m}\cdot{}^\circ\text{C})\frac{T_1 - 60^\circ\text{C}}{0.06\,\text{m}} = 0 \longrightarrow T_1 = 51.6^\circ\text{C}$$

Other nodal temperatures are determined from the general interior node relation as follows:

$$m = 1: \quad T_2 = 2T_1 - T_0 = 2\times 51.6 - 60 = 43.2^\circ\text{C}$$
$$m = 2: \quad T_3 = 2T_2 - T_1 = 2\times 43.2 - 51.6 = 34.8^\circ\text{C}$$
$$m = 3: \quad T_4 = 2T_3 - T_2 = 2\times 34.8 - 43.2 = 26.4^\circ\text{C}$$
$$m = 4: \quad T_5 = 2T_4 - T_3 = 2\times 26.4 - 34.8 = \mathbf{18.0^\circ C}$$

Therefore, the temperature of the other surface will be 18.0°C

Discussion This problem can be solved analytically by solving the differential equation as described in Chap. 2, and the analytical (exact) solution can be used to check the accuracy of the numerical solution above.

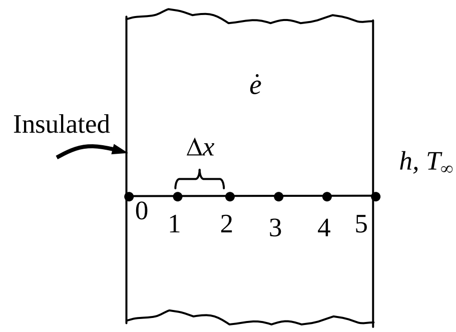
5-29 A uranium plate is subjected to insulation on one side and convection on the other. The finite difference formulation of this problem is to be obtained, and the nodal temperatures under steady conditions are to be determined.

Assumptions **1** Heat transfer through the wall is steady since there is no indication of change with time. **2** Heat transfer is one-dimensional since the plate is large relative to its thickness. **3** Thermal conductivity is constant. **4** Radiation heat transfer is negligible.

Properties The thermal conductivity is given to be k = 28 W/m·°C.

Analysis The number of nodes is specified to be M = 6. Then the nodal spacing Δx becomes

$$\Delta x = \frac{L}{M-1} = \frac{0.05\,\text{m}}{6\text{-}1} = 0.01\,\text{m}$$



This problem involves 6 unknown nodal temperatures, and thus we need to have 6 equations to determine them uniquely. Node 0 is on insulated boundary, and thus we can treat it as an interior note by using the mirror image concept. Nodes 1, 2, 3, and 4 are interior nodes, and thus for them we can use the general finite difference relation expressed as

$$\frac{T_{m-1} - 2T_m + T_{m+1}}{\Delta x^2} + \frac{\dot{e}_m}{k} = 0, \quad \text{for } m = 0, 1, 2, 3, \text{ and } 4$$

Finally, the finite difference equation for node 5 on the right surface subjected to convection is obtained by applying an energy balance on the half volume element about node 5 and taking the direction of all heat transfers to be towards the node under consideration:

$$\text{Node 0 (Left surface - insulated)}: \quad \frac{T_1 - 2T_0 + T_1}{\Delta x^2} + \frac{\dot{e}}{k} = 0$$

$$\text{Node 1 (interior)}: \quad \frac{T_0 - 2T_1 + T_2}{\Delta x^2} + \frac{\dot{e}}{k} = 0$$

$$\text{Node 2 (interior)}: \quad \frac{T_1 - 2T_2 + T_3}{\Delta x^2} + \frac{\dot{e}}{k} = 0$$

$$\text{Node 3 (interior)}: \quad \frac{T_2 - 2T_3 + T_4}{\Delta x^2} + \frac{\dot{e}}{k} = 0$$

$$\text{Node 4 (interior)}: \quad \frac{T_3 - 2T_4 + T_5}{\Delta x^2} + \frac{\dot{e}}{k} = 0$$

$$\text{Node 5 (right surface - convection)}: \quad h(T_\infty - T_5) + k\frac{T_4 - T_5}{\Delta x} + \dot{e}(\Delta x/2) = 0$$

where

$$\Delta x = 0.01\,\text{m}, \dot{e} = 6\times 10^5\ \text{W/m}^3,\ k = 28\ \text{W/m}\cdot\text{°C},\ h = 60\ \text{W/m}^2\cdot\text{°C},\ \text{and } T_\infty = 30\text{°C}.$$

This system of 6 equations with six unknown temperatures constitute the finite difference formulation of the problem.

(*b*) The 6 nodal temperatures under steady conditions are determined by solving the 6 equations above simultaneously with an equation solver to be

T_0 = **556.8°C**, T_1 = **555.7°C**, T_2 = **552.5°C**, T_3 = **547.1°C**, T_4 = **539.6°C**, and T_5 = **530.0°C**

Discussion This problem can be solved analytically by solving the differential equation as described in Chap. 2, and the analytical (exact) solution can be used to check the accuracy of the numerical solution above.

5-30 Prob. 5-29 is reconsidered. The nodal temperatures under steady conditions are to be determined.

Analysis The problem is solved using EES, and the solution is given below.

```
"GIVEN"
e_gen=6e5 [W/m^3] "heat generation"
dx=0.01 [m] "mesh size"
h=60 [W/m^2-K] "convection coefficient"
k=28 [W/m-K] "thermal conductivity"
T_inf=30 [C] "ambient temperature"

"ANALYSIS"
"Using the finite difference method, the nodal temperatures can be determined"
(T_1-T_0)/dx^2+e_gen/(2*k)=0 "for node 0"
(T_0-2*T_1+T_2)/dx^2+e_gen/k=0 "for node 1"
(T_1-2*T_2+T_3)/dx^2+e_gen/k=0 "for node 2"
(T_2-2*T_3+T_4)/dx^2+e_gen/k=0 "for node 3"
(T_3-2*T_4+T_5)/dx^2+e_gen/k=0 "for node 4"
h*(T_inf-T_5)+k*(T_4-T_5)/dx+e_gen*dx/2=0 "for node 5"
```

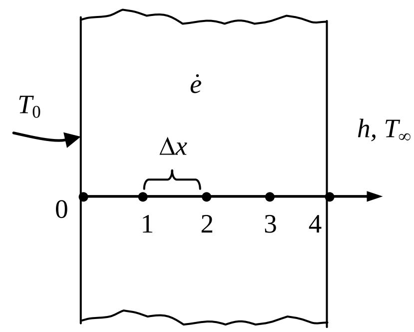
The nodal temperatures are determined to be

T_0 = **556.8°C**, T_1 = **555.7°C**, T_2 = **552.5°C**, T_3 = **547.1°C**, T_4 = **539.6°C**, T_5 = **530.0°C**

5-31 A plane wall is subjected to specified temperature on one side and convection on the other. The finite difference formulation of this problem is to be obtained, and the nodal temperatures under steady conditions as well as the rate of heat transfer through the wall are to be determined.

Assumptions **1** Heat transfer through the wall is given to be steady and one-dimensional. **2** Thermal conductivity is constant. **3** There is no heat generation. **4** Radiation heat transfer is negligible.

Properties The thermal conductivity is given to be k = 2.3 W/m·°C.



Analysis The nodal spacing is given to be Δx=0.1 m. Then the number of nodes M becomes

$$M = \frac{L}{\Delta x} + 1 = \frac{0.4\,\text{m}}{0.1\,\text{m}} + 1 = 5$$

The left surface temperature is given to be T_0 = 95°C. This problem involves 4 unknown nodal temperatures, and thus we need to have 4 equations to determine them uniquely. Nodes 1, 2, and 3 are interior nodes, and thus for them we can use the general finite difference relation expressed as

$$\frac{T_{m-1} - 2T_m + T_{m+1}}{\Delta x^2} + \frac{\dot{e}_m}{k} = 0 \;\rightarrow\; T_{m-1} - 2T_m + T_{m+1} = 0 \quad (\text{since } \dot{e} = 0), \quad \text{for } m = 0, 1, 2, \text{and } 3$$

The finite difference equation for node 4 on the right surface subjected to convection is obtained by applying an energy balance on the half volume element about node 4 and taking the direction of all heat transfers to be towards the node under consideration:

$$\begin{aligned}
&\text{Node 1 (interior):} && T_0 - 2T_1 + T_2 = 0 \\
&\text{Node 2 (interior):} && T_1 - 2T_2 + T_3 = 0 \\
&\text{Node 3 (interior):} && T_2 - 2T_3 + T_4 = 0 \\
&\text{Node 4 (right surface - convection):} && h(T_\infty - T_4) + k\frac{T_3 - T_4}{\Delta x} = 0
\end{aligned}$$

where

$$\Delta x = 0.1\,\text{m}, \; k = 2.3\,\text{W/m}\cdot{}^\circ\text{C}, \; h = 18\,\text{W/m}^2\cdot{}^\circ\text{C}, T_0 = 95^\circ\text{C} \text{ and } T_\infty = 15^\circ\text{C}.$$

The system of 4 equations with 4 unknown temperatures constitute the finite difference formulation of the problem.

(b) The nodal temperatures under steady conditions are determined by solving the 4 equations above simultaneously with an equation solver to be

T_1 = **79.8°C**, T_2 = **64.7°C**, T_3 = **49.5°C**, and T_4 = **34.4°C**

(c) The rate of heat transfer through the wall is simply convection heat transfer at the right surface,

$$\dot{Q}_{\text{wall}} = \dot{Q}_{\text{conv}} = hA(T_4 - T_\infty) = (18\,\text{W/m}^2.^\circ\text{C})(20\,\text{m}^2)(34.37\text{-}15)^\circ\text{C} = \mathbf{6970\ W}$$

Discussion This problem can be solved analytically by solving the differential equation as described in Chap. 2, and the analytical (exact) solution can be used to check the accuracy of the numerical solution above.

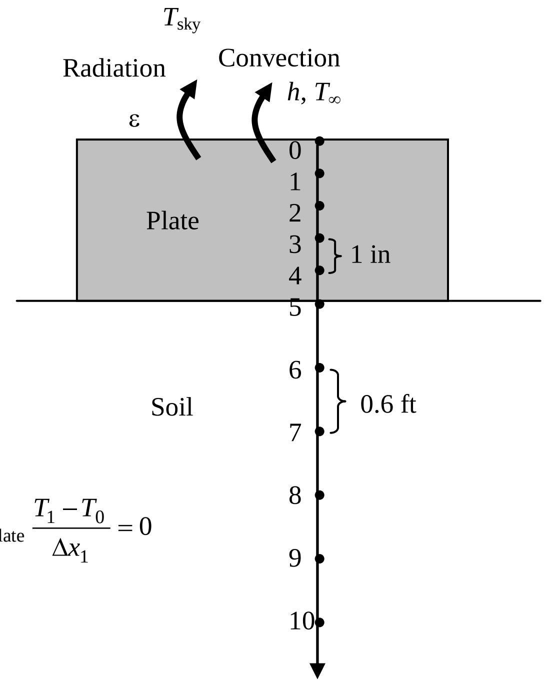
5-32E A large plate lying on the ground is subjected to convection and radiation. Finite difference formulation is to be obtained, and the top and bottom surface temperatures under steady conditions are to be determined.

Assumptions **1** Heat transfer through the plate is given to be steady and one-dimensional. **2** There is no heat generation in the plate and the soil. **3** Thermal contact resistance at plate-soil interface is negligible.

Properties The thermal conductivity of the plate and the soil are given to be k_{plate} = 7.2 Btu/h·ft·°F and k_{soil} = 0.49 Btu/h·ft·°F.

Analysis The nodal spacing is given to be Δx_1=1 in. in the plate, and be Δx_2=0.6 ft in the soil. Then the number of nodes becomes

$$M = \left(\frac{L}{\Delta x}\right)_{\text{plate}} + \left(\frac{L}{\Delta x}\right)_{\text{soil}} + 1 = \frac{5\text{ in}}{1\text{ in}} + \frac{3\text{ ft}}{0.6\text{ ft}} + 1 = 11$$



The temperature at node 10 (bottom of thee soil) is given to be T_{10} =50°F. Nodes 1, 2, 3, and 4 in the plate and 6, 7, 8, and 9 in the soil are interior nodes, and thus for them we can use the general finite difference relation expressed as

$$\frac{T_{m-1} - 2T_m + T_{m+1}}{\Delta x^2} + \frac{\dot{e}_m}{k} = 0 \rightarrow T_{m-1} - 2T_m + T_{m+1} = 0 \quad (\text{since } \dot{e} = 0)$$

The finite difference equation for node 0 on the left surface and node 5 at the interface are obtained by applying an energy balance on their respective volume elements and taking the direction of all heat transfers to be towards the node under consideration:

Node 0 (top surface) : $h(T_\infty - T_0) + \varepsilon\sigma[T_{sky}^4 - (T_0 + 460)^4] + k_{\text{plate}}\dfrac{T_1 - T_0}{\Delta x_1} = 0$

Node 1 (interior) : $T_0 - 2T_1 + T_2 = 0$

Node 2 (interior) : $T_1 - 2T_2 + T_3 = 0$

Node 3 (interior) : $T_2 - 2T_3 + T_4 = 0$

Node 4 (interior) : $T_3 - 2T_4 + T_5 = 0$

Node 5 (interface) : $k_{\text{plate}}\dfrac{T_4 - T_5}{\Delta x_1} + k_{\text{soil}}\dfrac{T_6 - T_5}{\Delta x_2} = 0$

Node 6 (interior) : $T_5 - 2T_6 + T_7 = 0$

Node 7 (interior) : $T_6 - 2T_7 + T_8 = 0$

Node 8 (interior) : $T_7 - 2T_8 + T_9 = 0$

Node 9 (interior) : $T_8 - 2T_9 + T_{10} = 0$

where

Δx_1=1/12 ft, Δx_2=0.6 ft, k_{plate} = 7.2 Btu/h·ft·°F, k_{soil} = 0.49 Btu/h·ft·°F,

h = 3.5 Btu/h·ft²·°F, T_{sky} =510 R, ε = 0.6, T_∞ = 80°F, and T_{10} =50°F.

This system of 10 equations with 10 unknowns constitute the finite difference formulation of the problem.

(*b*) The temperatures are determined by solving equations above to be

T_0 = **74.71°F**, T_1 =74.67°F, T_2 =74.62°F, T_3 =74.58°F, T_4 =74.53°F,

T_5 = **74.48°F**, T_6 =69.6°F, T_7 =64.7°F, T_8 =59.8°F, T_9 =54.9°F

Discussion Note that the plate is essentially isothermal at about 74.6°F. Also, the temperature in each layer varies linearly and thus we could solve this problem by considering 3 nodes only (one at the interface and two at the boundaries).

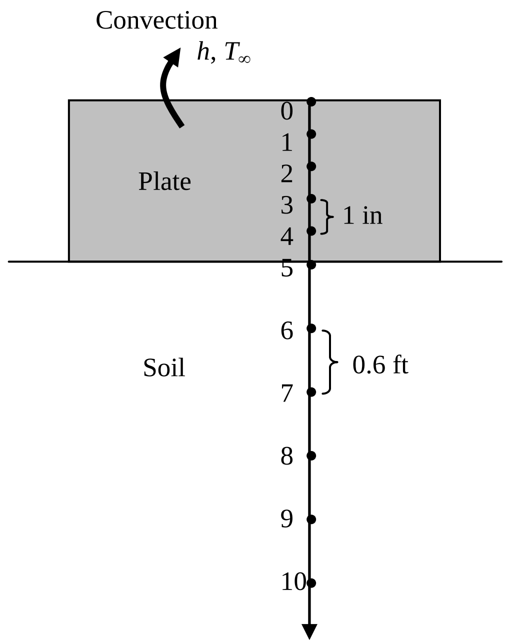
5-33E A large plate lying on the ground is subjected to convection from its exposed surface. The finite difference formulation of this problem is to be obtained, and the top and bottom surface temperatures under steady conditions are to be determined.

Assumptions **1** Heat transfer through the plate is given to be steady and one-dimensional. **2** There is no heat generation in the plate and the soil. **3** The thermal contact resistance at the plate-soil interface is negligible. **4** Radiation heat transfer is negligible.

Properties The thermal conductivity of the plate and the soil are given to be k_{plate} = 7.2 Btu/h·ft·°F and k_{soil} = 0.49 Btu/h·ft·°F.

Analysis The nodal spacing is given to be Δx_1=1 in. in the plate, and be Δx_2=0.6 ft in the soil. Then the number of nodes becomes

$$M = \left(\frac{L}{\Delta x}\right)_{\text{plate}} + \left(\frac{L}{\Delta x}\right)_{\text{soil}} + 1 = \frac{5\text{ in}}{1\text{ in}} + \frac{3\text{ ft}}{0.6\text{ ft}} + 1 = 11$$



The temperature at node 10 (bottom of thee soil) is given to be T_{10} =50°F. Nodes 1, 2, 3, and 4 in the plate and 6, 7, 8, and 9 in the soil are interior nodes, and thus for them we can use the general finite difference relation expressed as

$$\frac{T_{m-1} - 2T_m + T_{m+1}}{\Delta x^2} + \frac{\dot{e}_m}{k} = 0 \rightarrow T_{m-1} - 2T_m + T_{m+1} = 0 \quad (\text{since } \dot{e} = 0)$$

The finite difference equation for node 0 on the left surface and node 5 at the interface are obtained by applying an energy balance on their respective volume elements and taking the direction of all heat transfers to be towards the node under consideration:

$$\text{Node 0 (top surface)}: \quad h(T_\infty - T_0) + k_{\text{plate}}\frac{T_1 - T_0}{\Delta x_1} = 0$$

$$\text{Node 1 (interior)}: \quad T_0 - 2T_1 + T_2 = 0$$

$$\text{Node 2 (interior)}: \quad T_1 - 2T_2 + T_3 = 0$$

$$\text{Node 3 (interior)}: \quad T_2 - 2T_3 + T_4 = 0$$

$$\text{Node 4 (interior)}: \quad T_3 - 2T_4 + T_5 = 0$$

$$\text{Node 5 (interface)}: \quad k_{\text{plate}}\frac{T_4 - T_5}{\Delta x_1} + k_{\text{soil}}\frac{T_6 - T_5}{\Delta x_2} = 0$$

$$\text{Node 6 (interior)}: \quad T_5 - 2T_6 + T_7 = 0$$

$$\text{Node 7 (interior)}: \quad T_6 - 2T_7 + T_8 = 0$$

$$\text{Node 8 (interior)}: \quad T_7 - 2T_8 + T_9 = 0$$

$$\text{Node 9 (interior)}: \quad T_8 - 2T_9 + T_{10} = 0$$

where

Δx_1=1/12 ft, Δx_2=0.6 ft, k_{plate} = 7.2 Btu/h·ft·°F, k_{soil} = 0.49 Btu/h·ft·°F,

h = 3.5 Btu/h·ft²·°F, $T_\infty = 80°\text{F}$, and T_{10} =50°F.

This system of 10 equations with 10 unknowns constitute the finite difference formulation of the problem.

(*b*) The temperatures are determined by solving equations above to be

T_0 = **78.67°F**, T_1 =78.62°F, T_2 =78.57°F, T_3 =78.51°F, T_4 =78.46°F,

T_5 = **78.41°F**, T_6 =72.7°F, T_7 =67.0°F, T_8 =61.4°F, T_9 =55.7°F

Discussion Note that the plate is essentially isothermal at about 78.6°F. Also, the temperature in each layer varies linearly and thus we could solve this problem by considering 3 nodes only (one at the interface and two at the boundaries).

5-34 A steel plate with no internal heat generation is subjected to a uniform heat flux on its top surface while the bottom surface is cooled convectively by a fluid at 10°C and having h = 150 W/m^2·K. Using finite difference formulation, the temperature at the midpoint of the plate is to be determined.

Assumptions **1** Steady state 1-D heat transfer in lateral direction. **2** Constant thermal conductivity of the steel plate.

Properties Thermal conductivity of the steel plate is given as 35 W/m·K.

Analysis To discretize the plate of thickness 0.1 m into four equal parts, each part must be of length 0.025 m i.e.,

$$\Delta x = 0.025m$$

And hence the number of nodes is

$$M = 1 + \frac{L}{\Delta x} = 1 + \frac{0.1}{0.025} = 5$$

This problem involves 5 unknown nodal temperatures and hence we need 5 equations to determine these temperatures. The steel plate thickness is discretized such that the node 0 is on the bottom of the plate exposed to convective environment while the node 4 is on the top surface exposed to the uniform heat flux. Nodes 1, 2 and 3 are the internal nodes and their temperature can be expressed using general form of the finite difference formulation.

$$\frac{T_{m-1} - 2T_m + T_{m+1}}{\Delta x^2} + \frac{\dot{e}_m}{k} = 0 \quad \text{for } m = 0, 1, 2 \text{ and } 3$$

The finite difference formulation at the top and bottom surfaces can be obtained by applying energy balance on the half volume element around these nodes and considering all heat transfers towards these nodes. The finite difference equations for different nodes without internal heat generation are as follows.

Node 0 (Bottom node): $hA(T_\infty - T_0) + kA\left(\frac{T_1 - T_0}{\Delta x}\right) = 0$

Node 1 (Interior node): $\frac{T_0 - 2T_1 + T_2}{\Delta x} = 0$

Node 2 (Interior node): $\frac{T_1 - 2T_2 + T_3}{\Delta x} = 0$

Node 3 (Interior node): $\frac{T_2 - 2T_3 + T_4}{\Delta x} = 0$

Node 4 (Top node): $\dot{q}_0 A + kA\left(\frac{T_3 - T_4}{\Delta x}\right) = 0$

where

$$\Delta x = 0.025\,\text{m}, \text{h} = 150\,\text{W/m}^2\cdot\text{K}, \text{k} = 35\,\text{W/m}\cdot\text{K}, \dot{\text{q}}_0 = 5500\,\text{W/m}^2 \text{ and } \text{T}_\infty = 10^\circ\text{C}$$

Solving these five equations for five unknowns using EES or any other software gives

T_0 = **46.67°C**, T_1 = **50.6°C**, T_2 = **54.52°C**, T_3 = **58.45°C**, T_4 = **62.38°C**.

5-35 A composite wall made up of two different materials is subjected to constant temperature and radiation boundary condition. The finite difference formulations and the temperature distribution across the wall thickness is to be determined

Assumptions **1** 1-D steady state heat conduction. **2** No internal heat generation in material B. **3** Constant thermal conductivities. **4** Perfect contact at the material A and B interface.

Properties Thermal conductivity of material A is k = 45 W/m·K and that of material B is k = 28 W/m·K.

Analysis Using a nodal spacing of 2.5 cm, the composite wall of thickness 20 cm (10 cm thick material A and B each) can be discretized into 8 equal parts (4 parts of material A and B each).

And hence the number of nodes is

$$M = 1 + \frac{L}{\Delta x} = 1 + \frac{0.2}{0.025} = 9$$

This problem involves 9 unknown nodal temperatures and hence we need 9 equations to determine these temperatures. The composite wall thickness is discretized such that the node 1 is on the left side boundary of the wall exposed to a constant heat flux while node 9 is placed such that it is on the right boundary of the composite wall. Node 5 is at the interface of material A and B.

The finite difference equations at all nodes are as follows

Node 1: (Left boundary node) $\dot{q} + k_A \frac{(T_2 - T_1)}{\Delta x} + \dot{e}_m \frac{\Delta x}{2} = 0$

Node 2: (Internal node) $k_A \frac{(T_1 - T_2)}{\Delta x} + k_A \frac{(T_3 - T_2)}{\Delta x} + \dot{e}_m \Delta x = 0$

Node 3: (Internal node) $k_A \frac{(T_2 - T_3)}{\Delta x} + k_A \frac{(T_4 - T_3)}{\Delta x} + \dot{e}_m \Delta x = 0$

Node 4: (Internal node) $k_A \frac{(T_3 - T_4)}{\Delta x} + k_A \frac{(T_5 - T_4)}{\Delta x} + \dot{e}_m \Delta x = 0$

Node 5: (Interface node) $k_A \frac{(T_4 - T_5)}{\Delta x} + \dot{e}_m \frac{\Delta x}{2} + k_B \frac{(T_6 - T_5)}{\Delta x} = 0$

Node 6: (Internal node) $k_B \frac{(T_5 - T_6)}{\Delta x} + k_B \frac{(T_7 - T_6)}{\Delta x} = 0$

Node 7: (Internal node) $k_B \frac{(T_6 - T_7)}{\Delta x} + k_B \frac{(T_8 - T_7)}{\Delta x} = 0$

Node 8: (Internal node) $k_B \frac{(T_7 - T_{87})}{\Delta x} + k_B \frac{(T_9 - T_8)}{\Delta x} = 0$

Node 9: (Right boundary node) $h(T_\infty - T_9) + \varepsilon\sigma(T_{surr}^4 - T_9^4) + k_B \frac{(T_8 - T_9)}{\Delta x} = 0$

T_1 = **216.9°C**, T_2 = **213.9°C**, T_3 = **209.9°C**, T_4 = **205°C**, T_5 = **199.1°C**,

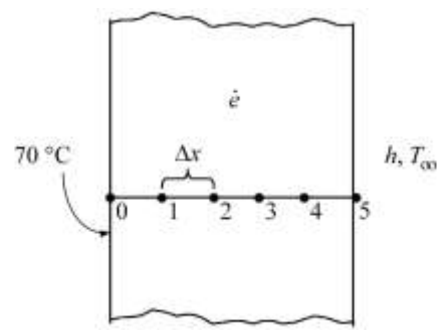
T_6 = **188.8°C**, T_7 = **178.6°C**, T_8 = **168.3°C**, T_9 = **158°C**.

5-36 A stainless steel plane wall experiencing a uniform heat generation is subjected to constant temperature on one side and convection on the other. The finite difference equations and the nodal temperatures are to be determined.

Assumptions **1** Heat transfer through the wall is steady and one-dimensional. **2** Thermal properties are constant. **3** Heat transfer by radiation is negligible.

Properties The thermal conductivity is given as 15.1 W/m·K.

Analysis (*a*) The nodal spacing is given to be Δx = 2 cm. Then the number of nodes M becomes



$$M = \frac{L}{\Delta x} + 1 = \frac{1\,\text{m}}{0.2\,\text{m}} + 1 = 6$$

The left surface temperature is given to be T_0 = 70°C. There are 5 unknown nodal temperatures, thus we need to have 5 equations to determine them uniquely. Nodes 1, 2, 3, and 4 are interior nodes, and we can use the general finite difference relation expressed as

$$\frac{T_{m-1} - 2T_m + T_{m+1}}{\Delta x^2} + \frac{\dot{e}_m}{k} = 0 \quad \rightarrow \quad T_{m-1} - 2T_m + T_{m+1} + \frac{\dot{e}_m}{k}\Delta x^2 = 0$$

The finite difference equation for node 5 on the right surface subjected to convection is obtained by applying an energy balance on the half volume element about that node:

$$k\frac{T_4 - T_5}{\Delta x} + \dot{e}_5\frac{\Delta x}{2} + h(T_\infty - T_5) = 0 \quad \rightarrow \quad T_4 - \left(1 + \frac{h}{k}\Delta x\right)T_5 + \frac{\Delta x^2}{2k}\dot{e}_5 + \frac{h}{k}\Delta x T_\infty = 0$$

Then

m = 1: $T_0 - 2T_1 + T_2 + (\dot{e}_1 / k)\Delta x^2 = 0$

m = 2: $T_1 - 2T_2 + T_3 + (\dot{e}_2 / k)\Delta x^2 = 0$

m = 3: $T_2 - 2T_3 + T_4 + (\dot{e}_3 / k)\Delta x^2 = 0$

m = 4: $T_3 - 2T_4 + T_5 + (\dot{e}_4 / k)\Delta x^2 = 0$

m = 5: $T_4 - (1 + h\Delta x / k)T_5 + (\Delta x^2 \dot{e}_5)/(2k) + (h\Delta x / k)T_\infty = 0$

(*b*) The nodal temperatures under steady conditions are determined by solving the 5 equations above simultaneously with an equation solver. Copy the following lines and paste on a blank EES screen to solve the above equations:

```
e_gen=1000
h=250
k=15.1
Dx=0.2
T_inf=0
T_0=70
T_0-2*T_1+T_2+(e_gen/k)*Dx^2=0
T_1-2*T_2+T_3+(e_gen/k)*Dx^2=0
T_2-2*T_3+T_4+(e_gen/k)*Dx^2=0
T_3-2*T_4+T_5+(e_gen/k)*Dx^2=0
T_4-(1+h*Dx/k)*T_5+(Dx^2*e_gen)/(2*k)+(h*Dx/k)*T_inf=0
```

Solving by EES software, we get

$$T_1 = \mathbf{62.5°C}, \quad T_2 = \mathbf{52.3°C}, \quad T_3 = \mathbf{39.5°C}, \quad T_4 = \mathbf{24.0°C}, \quad T_5 = \mathbf{5.87°C}$$

Discussion For a very large value of convection heat transfer coefficient (e.g. 20000 W/m^2·K), the right surface temperature would become approximately the same as the ambient fluid temperature ($T_5 \approx T_\infty$).

5-37 For a 0.1 m thick stainless steel plate exposed to a constant heat flux and convection environment, temperature distribution is to be determined for a case of variable thermal conductivity.

Assumptions **1** One-dimensional steady state heat conduction. **2** No internal heat generation.

Properties The thermal conductivity of the stainless steel plate is given as $k(T) = k_o(1+\beta T)$ where k_o = 48 W/m·K and β = 9.21x10^{-4} ^{o}C^{-1}

Analysis The one-dimensional first derivative of the finite difference formulation for an interior node 'm' in between node 'm-1' and 'm+2' is expressed as

$$k(T)\frac{dT}{dx}\bigg|_{m-\frac{1}{2}} = k_o\left[1+\beta\frac{(T_{m-1}+T_m)}{2}\right]\frac{(T_{m-1}-T_m)}{\Delta x} \quad \text{and} \quad k(T)\frac{dT}{dx}\bigg|_{m+\frac{1}{2}} = k_o\left[1+\beta\frac{(T_{m+1}+T_m)}{2}\right]\frac{(T_{m+1}-T_m)}{\Delta x}$$

Now accounting for the internal heat generation and combining these two equations to get the finite difference equation for the interior node 'm' gives

$$(T_{m+1}-T_m)\left[1+\frac{\beta}{2}(T_{m+1}+T_m)\right]+(T_{m-1}-T_m)\left[1+\frac{\beta}{2}(T_{m-1}+T_m)\right]+\dot{e}_m\frac{\Delta x^2}{k_o}=0$$

$$\therefore (T_{m-1}-2T_m+T_{m+1})+\frac{\beta}{2}\left[T_{m-1}^2-2T_m^2+T_{m+1}^2\right]+\dot{e}_m\frac{\Delta x^2}{k_o}=0$$

For the left boundary node exposed to constant heat flux we replace thermal conductivity 'k' in Eq. (5-22) by $k(T) = k_o(1+\beta T)$ resulting into following equation

$$\therefore (T_{m+1}-T_m)+\frac{\beta}{2}\left(T_{m+1}^2-T_m^2\right)+\frac{\dot{q}\Delta x}{k_o}+\frac{\dot{e}_m\Delta x^2}{2k_o}=0$$

For the right boundary node exposed to convection environment we replace thermal conductivity 'k' in Eq. (5-24) by $k(T) = k_o(1+\beta T)$ resulting into following equation

$$\therefore (T_{m-1}-T_m)+\frac{\beta}{2}\left(T_{m-1}^2-T_m^2\right)+\frac{h\Delta x}{k_o}(T_\infty-T_m)+\frac{\dot{e}_m\Delta x^2}{2k_o}=0$$

The finite difference equations for the interior and boundary nodes based on the above equations are expressed as follows

Node 0: (Left boundary node) $\quad (T_1-T_0)+\frac{\beta}{2}\left(T_1^2-T_0^2\right)+\frac{\dot{q}\Delta x}{k_o}+\frac{\dot{e}_m\Delta x^2}{2k_o}=0$

Node 1: (Internal node) $\quad (T_0-2T_1+T_2)+\frac{\beta}{2}\left(T_0^2-2T_1^2+T_2^2\right)+\frac{\dot{e}_m\Delta x^2}{k_o}=0$

Node 2: (Internal node) $\quad (T_1-2T_2+T_3)+\frac{\beta}{2}\left(T_1^2-2T_2^2+T_3^2\right)+\frac{\dot{e}_m\Delta x^2}{k_o}=0$

Node 3: (Internal node) $\quad (T_2-2T_3+T_4)+\frac{\beta}{2}\left(T_2^2-2T_3^2+T_4^2\right)+\frac{\dot{e}_m\Delta x^2}{k_o}=0$

Node 4: (Internal node) $\quad (T_3-2T_4+T_5)+\frac{\beta}{2}\left(T_3^2-2T_4^2+T_5^2\right)+\frac{\dot{e}_m\Delta x^2}{k_o}=0$

Node 5: (Right boundary node) $\quad (T_4-T_5)+\frac{\beta}{2}\left(T_4^2-T_5^2\right)+\frac{h\Delta x}{k_o}(T_\infty-T_5)+\frac{\dot{e}_m\Delta x^2}{2k_o}=0$

The temperature distribution in the stainless steel plate is found by solving these 5 equations simultaneously in EES or any other software.

```
"Given data"
k_o = 48 [W/mC] " Thermal conductivity"
beta = 9.21e-4 [C^-1] “Temperature coefficient of thermal conductivity”
e_dot = 8e5 [W/m^3] " Internal heat generation per unit volume"
q_dot = 2000 [W/m^2] "Heat flux at left boundary"
DELTAx = 0.02 [m] " Mesh size"
h = 400 [W/m^2C] "Convective heat transfer coefficient at right boundary"
T_infi = 0 [C] " Conective environment temperature"

"Finite difference equations"
"Node 0"(T[1]-T[0])+beta/2*(T[1]^2-T[0]^2)+q_dot*DELTAx/k_o+e_dot*DELTAx^2/(2*k_o) = 0
"Node 1"(T[0]-2*T[1]+T[2])+beta/2*(T[0]^2-2*T[1]^2+T[2]^2)+e_dot*DELTAx^2/k_o = 0
"Node 2"(T[1]-2*T[2]+T[3])+beta/2*(T[1]^2-2*T[2]^2+T[3]^2)+e_dot*DELTAx^2/k_o = 0
"Node 3"(T[2]-2*T[3]+T[4])+beta/2*(T[2]^2-2*T[3]^2+T[4]^2)+e_dot*DELTAx^2/k_o = 0
"Node 4"(T[3]-2*T[4]+T[5])+beta/2*(T[3]^2-2*T[4]^2+T[5]^2)+e_dot*DELTAx^2/k_o = 0
"Node 5"(T[4]-T[5])+beta/2*(T[4]^2-T[5]^2)+e_dot*DELTAx^2/(2*k_o)+h*DELTAx/k_o*(T_infi-T[5]) = 0
```

T_0 = **276.6 °C**, T_1 = **273.3 °C**, T_2 = **264.6 °C**, T_3 = **250.5 °C**, T_4 = **230.7 °C**, T_5 = **205 °C.**

Discussion Thermal conductivity as a function of temperature must be accounted for during finite difference formulations for high temperature applications such as metallurgical processes and thermal power generation.

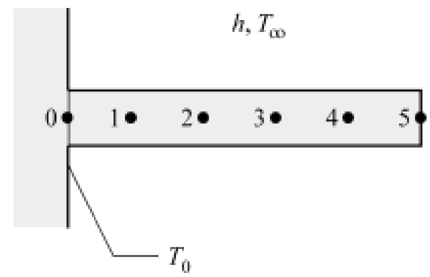
5-38 Straight rectangular fins are attached to a plane wall. For a single fin, (*a*) the finite difference equations, (*b*) the nodal temperatures, and (*c*) heat transfer rate are to be determined. The heat transfer rate is also to be compared with analytical solution.

Assumptions **1** Heat transfer along the fin is steady and one-dimensional. **2** Thermal properties are constant. **3** Heat transfer by radiation is negligible.

Properties The thermal conductivity is given as 235 W/m·K.

Analysis (*a*) The nodal spacing is given to be Δx = 10 cm. Then the number of nodes M becomes

$$M = \frac{L}{\Delta x} + 1 = \frac{50\text{ mm}}{10\text{ mm}} + 1 = 6$$



The base temperature at node 0 is given to be T_0 = 350°C. There are 5 unknown nodal temperatures, thus we need to have 5 equations to determine them uniquely. Nodes 1, 2, 3, and 4 are interior nodes, and we can use the general finite difference relation expressed as

$$kA\frac{T_{m-1} - T_m}{\Delta x} + kA\frac{T_{m+1} - T_m}{\Delta x} + h(p\Delta x)(T_\infty - T_m) = 0$$

$$T_{m-1} - \left(2 + \frac{hp\Delta x^2}{kA}\right)T_m + T_{m+1} + \frac{hp\Delta x^2}{kA}T_\infty = 0$$

where

$$\frac{hp\Delta x^2}{kA} = \frac{h(2t+2w)\Delta x^2}{k(wt)} = \frac{(154\text{ W/m}^2\cdot\text{K})2(0.005\text{ m} + 0.1\text{ m})(0.01\text{ m})^2}{(235\text{ W/m}\cdot\text{K})(0.005\text{ m})(0.1\text{ m})} = 0.0275$$

The finite difference equation for node 5 at the fin tip (convection boundary) is obtained by applying an energy balance on the half volume element about that node:

$$kA\frac{T_4 - T_5}{\Delta x} + h\left(\frac{p\Delta x}{2} + A\right)(T_\infty - T_5) = 0$$

$$T_4 - \left[1 + \frac{h\Delta x}{kA}\left(\frac{p\Delta x}{2} + A\right)\right]T_5 + \frac{h\Delta x}{kA}\left(\frac{p\Delta x}{2} + A\right)T_\infty = 0$$

where

$$\frac{h\Delta x}{kA}\left(\frac{p\Delta x}{2} + A\right) = \frac{h\Delta x}{k}\left[\frac{(t+w)\Delta x}{wt} + 1\right] = 0.0203$$

Then,

m = 1: $\quad T_0 - 2.0275T_1 + T_2 + 0.0275T_\infty = 0$

m = 2: $\quad T_1 - 2.0275T_2 + T_3 + 0.0275T_\infty = 0$

m = 3: $\quad T_2 - 2.0275T_3 + T_4 + 0.0275T_\infty = 0$

m = 4: $\quad T_3 - 2.0275T_4 + T_5 + 0.0275T_\infty = 0$

m = 5: $\quad T_4 - 1.0203T_5 + 0.0203T_\infty = 0$

(*b*) The nodal temperatures under steady conditions are determined by solving the 5 equations above simultaneously with an equation solver. Copy the following lines and paste on a blank EES screen to solve the above equations:

```
T_0=350
T_0-2.0275*T_1+T_2+0.0275*25=0
T_1-2.0275*T_2+T_3+0.0275*25=0
T_2-2.0275*T_3+T_4+0.0275*25=0
T_3-2.0275*T_4+T_5+0.0275*25=0
```

```
T_4-1.0203*T_5+0.0203*25=0
```

Solving by EES software, we get

$$T_1 = \mathbf{316.6^\circ C}, \quad T_2 = \mathbf{291.2^\circ C}, \quad T_3 = \mathbf{273.2^\circ C}, \quad T_4 = \mathbf{261.9^\circ C}, \quad T_5 = \mathbf{257.2^\circ C}$$

(*c*) The rate of heat transfer from a single fin is simply the sum of the heat transfer from the nodal elements,

$$\begin{aligned}\dot{Q}_{\text{fin, num}} &= \sum_{m=0}^{5} \dot{Q}_{\text{element}, m} = \sum_{m=0}^{5} hA_{\text{surface}, m}(T_m - T_\infty) \\ &= hp\frac{\Delta x}{2}(T_0 - T_\infty) + hp\Delta x(T_1 + T_2 + T_3 + T_4 - 4T_\infty) + h\left(p\frac{\Delta x}{2} + A\right)(T_5 - T_\infty) \\ &= \mathbf{445\ W}\end{aligned}$$

For straight rectangular fins, the analytical solution from Chapter 3 for the heat transfer rate is,

$$\dot{Q}_{\text{fin, exact}} = \eta_{\text{fin}} hA_{\text{fin}}(T_b - T_\infty) = (0.813)(154\ \text{W/m}^2\cdot\text{K})(0.0105\ \text{m}^2)(350-25)\ ^\circ\text{C} = \mathbf{427\ W}$$

where

$$m = \sqrt{\frac{2h}{kt}} = 16.19\ \text{m}^{-1}$$

$$L_c = L + t/2 = 0.0525\ \text{m}$$

$$A_{\text{fin}} = 2wL_c = 0.0105\ \text{m}^2$$

$$\eta_{\text{fin}} = \frac{\tanh mL_c}{mL_c} = 0.813$$

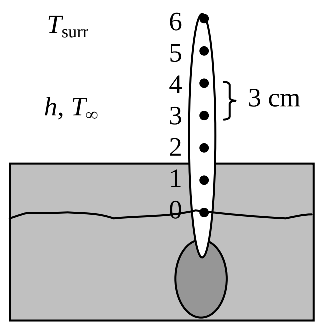
Discussion The comparison between the analytical and numerical solutions is within ±4.3% agreement. One way to increase the accuracy of the numerical solution is by reducing the nodal spacing, thereby increasing the number of nodes.

5-39 The handle of a stainless steel spoon partially immersed in boiling water loses heat by convection and radiation. The finite difference formulation of the problem is to be obtained, and the tip temperature of the spoon as well as the rate of heat transfer from the exposed surfaces are to be determined.

Assumptions **1** Heat transfer through the handle of the spoon is given to be steady and one-dimensional. **2** Thermal conductivity and emissivity are constant. **3** Convection heat transfer coefficient is constant and uniform.

Properties The thermal conductivity and emissivity are given to be k = 15.1 W/m·°C and ε = 0.6.

Analysis The nodal spacing is given to be Δx=3 cm. Then the number of nodes M becomes



$$M = \frac{L}{\Delta x} + 1 = \frac{18\text{ cm}}{3\text{ cm}} + 1 = 7$$

The base temperature at node 0 is given to be T_0 = 95°C. This problem involves 6 unknown nodal temperatures, and thus we need to have 6 equations to determine them uniquely. Nodes 1, 2, 3, 4, and 5 are interior nodes, and thus for them we can use the general finite difference relation expressed as

$$kA\frac{T_{m-1} - T_m}{\Delta x} + kA\frac{T_{m+1} - T_m}{\Delta x} + h(p\Delta x)(T_\infty - T_m) + \varepsilon\sigma(p\Delta x)[T_{surr}^4 - (T_m + 273)^4] = 0$$

or $\quad T_{m-1} - 2T_m + T_{m+1} + h(p\Delta x^2 / kA)(T_\infty - T_m) + \varepsilon\sigma(p\Delta x^2 / kA)[T_{surr}^4 - (T_m + 273)^4] = 0$, m = 1,2,3,4,5

The finite difference equation for node 6 at the fin tip is obtained by applying an energy balance on the half volume element about node 6. Then,

$$m=1:\ T_0 - 2T_1 + T_2 + h(p\Delta x^2 / kA)(T_\infty - T_1) + \varepsilon\sigma(p\Delta x^2 / kA)[T_{surr}^4 - (T_1 + 273)^4] = 0$$

$$m=2:\ T_1 - 2T_2 + T_3 + h(p\Delta x^2 / kA)(T_\infty - T_2) + \varepsilon\sigma(p\Delta x^2 / kA)[T_{surr}^4 - (T_2 + 273)^4] = 0$$

$$m=3:\ T_2 - 2T_3 + T_4 + h(p\Delta x^2 / kA)(T_\infty - T_3) + \varepsilon\sigma(p\Delta x^2 / kA)[T_{surr}^4 - (T_3 + 273)^4] = 0$$

$$m=4:\ T_3 - 2T_4 + T_5 + h(p\Delta x^2 / kA)(T_\infty - T_4) + \varepsilon\sigma(p\Delta x^2 / kA)[T_{surr}^4 - (T_4 + 273)^4] = 0$$

$$m=5:\ T_4 - 2T_5 + T_6 + h(p\Delta x^2 / kA)(T_\infty - T_5) + \varepsilon\sigma(p\Delta x^2 / kA)[T_{surr}^4 - (T_5 + 273)^4] = 0$$

$$\text{Node 6: } kA\frac{T_5 - T_6}{\Delta x} + h(p\Delta x / 2 + A)(T_\infty - T_6) + \varepsilon\sigma(p\Delta x / 2 + A)[T_{surr}^4 - (T_6 + 273)^4] = 0$$

where

$$\Delta x = 0.03\text{ m},\ k = 15.1\text{ W/m}\cdot{}^\circ\text{C},\ \varepsilon = 0.6,\ T_\infty = 25^\circ\text{C}, T_0 = 95^\circ\text{C},\ T_{surr} = 295\text{ K},\ h = 13\text{ W/m}^2\cdot{}^\circ\text{C}$$

and $\quad A = (1\text{ cm})(0.2\text{ cm}) = 0.2\text{ cm}^2 = 0.2\times10^{-4}\text{ m}^2$ and $p = 2(1 + 0.2\text{ cm}) = 2.4\text{ cm} = 0.024\text{ m}$

The system of 6 equations with 6 unknowns constitute the finite difference formulation of the problem.

(b) The nodal temperatures under steady conditions are determined by solving the 6 equations above simultaneously with an equation solver to be

T_1 = 49.0°C, T_2 = 33.0°C, T_3 = 27.4°C, T_4 = 25.5°C, T_5 = 24.8°C, and T_6 = **24.6°C**,

(c) The total rate of heat transfer from the spoon handle is simply the sum of the heat transfer from each nodal element, and is determined from

$$\dot{Q}_{fin} = \sum_{m=0}^{6}\dot{Q}_{element,\,m} = \sum_{m=0}^{6} hA_{surface,m}(T_m - T_\infty) + \sum_{m=0}^{6}\varepsilon\sigma A_{surface,m}[(T_m + 273)^4 - T_{surr}^4] = \mathbf{0.923\ \ W}$$

where $A_{\text{surface, m}}$ $=p\Delta x/2$ for node 0, $A_{\text{surface, m}}$ $=p\Delta x/2$+A for node 6, and $A_{\text{surface, m}}$ $=p\Delta x$ for other nodes.

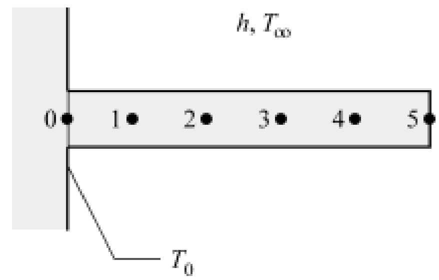
5-40 A circular fin of uniform cross section is attached to a wall. The finite difference equations for all nodes are to be obtained, the nodal temperatures along the fin and the heat transfer rate are to be determined and compared with analytical solutions.

Assumptions **1** Heat transfer along the fin is steady and one-dimensional. **2** Thermal properties are constant. **3** Heat transfer by radiation is negligible.

Properties The thermal conductivity of the fin is given as 240 W/m·K.

Analysis (*a*) The nodal spacing is given to be Δx = 10 mm. Then the number of nodes M becomes

$$M = \frac{L}{\Delta x} + 1 = \frac{50\text{ mm}}{10\text{ mm}} + 1 = 6$$



The base temperature at node 0 is given to be T_0 = 350°C. There are 5 unknown nodal temperatures, thus we need to have 5 equations to determine them uniquely. Nodes 1, 2, 3, and 4 are interior nodes, and we can use the general finite difference relation expressed as

$$kA\frac{T_{m-1} - T_m}{\Delta x} + kA\frac{T_{m+1} - T_m}{\Delta x} + h(p\Delta x)(T_\infty - T_m) = 0$$

$$T_{m-1} - 2T_m + T_{m+1} + \frac{hp\Delta x^2}{kA}(T_\infty - T_m) = 0$$

where

$$\frac{hp\Delta x^2}{kA} = \frac{4h\Delta x^2}{kD} = \frac{4(250\text{ W/m}^2\cdot\text{K})(0.01\text{ m})^2}{(240\text{ W/m}\cdot\text{K})(0.01\text{ m})} = 0.04167$$

The finite difference equation for node 5 at the fin tip (convection boundary) is obtained by applying an energy balance on the half volume element about that node:

$$kA\frac{T_4 - T_5}{\Delta x} + h\left(\frac{p\Delta x}{2} + A\right)(T_\infty - T_5) = 0 \quad \rightarrow \quad T_4 - T_5 + \frac{h\Delta x}{kA}\left(\frac{p\Delta x}{2} + A\right)(T_\infty - T_5) = 0$$

where

$$\frac{h\Delta x}{kA}\left(\frac{p\Delta x}{2} + A\right) = \frac{h\Delta x}{k}\left(\frac{2\Delta x}{D} + 1\right) = 0.03125$$

Then,

m = 1: $\quad T_0 - 2T_1 + T_2 + 0.04167(T_\infty - T_1) = 0$

m = 2: $\quad T_1 - 2T_2 + T_3 + 0.04167(T_\infty - T_2) = 0$

m = 3: $\quad T_2 - 2T_3 + T_4 + 0.04167(T_\infty - T_3) = 0$

m = 4: $\quad T_3 - 2T_4 + T_5 + 0.04167(T_\infty - T_4) = 0$

m = 5: $\quad T_4 - T_5 + 0.03125(T_\infty - T_5) = 0$

(*b*) The nodal temperatures under steady conditions are determined by solving the 5 equations above simultaneously with an equation solver. Copy the following lines and paste on a blank EES screen to solve the above equations:

```
T_0=350
T_0-2*T_1+T_2+0.04167*(25-T_1)=0
T_1-2*T_2+T_3+0.04167*(25-T_2)=0
T_2-2*T_3+T_4+0.04167*(25-T_3)=0
T_3-2*T_4+T_5+0.04167*(25-T_4)=0
T_4-T_5+0.03125*(25-T_5)=0
```

Solving by EES software, we get

$$T_1 = \mathbf{304.1°C},\ T_2 = \mathbf{269.9°C},\ T_3 = \mathbf{245.9°C},\ T_4 = \mathbf{231.0°C},\ T_5 = \mathbf{224.8°C}$$

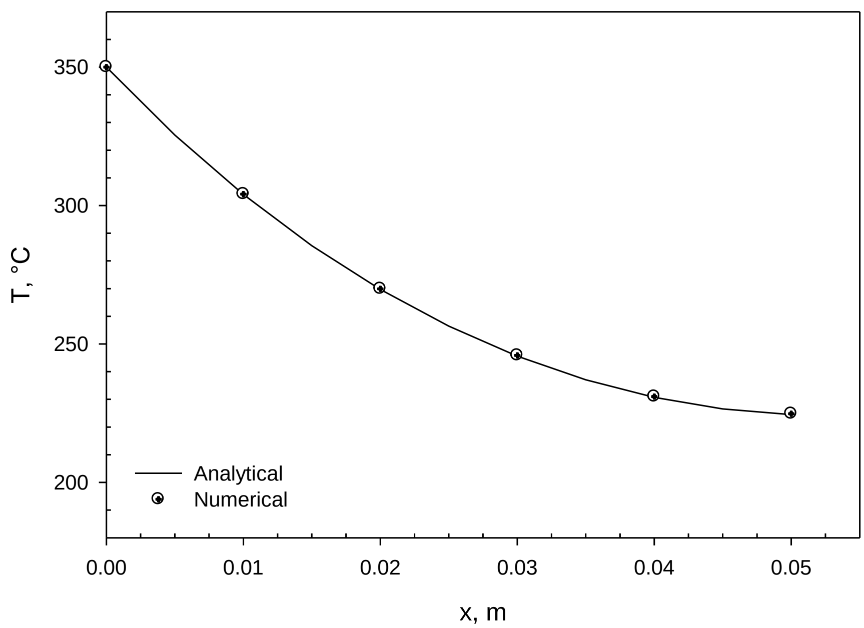
From Chapter 3, the analytical solution for the temperature variation along the fin (for convection from fin tip) is given as

$$\frac{T(x) - T_\infty}{T_b - T_\infty} = \frac{\cosh m(L-x) + (h/mk)\sinh m(L-x)}{\cosh mL + (h/mk)\sinh mL}$$

The nodal temperatures for analytical and numerical solutions are tabulated in the following table:

x, m	$T(x)$,°C	
	Analytical	Numerical
0	350.0	350.0
0.01	304.0	304.1
0.02	269.7	269.9
0.03	245.6	245.9
0.04	230.7	231.0
0.05	224.5	224.8

The comparison of the analytical and numerical solutions is shown in the following figure:



(c) The rate of heat transfer from a single fin is simply the sum of the heat transfer from the nodal elements,

$$\dot{Q}_{\text{fin}} = \sum_{m=0}^{5} \dot{Q}_{\text{element},\, m} = \sum_{m=0}^{5} hA_{\text{surface},\, m}(T_m - T_\infty)$$

$$= hp\frac{\Delta x}{2}(T_0 - T_\infty) + hp\Delta x(T_1 + T_2 + T_3 + T_4 - 4T_\infty) + h\left(p\frac{\Delta x}{2} + A\right)(T_5 - T_\infty)$$

$$= \mathbf{99.2\ W}$$

From Chapter 3, the analytical solution for the heat transfer rate of fin with convection from the tip is,

$$\dot{Q}_{\text{conv tip}} = \sqrt{hpkA_c}\,(T_b - T_\infty)\frac{\sinh mL + (h/mk)\cosh mL}{\cosh mL + (h/mk)\sinh mL}$$

$$= (0.3848\ \text{W/°C})(350\ \text{°C} - 25\ \text{°C})(0.7901)$$

$$= \mathbf{98.8\ W}$$

where

$$m = \sqrt{\frac{hp}{kA_c}} = 20.41\ \text{m}^{-1}, \qquad p = \pi D = 0.03142\ \text{m}, \qquad A_c = \frac{\pi D^2}{4} = 7.854\times 10^{-5}\ \text{m}^2$$

Discussion For part (b), the comparison between the analytical and numerical solutions is excellent, with agreement within ±0.15%. For part (c), the comparison between the analytical and numerical solutions is within ±0.5%.

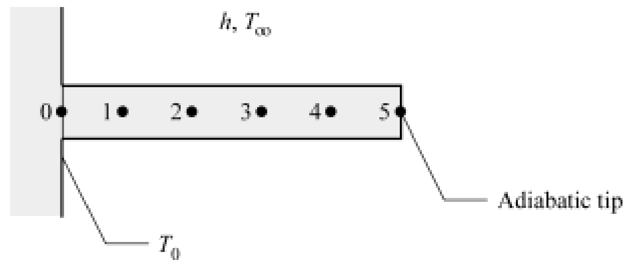
5-41 A circular aluminum fin of uniform cross section with adiabatic tip is attached to a wall. The finite difference equations for all nodes are to be obtained and solved using Gauss-Seidel iterative method, and the nodal temperatures along the fin are to be determined and compared with analytical solution.

Assumptions **1** Heat transfer along the fin is steady and one-dimensional. **2** Thermal properties are constant. **3** Heat transfer by radiation is negligible.

Properties The thermal conductivity of the fin is given as 237 W/m·K.

Analysis (*a*) The nodal spacing is given to be Δx = 10 mm. Then the number of nodes M becomes

$$M = \frac{L}{\Delta x} + 1 = \frac{5\text{ cm}}{1\text{ cm}} + 1 = 6$$



The base temperature at node 0 is given to be T_0 = 300°C. There are 5 unknown nodal temperatures, thus we need to have 5 equations to determine them uniquely. Nodes 1, 2, 3, and 4 are interior nodes, and we can use the general finite difference relation expressed in explicit form as

$$kA\frac{T_{m-1} - T_m}{\Delta x} + kA\frac{T_{m+1} - T_m}{\Delta x} + h(p\Delta x)(T_\infty - T_m) = 0$$

$$T_m = \left(2 + \frac{hp\Delta x^2}{kA}\right)^{-1}\left(T_{m-1} + T_{m+1} + \frac{hp\Delta x^2}{kA}T_\infty\right)$$

The finite difference equation for node 5 at the fin tip (adiabatic) is obtained by applying an energy balance on the half volume element about that node:

$$2kA\frac{T_4 - T_5}{\Delta x} + h(p\Delta x)(T_\infty - T_5) = 0 \quad \rightarrow \quad T_5 = \left(2 + \frac{hp\Delta x^2}{kA}\right)^{-1}\left(2T_4 + \frac{hp\Delta x^2}{kA}T_\infty\right)$$

Then,

m = 1: $\quad T_1 = 0.4938T_0 + 0.4938T_2 + 0.1875$

m = 2: $\quad T_2 = 0.4938T_1 + 0.4938T_3 + 0.1875$

m = 3: $\quad T_3 = 0.4938T_2 + 0.4938T_4 + 0.1875$

m = 4: $\quad T_4 = 0.4938T_3 + 0.4938T_5 + 0.1875$

m = 5: $\quad T_5 = 0.9876T_4 + 0.1875$

(*b*) By letting the initial guesses as $T_1 = T_2 = T_3 = T_4 = T_5 = 250\,°\text{C}$, the results obtained from successive iterations are listed in the following table:

Iteration	Nodal temperature,°C				
	T_1	T_2	T_3	T_4	T_5
1	271.8	257.8	251.0	247.6	244.7
2	275.6	260.2	250.9	244.9	242.1
3	276.8	260.8	249.9	243.1	240.3
4	277.1	260.4	248.8	241.7	238.9
5	276.9	259.8	247.8	240.5	237.7
6	276.6	259.2	246.9	239.5	236.7
7	276.3	258.6	246.1	238.6	235.9
8	276.0	258.0	245.4	237.9	235.1
...	...	...	...	...	...
52	273.7	253.9	240.1	232.0	229.3

Hence, the converged nodal temperatures are

$$T_1 = \mathbf{273.7°C},\ \ T_2 = \mathbf{253.9°C},\ \ T_3 = \mathbf{240.1°C},\ \ T_4 = \mathbf{232.0°C},\ \ T_5 = \mathbf{229.3°C}$$

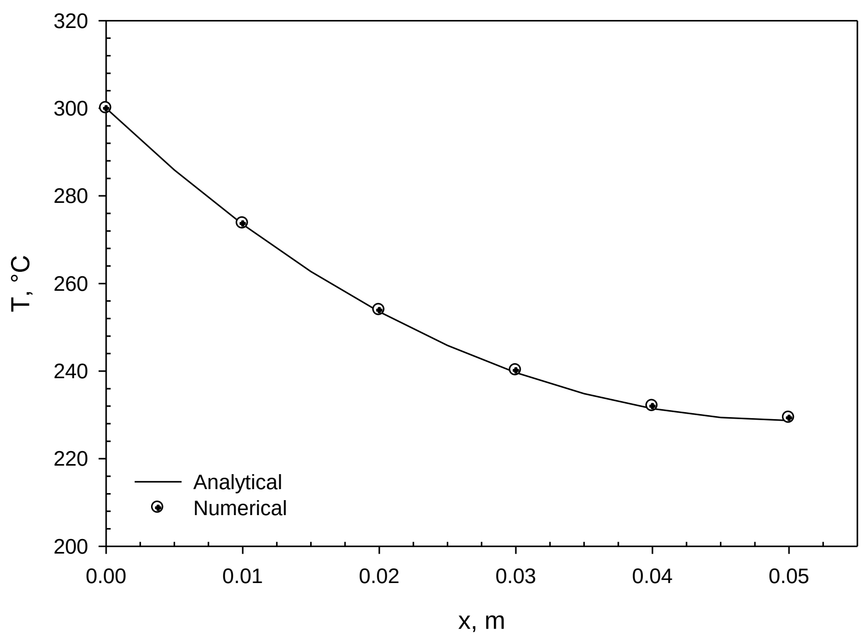
From Chapter 3, the analytical solution for the temperature variation along the fin (for adiabatic tip) is given as

$$\frac{T(x) - T_\infty}{T_b - T_\infty} = \frac{\cosh m(L-x)}{\cosh mL}$$

The nodal temperatures for analytical and numerical solutions are tabulated in the following table:

x, m	*T*(*x*),°C	
	Analytical	Numerical
0	300.0	300.0
0.01	273.5	273.7
0.02	253.5	253.9
0.03	239.6	240.1
0.04	231.4	232.0
0.05	228.7	229.3

The comparison of the analytical and numerical solutions is shown in the following figure:



Discussion The comparison between the analytical and numerical solutions is excellent, with agreement within ±0.3%.

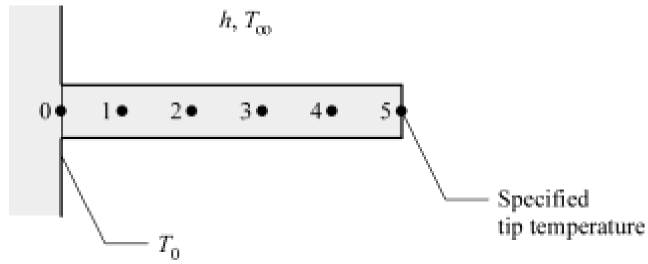
5-42 A circular fin of uniform cross section is attached to a wall with the fin tip temperature specified as 250°C. The finite difference equations for all nodes are to be obtained and the nodal temperatures along the fin are to be determined and compared with analytical solution.

Assumptions **1** Heat transfer along the fin is steady and one-dimensional. **2** Thermal properties are constant. **3** Heat transfer by radiation is negligible.

Properties The thermal conductivity of the fin is given as 240 W/m·K.

Analysis (*a*) The nodal spacing is given to be Δx = 10 mm. Then the number of nodes *M* becomes

$$M = \frac{L}{\Delta x} + 1 = \frac{50\text{ mm}}{10\text{ mm}} + 1 = 6$$



The base temperature at node 0 is given to be T_0 =350°C and the tip temperature at node 5 is given as T_5 = 200°C. There are 4 unknown nodal temperatures, thus we need to have 4 equations to determine them uniquely. Nodes 1, 2, 3, and 4 are interior nodes, and we can use the general finite difference relation expressed as

$$kA\frac{T_{m-1} - T_m}{\Delta x} + kA\frac{T_{m+1} - T_m}{\Delta x} + h(p\Delta x)(T_\infty - T_m) = 0$$

$$T_{m-1} - 2T_m + T_{m+1} + \frac{hp\Delta x^2}{kA}(T_\infty - T_m) = 0$$

where

$$\frac{hp\Delta x^2}{kA} = \frac{4h\Delta x^2}{kD} = \frac{4(250\text{ W/m}^2\cdot\text{K})(0.01\text{ m})^2}{(240\text{ W/m}\cdot\text{K})(0.01\text{ m})} = 0.04167$$

Then,

m = 1: $\quad T_0 - 2T_1 + T_2 + 0.04167(T_\infty - T_1) = 0$

m = 2: $\quad T_1 - 2T_2 + T_3 + 0.04167(T_\infty - T_2) = 0$

m = 3: $\quad T_2 - 2T_3 + T_4 + 0.04167(T_\infty - T_3) = 0$

m = 4: $\quad T_3 - 2T_4 + T_5 + 0.04167(T_\infty - T_4) = 0$

(*b*) The nodal temperatures under steady conditions are determined by solving the 4 equations above simultaneously with an equation solver. Copy the following lines and paste on a blank EES screen to solve the above equations:

```
T_0=350
T_5=200
T_0-2*T_1+T_2+0.04167*(25-T_1)=0
T_1-2*T_2+T_3+0.04167*(25-T_2)=0
T_2-2*T_3+T_4+0.04167*(25-T_3)=0
T_3-2*T_4+T_5+0.04167*(25-T_4)=0
```

Solving by EES software, we get

$$T_1 = \mathbf{299.9\,°C},\ \ T_2 = \mathbf{261.3\,°C},\ \ T_3 = \mathbf{232.5\,°C},\ \ T_4 = \mathbf{212.3\,°C}$$

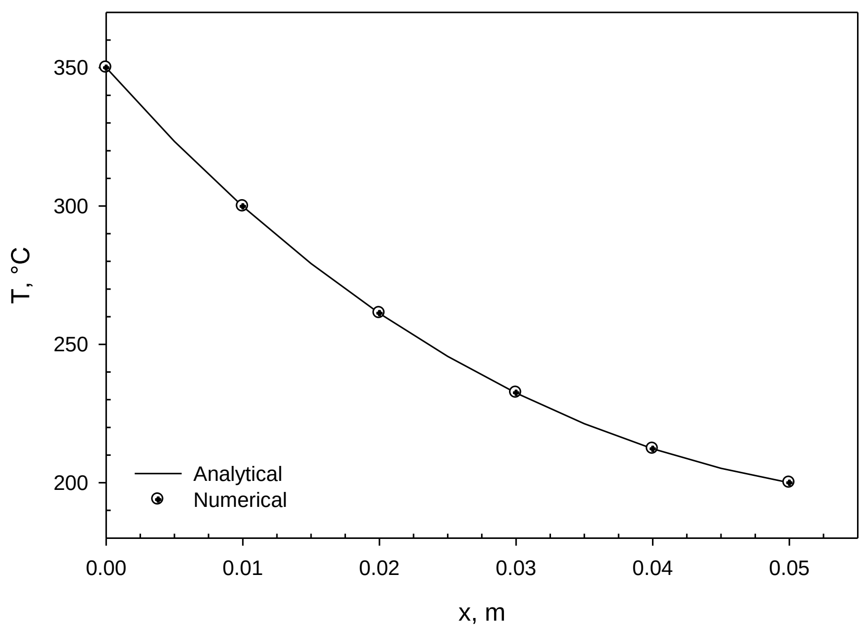
From Chapter 3, the analytical solution for the temperature variation along the fin (for specified tip temperature) is given as

$$\frac{T(x) - T_\infty}{T_b - T_\infty} = \frac{(T_L - T_\infty)/(T_b - T_\infty)\sinh mx + \sinh m(L - x)}{\sinh mL}$$

The nodal temperatures for analytical and numerical solutions are tabulated in the following table:

x, m	$T(x)$,°C	
	Analytical	Numerical
0	350.0	350.0
0.01	299.8	299.9
0.02	261.2	261.3
0.03	232.4	232.5
0.04	212.3	212.3
0.05	200.0	200.0

The comparison of the analytical and numerical solutions is shown in the following figure:



Discussion The comparison between the analytical and numerical solutions is excellent, with agreement within ±0.05%.

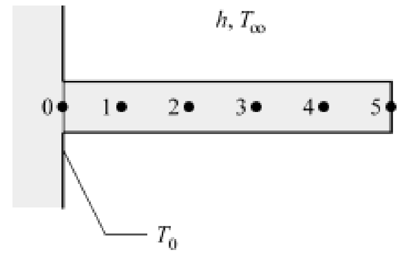
5-43 A DC motor delivers mechanical power to a rotating stainless steel shaft. With a uniform nodal spacing of 5 cm along shaft, the finite difference equations and the nodal temperatures are to be determined.

Assumptions **1** Heat transfer along the shaft is steady and one-dimensional. **2** Thermal properties are constant. **3** Heat transfer by radiation is negligible.

Properties The thermal conductivity of the shaft is given as 15.1 W/m·K.

Analysis (*a*) The nodal spacing is given to be Δx = 5 cm. Then the number of nodes *M* becomes

$$M = \frac{L}{\Delta x} + 1 = \frac{25\text{ cm}}{5\text{ cm}} + 1 = 6$$



The base temperature at node 0 is given to be T_0 = 90°C. There are 5 unknown nodal temperatures, thus we need to have 5 equations to determine them uniquely. Nodes 1, 2, 3, and 4 are interior nodes, and we can use the general finite difference relation expressed as

$$kA\frac{T_{m-1} - T_m}{\Delta x} + kA\frac{T_{m+1} - T_m}{\Delta x} + h(p\Delta x)(T_\infty - T_m) = 0$$

$$T_{m-1} - \left(2 + \frac{hp\Delta x^2}{kA}\right)T_m + T_{m+1} + \frac{hp\Delta x^2}{kA}T_\infty = 0$$

where $\dfrac{hp\Delta x^2}{kA} = \dfrac{4h\Delta x^2}{kD} = \dfrac{4(25\text{ W/m}^2\cdot\text{K})(0.05\text{ m})^2}{(15.5\text{ W/m}\cdot\text{K})(0.025\text{ m})} = 0.6452$

The finite difference equation for node 5 at the fin tip (convection boundary) is obtained by applying an energy balance on the half volume element about that node:

$$kA\frac{T_4 - T_5}{\Delta x} + h\left(\frac{p\Delta x}{2} + A\right)(T_\infty - T_5) = 0$$

$$T_4 - \left[1 + \frac{h\Delta x}{kA}\left(\frac{p\Delta x}{2} + A\right)\right]T_5 + \frac{h\Delta x}{kA}\left(\frac{p\Delta x}{2} + A\right)T_\infty = 0$$

where $\dfrac{h\Delta x}{kA}\left(\dfrac{p\Delta x}{2} + A\right) = \dfrac{h\Delta x}{k}\left(\dfrac{2\Delta x}{D} + 1\right) = 0.4032$

Then,

$m = 1$: $T_0 - 2.6452T_1 + T_2 + 0.6452T_\infty = 0$

$m = 2$: $T_1 - 2.6452T_2 + T_3 + 0.6452T_\infty = 0$

$m = 3$: $T_2 - 2.6452T_3 + T_4 + 0.6452T_\infty = 0$

$m = 4$: $T_3 - 2.6452T_4 + T_5 + 0.6452T_\infty = 0$

$m = 5$: $T_4 - 1.4032T_5 + 0.4032T_\infty = 0$

The nodal temperatures under steady conditions are determined by solving the 5 equations above simultaneously with an equation solver. Copy the following lines and paste on a blank EES screen to solve the above equations:

```
T_0=90
T_0-2.6452*T_1+T_2+0.6452*20=0
T_1-2.6452*T_2+T_3+0.6452*20=0
T_2-2.6452*T_3+T_4+0.6452*20=0
T_3-2.6452*T_4+T_5+0.6452*20=0
T_4-1.4032*T_5+0.4032*20=0
```

Solving by EES software, we get

$T_1 = \mathbf{52.03°C}$, $T_2 = \mathbf{34.72°C}$, $T_3 = \mathbf{26.92°C}$, $T_4 = \mathbf{23.58°C}$, $T_5 = \mathbf{22.55°C}$

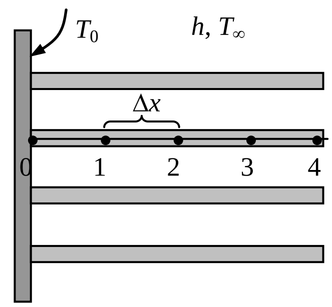
Discussion The nodal temperatures along the motor shaft can be compared with the analytical solution from Chapter 3 for fin with convection fin tip boundary condition:

$$\frac{T(x) - T_\infty}{T_b - T_\infty} = \frac{\cosh m(L-x) + (h/mk)\sinh m(L-x)}{\cosh mL + (h/mk)\sinh mL}$$

5-44 One side of a hot vertical plate is to be cooled by attaching aluminum fins of rectangular profile. The finite difference formulation of the problem for all nodes is to be obtained, and the nodal temperatures, the rate of heat transfer from a single fin and from the entire surface of the plate are to be determined.

Assumptions **1** Heat transfer along the fin is given to be steady and one-dimensional. **2** The thermal conductivity is constant. **3** Combined convection and radiation heat transfer coefficient is constant and uniform.

Properties The thermal conductivity is given to be k = 237 W/m·°C.



Analysis (a) The nodal spacing is given to be Δx=0.5 cm. Then the number of nodes M becomes

$$M = \frac{L}{\Delta x} + 1 = \frac{2\,\text{cm}}{0.5\,\text{cm}} + 1 = 5$$

The base temperature at node 0 is given to be T_0 = 80°C. This problem involves 4 unknown nodal temperatures, and thus we need to have 4 equations to determine them uniquely. Nodes 1, 2, and 3 are interior nodes, and thus for them we can use the general finite difference relation expressed as

$$kA\frac{T_{m-1} - T_m}{\Delta x} + kA\frac{T_{m+1} - T_m}{\Delta x} + h(p\Delta x)(T_\infty - T_m) = 0 \;\rightarrow\; T_{m-1} - 2T_m + T_{m+1} + h(p\Delta x^2 / kA)(T_\infty - T_m) = 0$$

The finite difference equation for node 4 at the fin tip is obtained by applying an energy balance on the half volume element about that node. Then,

$$m=1:\; T_0 - 2T_1 + T_2 + h(p\Delta x^2 / kA)(T_\infty - T_1) = 0$$

$$m=2:\; T_1 - 2T_2 + T_3 + h(p\Delta x^2 / kA)(T_\infty - T_2) = 0$$

$$m=3:\; T_2 - 2T_3 + T_4 + h(p\Delta x^2 / kA)(T_\infty - T_3) = 0$$

$$\text{Node 4:}\qquad kA\frac{T_3 - T_4}{\Delta x} + h(p\Delta x / 2 + A)(T_\infty - T_4) = 0$$

where $\quad \Delta x = 0.005\,\text{m},\; k = 237\ \text{W/m}\cdot{}^\circ\text{C},\; T_\infty = 35^\circ\text{C}, T_0 = 80^\circ\text{C},\; h = 30\ \text{W/m}^2\cdot{}^\circ\text{C}$

and $\quad A = (3\,\text{m})(0.003\,\text{m}) = 0.009\,\text{m}^2$ and $p = 2(3 + 0.003\,\text{m}) = 6.006\,\text{m}$.

This system of 4 equations with 4 unknowns constitute the finite difference formulation of the problem.

(b) The nodal temperatures under steady conditions are determined by solving the 4 equations above simultaneously with an equation solver to be

T_1 = **79.64°C**, T_2 = **79.38°C**, T_3 = **79.21°C**, T_4 = **79.14°C**

(c) The rate of heat transfer from a single fin is simply the sum of the heat transfer from each nodal element,

$$\dot{Q}_{\text{fin}} = \sum_{m=0}^{4} \dot{Q}_{\text{element},\,m} = \sum_{m=0}^{4} hA_{\text{surface},m}(T_m - T_\infty)$$
$$= hp(\Delta x / 2)(T_0 - T_\infty) + hp\Delta x(T_1 + T_2 + T_3 - 3T_\infty) + h(p\Delta x / 2 + A)(T_4 - T_\infty) = \mathbf{172\ W}$$

(d) The number of fins on the surface is

$$\text{No. of fins} = \frac{\text{Plate height}}{\text{Fin thickness} + \text{fin spacing}} = \frac{2\,\text{m}}{(0.003 + 0.004)\,\text{m}} = 286\,\text{fins}$$

Then the rate of heat tranfer from the fins, the unfinned portion, and the entire finned surface become

$$\dot{Q}_{\text{fin, total}} = (\text{No. of fins})\dot{Q}_{\text{fin}} = 286(172\ \text{W}) = 49{,}192\ \text{W}$$

$$\dot{Q}_{\text{unfinned}} = hA_{\text{unfinned}}(T_0 - T_\infty) = (30\ \text{W/m}^2\cdot{}^\circ\text{C})(286 \times 3\,\text{m} \times 0.004\,\text{m})(80 - 35)^\circ\text{C} = 4633\ \text{W}$$

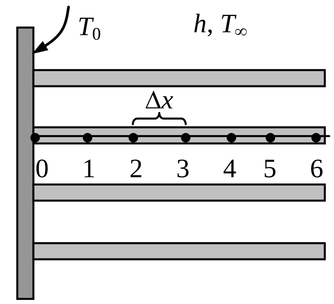
$$\dot{Q}_{\text{total}} = \dot{Q}_{\text{fin, total}} + \dot{Q}_{\text{unfinned}} = 49{,}192 + 4633 = 53{,}825\ \text{W} \cong \mathbf{53.8\,kW}$$

5-45 One side of a hot vertical plate is to be cooled by attaching aluminum pin fins. The finite difference formulation of the problem for all nodes is to be obtained, and the nodal temperatures, the rate of heat transfer from a single fin and from the entire surface of the plate are to be determined.

Assumptions **1** Heat transfer along the fin is given to be steady and one-dimensional. **2** The thermal conductivity is constant. **3** Combined convection and radiation heat transfer coefficient is constant and uniform.

Properties The thermal conductivity is given to be k = 237 W/m·°C.

Analysis (*a*) The nodal spacing is given to be Δx=0.5 cm. Then the number of nodes M becomes



$$M = \frac{L}{\Delta x} + 1 = \frac{3\,\text{cm}}{0.5\,\text{cm}} + 1 = 7$$

The base temperature at node 0 is given to be T_0 = 100°C. This problem involves 6 unknown nodal temperatures, and thus we need to have 6 equations to determine them uniquely. Nodes 1, 2, 3, 4, and 5 are interior nodes, and thus for them we can use the general finite difference relation expressed as

$$kA\frac{T_{m-1} - T_m}{\Delta x} + kA\frac{T_{m+1} - T_m}{\Delta x} + h(p\Delta x)(T_\infty - T_m) = 0 \;\rightarrow\; T_{m-1} - 2T_m + T_{m+1} + h(p\Delta x^2 / kA)(T_\infty - T_m) = 0$$

The finite difference equation for node 6 at the fin tip is obtained by applying an energy balance on the half volume element about that node. Then,

m= 1: $T_0 - 2T_1 + T_2 + h(p\Delta x^2 / kA)(T_\infty - T_1) = 0$

m= 2: $T_1 - 2T_2 + T_3 + h(p\Delta x^2 / kA)(T_\infty - T_2) = 0$

m= 3: $T_2 - 2T_3 + T_4 + h(p\Delta x^2 / kA)(T_\infty - T_3) = 0$

m= 4: $T_3 - 2T_4 + T_5 + h(p\Delta x^2 / kA)(T_\infty - T_4) = 0$

m= 5: $T_4 - 2T_5 + T_6 + h(p\Delta x^2 / kA)(T_\infty - T_5) = 0$

Node 6: $kA\dfrac{T_5 - T_6}{\Delta x} + h(p\Delta x / 2 + A)(T_\infty - T_6) = 0$

where $\Delta x = 0.005\,\text{m},\; k = 237\ \text{W/m·°C},\; T_\infty = 30°\text{C}, T_0 = 100°\text{C},\; h = 35\ \text{W/m}^2\cdot°\text{C}$

and
$A = \pi D^2 / 4 = \pi(0.25\,\text{cm})^2/4 = 0.0491\,\text{cm}^2 = 0.0491\times 10^{-4}\ \text{m}^2$
$p = \pi D = \pi(0.0025\,\text{m}) = 0.00785\,\text{m}$

(*b*) The nodal temperatures under steady conditions are determined by solving the 6 equations above simultaneously with an equation solver to be

T_1 = **97.9°C**, T_2 = **96.1°C**, T_3 = **94.7°C**, T_4 = **93.8°C**, T_5 = **93.1°C**, T_6 = **92.9°C**

(*c*) The rate of heat transfer from a single fin is simply the sum of the heat transfer from the nodal elements,

$$\dot{Q}_{\text{fin}} = \sum_{m=0}^{6} \dot{Q}_{\text{element},\,m} = \sum_{m=0}^{6} hA_{\text{surface},m}(T_m - T_\infty)$$
$$= hp\Delta x / 2(T_0 - T_\infty) + hp\Delta x(T_1 + T_2 + T_3 + T_4 + T_5 - 5T_\infty) + h(p\Delta x / 2 + A)(T_6 - T_\infty) = \mathbf{0.5496\ \ W}$$

(*d*) The number of fins on the surface is

$$\text{No. of fins} = \frac{1\,\text{m}^2}{(0.006\,\text{m})(0.006\,\text{m})} = 27{,}778\,\text{fins}$$

Then the rate of heat tranfer from the fins, the unfinned portion, and the entire finned surface become

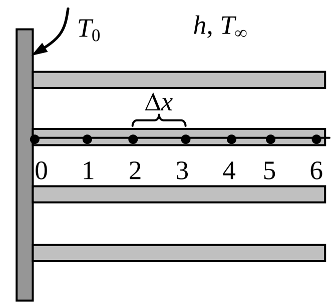
$$\dot{Q}_{\text{fin, total}} = (\text{No. of fins})\dot{Q}_{\text{fin}} = 27{,}778(0.5496\ \text{W}) = 15{,}267\ \text{W}$$
$$\dot{Q}_{\text{unfinned}} = hA_{\text{unfinned}}(T_0 - T_\infty) = (35\ \text{W/m}^2\cdot°\text{C})(1 - 27{,}778\times 0.0491\times 10^{-4}\ \text{m}^2)(100 - 30)°\text{C} = 2116\ \text{W}$$
$$\dot{Q}_{\text{total}} = \dot{Q}_{\text{fin, total}} + \dot{Q}_{\text{unfinned}} = 15{,}267 + 2116 = \mathbf{17{,}383\ W \cong 17.4\ \ kW}$$

5-46 One side of a hot vertical plate is to be cooled by attaching copper pin fins. The finite difference formulation of the problem for all nodes is to be obtained, and the nodal temperatures, the rate of heat transfer from a single fin and from the entire surface of the plate are to be determined.

Assumptions **1** Heat transfer along the fin is given to be steady and one-dimensional. **2** The thermal conductivity is constant. **3** Combined convection and radiation heat transfer coefficient is constant and uniform.

Properties The thermal conductivity is given to be k = 386 W/m·°C.

Analysis (*a*) The nodal spacing is given to be Δx=0.5 cm. Then the number of nodes M becomes



$$M = \frac{L}{\Delta x} + 1 = \frac{3\,\text{cm}}{0.5\,\text{cm}} + 1 = 7$$

The base temperature at node 0 is given to be T_0 = 100°C. This problem involves 6 unknown nodal temperatures, and thus we need to have 6 equations to determine them uniquely. Nodes 1, 2, 3, 4, and 5 are interior nodes, and thus for them we can use the general finite difference relation expressed as

$$kA\frac{T_{m-1} - T_m}{\Delta x} + kA\frac{T_{m+1} - T_m}{\Delta x} + h(p\Delta x)(T_\infty - T_m) = 0 \;\rightarrow\; T_{m-1} - 2T_m + T_{m+1} + h(p\Delta x^2 / kA)(T_\infty - T_m) = 0$$

The finite difference equation for node 6 at the fin tip is obtained by applying an energy balance on the half volume element about that node. Then,

$$m=1:\; T_0 - 2T_1 + T_2 + h(p\Delta x^2 / kA)(T_\infty - T_1) = 0$$

$$m=2:\; T_1 - 2T_2 + T_3 + h(p\Delta x^2 / kA)(T_\infty - T_2) = 0$$

$$m=3:\; T_2 - 2T_3 + T_4 + h(p\Delta x^2 / kA)(T_\infty - T_3) = 0$$

$$m=4:\; T_3 - 2T_4 + T_5 + h(p\Delta x^2 / kA)(T_\infty - T_4) = 0$$

$$m=5:\; T_4 - 2T_5 + T_6 + h(p\Delta x^2 / kA)(T_\infty - T_5) = 0$$

$$\text{Node 6:}\; kA\frac{T_5 - T_6}{\Delta x} + h(p\Delta x/2 + A)(T_\infty - T_6) = 0$$

where $\Delta x = 0.005\,\text{m},\; k = 386\,\text{W/m·°C},\; T_\infty = 30°\text{C}, T_0 = 100°\text{C},\; h = 35\,\text{W/m}^2\cdot°\text{C}$

and
$$A = \pi D^2/4 = \pi(0.25\,\text{cm})^2/4 = 0.0491\,\text{cm}^2 = 0.0491\times10^{-4}\,\text{m}^2$$
$$p = \pi D = \pi(0.0025\,\text{m}) = 0.00785\,\text{m}$$

(*b*) The nodal temperatures under steady conditions are determined by solving the 6 equations above simultaneously with an equation solver to be

T_1 = **98.6°C**, T_2 = **97.5°C**, T_3 = **96.7°C**, T_4 = **96.0°C**, T_5 = **95.7°C**, T_6 = **95.5°C**

(*c*) The rate of heat transfer from a single fin is simply the sum of the heat transfer from the nodal elements,

$$\dot{Q}_{\text{fin}} = \sum_{m=0}^{6}\dot{Q}_{\text{element},\,m} = \sum_{m=0}^{6} hA_{\text{surface},m}(T_m - T_\infty)$$
$$= hp\Delta x/2(T_0 - T_\infty) + hp\Delta x(T_1 + T_2 + T_3 + T_4 + T_5 - 5T_\infty) + h(p\Delta x/2 + A)(T_6 - T_\infty) = \mathbf{0.5641\ W}$$

(*d*) The number of fins on the surface is

$$\text{No. of fins} = \frac{1\,\text{m}^2}{(0.006\,\text{m})(0.006\,\text{m})} = 27{,}778\,\text{fins}$$

Then the rate of heat tranfer from the fins, the unfinned portion, and the entire finned surface become

$$\dot{Q}_{\text{fin, total}} = (\text{No. of fins})\dot{Q}_{\text{fin}} = 27{,}778(0.5641\,\text{W}) = 15{,}670\,\text{W}$$

$$\dot{Q}_{\text{unfinned}} = hA_{\text{unfinned}}(T_0 - T_\infty) = (35\,\text{W/m}^2\cdot°\text{C})(1 - 27{,}778\times0.0491\times10^{-4}\,\text{m}^2)(100 - 30)°\text{C} = 2116\,\text{W}$$

$$\dot{Q}_{\text{total}} = \dot{Q}_{\text{fin, total}} + \dot{Q}_{\text{unfinned}} = 15{,}670 + 2116 = \mathbf{17{,}786\ W \cong 17.8\,kW}$$

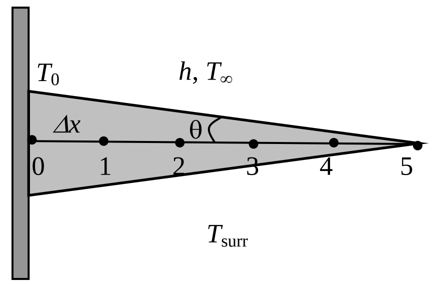
5-47 A long triangular fin attached to a surface is considered. The nodal temperatures, the rate of heat transfer, and the fin efficiency are to be determined numerically using 6 equally spaced nodes.

Assumptions **1** Heat transfer along the fin is given to be steady, and the temperature along the fin to vary in the *x* direction only so that $T = T(x)$. **2** Thermal conductivity is constant.

Properties The thermal conductivity is given to be k = 180 W/m·°C. The emissivity of the fin surface is 0.9.

Analysis The fin length is given to be L = 5 cm, and the number of nodes is specified to be M = 6. Therefore, the nodal spacing Δx is

$$\Delta x = \frac{L}{M-1} = \frac{0.05\text{ m}}{6\text{-}1} = 0.01\text{ m}$$



The temperature at node 0 is given to be T_0 = 180°C, and the temperatures at the remaining 5 nodes are to be determined. Therefore, we need to have 5 equations to determine them uniquely. Nodes 1, 2, 3, and 4 are interior nodes, and the finite difference formulation for a *general interior node m* is obtained by applying an energy balance on the volume element of this node. Noting that heat transfer is steady and there is no heat generation in the fin and assuming heat transfer to be into the medium from all sides, the energy balance can be expressed as

$$\sum_{\text{all sides}} \dot{Q} = 0 \rightarrow kA_{\text{left}}\frac{T_{m-1}-T_m}{\Delta x} + kA_{\text{right}}\frac{T_{m+1}-T_m}{\Delta x} + hA_{\text{conv}}(T_\infty - T_m) + \varepsilon\sigma A_{\text{surface}}[T_{\text{surr}}^4 - (T_m+273)^4\} = 0$$

Note that heat transfer areas are different for each node in this case, and using geometrical relations, they can be expressed as

$$A_{\text{left}} = (\text{Height}\times\text{width})_{@\,m-1/2} = 2w\big[L-(m-1/2)\Delta x\big]\tan\theta$$
$$A_{\text{right}} = (\text{Height}\times\text{width})_{@\,m+1/2} = 2w\big[L-(m+1/2)\Delta x\big]\tan\theta$$
$$A_{\text{surface}} = 2\times\text{Length}\times\text{width} = 2w(\Delta x/\cos\theta)$$

Substituting,

$$2kw[L-(m-0.5)\Delta x]\tan\theta\,\frac{T_{m-1}-T_m}{\Delta x} + 2kw[L-(m+0.5)\Delta x]\tan\theta\,\frac{T_{m+1}-T_m}{\Delta x}$$
$$+\,2w(\Delta x/\cos\theta)\{h(T_\infty - T_m) + \varepsilon\sigma[T_{\text{surr}}^4 - (T_m+273)^4]\} = 0$$

Dividing each term by $2kwL\tan\theta/\Delta x$ gives

$$\left[1-(m-1/2)\frac{\Delta x}{L}\right](T_{m-1}-T_m) + \left[1-(m+1/2)\frac{\Delta x}{L}\right](T_{m+1}-T_m) + \frac{h(\Delta x)^2}{kL\sin\theta}(T_\infty - T_m) + \frac{\varepsilon\sigma(\Delta x)^2}{kL\sin\theta}[T_{\text{surr}}^4 - (T_m+273)^4] = 0$$

Substituting,

m = 1: $$\left[1-0.5\frac{\Delta x}{L}\right](T_0-T_1) + \left[1-1.5\frac{\Delta x}{L}\right](T_2-T_1) + \frac{h(\Delta x)^2}{kL\sin\theta}(T_\infty - T_1) + \frac{\varepsilon\sigma(\Delta x)^2}{kL\sin\theta}[T_{\text{surr}}^4 - (T_1+273)^4] = 0$$

m = 2: $$\left[1-1.5\frac{\Delta x}{L}\right](T_1-T_2) + \left[1-2.5\frac{\Delta x}{L}\right](T_3-T_2) + \frac{h(\Delta x)^2}{kL\sin\theta}(T_\infty - T_2) + \frac{\varepsilon\sigma(\Delta x)^2}{kL\sin\theta}[T_{\text{surr}}^4 - (T_2+273)^4] = 0$$

m = 3: $$\left[1-2.5\frac{\Delta x}{L}\right](T_2-T_3) + \left[1-3.5\frac{\Delta x}{L}\right](T_4-T_3) + \frac{h(\Delta x)^2}{kL\sin\theta}(T_\infty - T_3) + \frac{\varepsilon\sigma(\Delta x)^2}{kL\sin\theta}[T_{\text{surr}}^4 - (T_3+273)^4] = 0$$

m = 4: $$\left[1-3.5\frac{\Delta x}{L}\right](T_3-T_4) + \left[1-4.5\frac{\Delta x}{L}\right](T_5-T_4) + \frac{h(\Delta x)^2}{kL\sin\theta}(T_\infty - T_4) + \frac{\varepsilon\sigma(\Delta x)^2}{kL\sin\theta}[T_{\text{surr}}^4 - (T_4+273)^4] = 0$$

An energy balance on the 5th node gives the 5th equation,

m = 5: $$2k\frac{\Delta x}{2}\tan\theta\,\frac{T_4-T_5}{\Delta x} + 2h\frac{\Delta x/2}{\cos\theta}(T_\infty - T_5) + 2\varepsilon\sigma\frac{\Delta x/2}{\cos\theta}[T_{\text{surr}}^4 - (T_5+273)^4] = 0$$

Solving the 5 equations above simultaneously for the 5 unknown nodal temperatures gives

T_1 = **177.0°C**, T_2 = **174.1°C**, T_3 = **171.2°C**, T_4 = **168.4°C**, and T_5 = **165.5°C**

(*b*) The total rate of heat transfer from the fin is simply the sum of the heat transfer from each volume element to the ambient, and for w = 1 m it is determined from

$$\dot{Q}_{\text{fin}} = \sum_{m=0}^{5} \dot{Q}_{\text{element},\,m} = \sum_{m=0}^{5} hA_{\text{surface},\,m}(T_m - T_\infty) + \sum_{m=0}^{5} \varepsilon\sigma A_{\text{surface},\,m}[(T_m + 273)^4 - T_{\text{surr}}^4]$$

Noting that the heat transfer surface area is $w\Delta x / \cos\theta$ for the boundary nodes 0 and 5, and twice as large for the interior nodes 1, 2, 3, and 4, we have

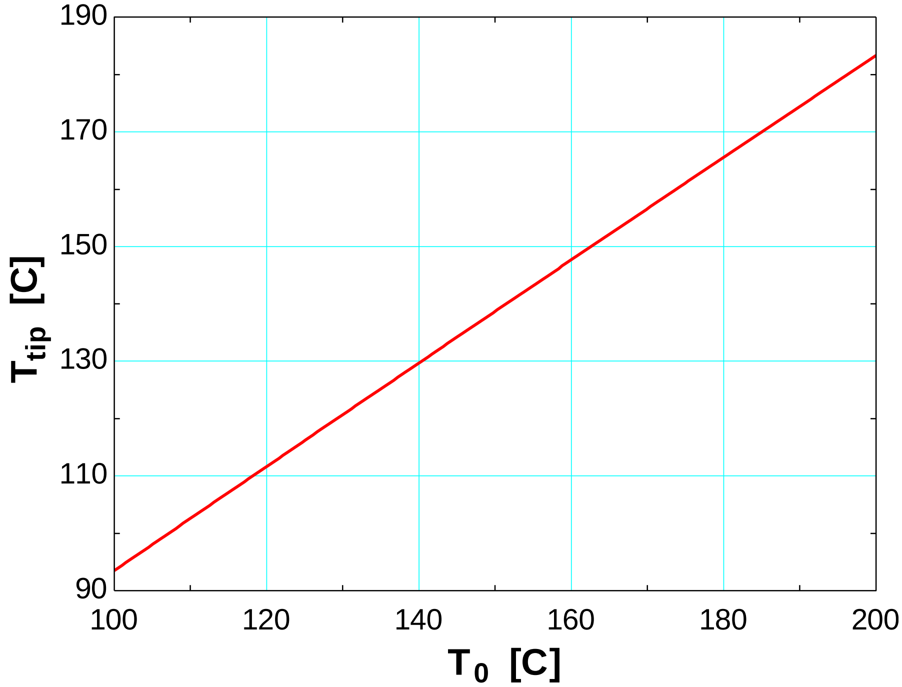
$$\begin{aligned}\dot{Q}_{\text{fin}} &= h\frac{w\Delta x}{\cos\theta}\left[(T_0 - T_\infty) + 2(T_1 - T_\infty) + 2(T_2 - T_\infty) + 2(T_3 - T_\infty) + 2(T_4 - T_\infty) + (T_5 - T_\infty)\right] \\ &\quad + \varepsilon\sigma\frac{w\Delta x}{\cos\theta}\{[(T_0 + 273)^4 - T_{\text{surr}}^4] + 2[(T_1 + 273)^4 - T_{\text{surr}}^4] + 2[(T_2 + 273)^4 - T_{\text{surr}}^4] + 2[(T_3 + 273)^4 - T_{\text{surr}}^4] \\ &\quad + 2[(T_4 + 273)^4 - T_{\text{surr}}^4] + [(T_5 + 273)^4 - T_{\text{surr}}^4]\} \\ &= \mathbf{537\ W}\end{aligned}$$

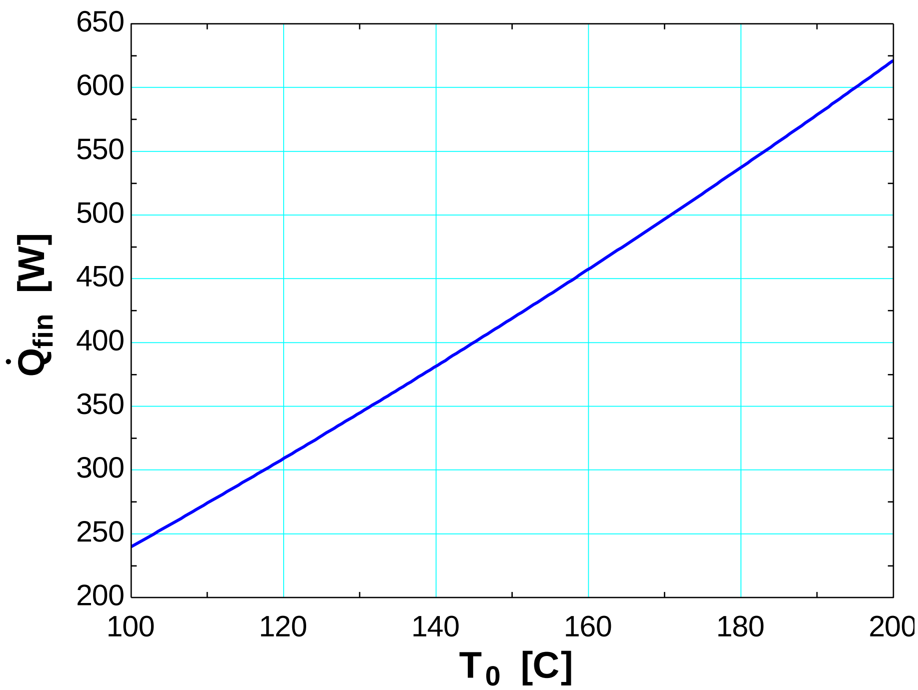
5-48 Prob. 5-47 is reconsidered. The effect of the fin base temperature on the fin tip temperature and the rate of heat transfer from the fin is to be investigated.

Analysis The problem is solved using EES, and the solution is given below.

```
"GIVEN"
k=180 [W/m-C]
L=0.05 [m]
b=0.01 [m]
w=1 [m]
T_0=180 [C]
T_infinity=25 [C]
h=25 [W/m^2-C]
T_surr=290 [K]
M=6
epsilon=0.9
tan(theta)=(0.5*b)/L
sigma=5.67E-8 [W/m^2-K^4] “Stefan-Boltzmann constant"

"ANALYSIS"
"(a)"
DELTAx=L/(M-1)
"Using the finite difference method, the five equations for the temperatures at 5 nodes are determined to be"
(1-0.5*DELTAx/L)*(T_0-T_1)+(1-1.5*DELTAx/L)*(T_2-T_1)+(h*DELTAx^2)/(k*L*sin(theta))*(T_infinity-
T_1)+(epsilon*sigma*DELTAX^2)/(k*L*sin(theta))*(T_surr^4-(T_1+273)^4)=0 "for mode 1"

(1-1.5*DELTAx/L)*(T_1-T_2)+(1-2.5*DELTAx/L)*(T_3-T_2)+(h*DELTAx^2)/(k*L*sin(theta))*(T_infinity-
T_2)+(epsilon*sigma*DELTAX^2)/(k*L*sin(theta))*(T_surr^4-(T_2+273)^4)=0 "for mode 2"

(1-2.5*DELTAx/L)*(T_2-T_3)+(1-3.5*DELTAx/L)*(T_4-T_3)+(h*DELTAx^2)/(k*L*sin(theta))*(T_infinity-
T_3)+(epsilon*sigma*DELTAX^2)/(k*L*sin(theta))*(T_surr^4-(T_3+273)^4)=0 "for mode 3"

(1-3.5*DELTAx/L)*(T_3-T_4)+(1-4.5*DELTAx/L)*(T_5-T_4)+(h*DELTAx^2)/(k*L*sin(theta))*(T_infinity-
T_4)+(epsilon*sigma*DELTAX^2)/(k*L*sin(theta))*(T_surr^4-(T_4+273)^4)=0 "for mode 4"

2*k*DELTAx/2*tan(theta)*(T_4-T_5)/DELTAx+2*h*(0.5*DELTAx)/cos(theta)*(T_infinity-
T_5)+2*epsilon*sigma*(0.5*DELTAx)/cos(theta)*(T_surr^4-(T_5+273)^4)=0 "for mode 5"

T_tip=T_5

"(b)"
Q_dot_fin=C+D "where"

C=h*(w*DELTAx)/cos(theta)*((T_0-T_infinity)+2*(T_1-T_infinity)+2*(T_2-T_infinity)+2*(T_3-T_infinity)+2*(T_4-
T_infinity)+(T_5-T_infinity))

D=epsilon*sigma*(w*DELTAx)/cos(theta)*(((T_0+273)^4-T_surr^4)+2*((T_1+273)^4-T_surr^4)+2*((T_2+273)^4-
T_surr^4)+2*((T_3+273)^4-T_surr^4)+2*((T_4+273)^4-T_surr^4)+((T_5+273)^4-T_surr^4))
```

T_0 [C]	T_{tip} [C]	$\dot{Q}_{fin}$ [W]
100	93.51	239.8
105	98.05	256.8
110	102.6	274
115	107.1	291.4
120	111.6	309
125	116.2	326.8
130	120.7	344.8
135	125.2	363.1
140	129.7	381.5
145	134.2	400.1
150	138.7	419
155	143.2	438.1
160	147.7	457.5
165	152.1	477.1
170	156.6	496.9
175	161.1	517
180	165.5	537.3
185	170	557.9
190	174.4	578.7
195	178.9	599.9
200	183.3	621.2

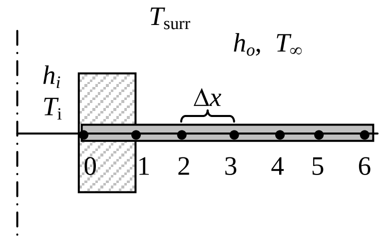




5-49 Two cast iron steam pipes are connected to each other through two 1-cm thick flanges, and heat is lost from the flanges by convection and radiation. The finite difference formulation of the problem for all nodes is to be obtained, and the temperature of the tip of the flange as well as the rate of heat transfer from the exposed surfaces of the flange are to be determined.

Assumptions **1** Heat transfer through the flange is stated to be steady and one-dimensional. **2** The thermal conductivity and emissivity are constants. **3** Convection heat transfer coefficient is constant and uniform.

Properties The thermal conductivity and emissivity are given to be k = 52 W/m·°C and ε = 0.8.



Analysis (*a*) The distance between nodes 0 and 1 is the thickness of the pipe, Δx_1=0.4 cm=0.004 m. The nodal spacing along the flange is given to be Δx_2=1 cm = 0.01 m. Then the number of nodes M becomes

$$M = \frac{L}{\Delta x} + 2 = \frac{5\,\text{cm}}{1\,\text{cm}} + 2 = 7$$

This problem involves 7 unknown nodal temperatures, and thus we need to have 7 equations to determine them uniquely. Noting that the total thickness of the flange is t = 0.02 m, the heat conduction area at any location along the flange is $A_{\text{cond}} = 2\pi rt$ where the values of radii at the nodes and between the nodes (the mid points) are

r_0 = 0.046 m, r_1=0.05 m, r_2=0.06 m, r_3=0.07 m, r_4=0.08 m, r_5=0.09 m, r_6=0.10 m

r_{01}=0.048 m, r_{12}=0.055 m, r_{23}=0.065 m, r_{34}=0.075 m, r_{45}=0.085 m, r_{56}=0.095 m

Then the finite difference equations for each node are obtained from the energy balance to be as follows:

Node 0: $$h_i(2\pi t r_0)(T_i - T_0) + k(2\pi t r_{01})\frac{T_1 - T_0}{\Delta x_1} = 0$$

Node 1: $$k(2\pi t r_{01})\frac{T_0 - T_1}{\Delta x_1} + k(2\pi t r_{12})\frac{T_2 - T_1}{\Delta x_2} + 2[2\pi t(r_1 + r_{12})/2)](\Delta x_2/2)\{h(T_\infty - T_1) + \varepsilon\sigma[T_{\text{surr}}^4 - (T_1 + 273)^4]\} = 0$$

Node 2: $$k(2\pi t r_{12})\frac{T_1 - T_2}{\Delta x_2} + k(2\pi t r_{23})\frac{T_3 - T_2}{\Delta x_2} + 2(2\pi t r_2 \Delta x_2)\{h(T_\infty - T_2) + \varepsilon\sigma[T_{\text{surr}}^4 - (T_2 + 273)^4]\} = 0$$

Node 3: $$k(2\pi t r_{23})\frac{T_2 - T_3}{\Delta x_2} + k(2\pi t r_{34})\frac{T_4 - T_3}{\Delta x_2} + 2(2\pi t r_3 \Delta x_2)\{h(T_\infty - T_3) + \varepsilon\sigma[T_{\text{surr}}^4 - (T_3 + 273)^4]\} = 0$$

Node 4: $$k(2\pi t r_{34})\frac{T_3 - T_4}{\Delta x_2} + k(2\pi t r_{45})\frac{T_5 - T_4}{\Delta x_2} + 2(2\pi t r_4 \Delta x_2)\{h(T_\infty - T_4) + \varepsilon\sigma[T_{\text{surr}}^4 - (T_4 + 273)^4]\} = 0$$

Node 5: $$k(2\pi t r_{45})\frac{T_4 - T_5}{\Delta x_2} + k(2\pi t r_{56})\frac{T_6 - T_5}{\Delta x_2} + 2(2\pi t r_5 \Delta x_2)\{h(T_\infty - T_5) + \varepsilon\sigma[T_{\text{surr}}^4 - (T_5 + 273)^4]\} = 0$$

Node 6: $$k(2\pi t r_{56})\frac{T_5 - T_6}{\Delta x_2} + 2[2\pi t(\Delta x_2/2)(r_{56} + r_6)/2 + 2\pi r_6 t]\{h(T_\infty - T_6) + \varepsilon\sigma[T_{\text{surr}}^4 - (T_6 + 273)^4]\} = 0$$

where $\Delta x_1 = 0.004\,\text{m},\ \Delta x_2 = 0.01\,\text{m}, k = 52\ \text{W/m}\cdot{}^\circ\text{C}, \varepsilon = 0.8,\ T_\infty = 8^\circ\text{C}, T_{in} = 200^\circ\text{C},\ T_{surr} = 290\ \text{K}$

and $h = 25\ \text{W/m}^2\cdot{}^\circ\text{C}, h_i = 180\ \text{W/m}^2\cdot{}^\circ\text{C}, \sigma = 5.67\times10^{-8}\ \text{W/m}^2\cdot\text{K}^4.$

The system of 7 equations with 7 unknowns constitutes the finite difference formulation of the problem.

(*b*) The nodal temperatures under steady conditions are determined by solving the 7 equations above simultaneously with an equation solver to be

T_0 = 119.7°C, T_1 = 118.6°C, T_2 = 116.3°C, T_3 = 114.3°C, T_4 = 112.7°C, T_5 = 111.2°C, and T_6 = **109.9°C**

(*c*) Knowing the inner surface temperature, the rate of heat transfer from the flange under steady conditions is simply the rate of heat transfer from the steam to the pipe at flange section

$$\dot{Q}_{\text{fin}} = \sum_{m=1}^{6}\dot{Q}_{\text{element},\,m} = \sum_{m=1}^{6} hA_{\text{surface},m}(T_m - T_\infty) + \sum_{m=1}^{6}\varepsilon\sigma A_{\text{surface},m}[(T_m + 273)^4 - T_{\text{surr}}^4] = \mathbf{83.6\ \ W}$$

where $A_{\text{surface, m}}$ are as given above for different nodes.

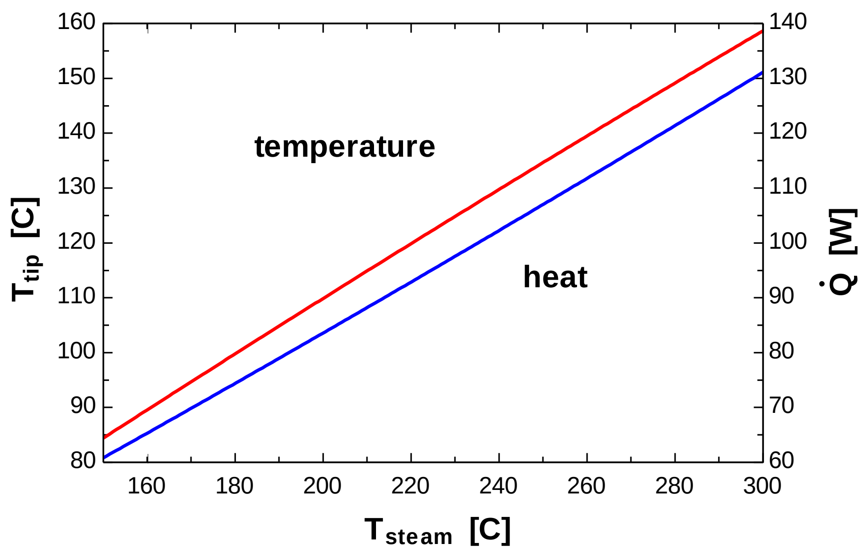
5-50 Prob. 5-49 is reconsidered. The effects of the steam temperature and the outer heat transfer coefficient on the flange tip temperature and the rate of heat transfer from the exposed surfaces of the flange are to be investigated.

Analysis The problem is solved using EES, and the solution is given below.

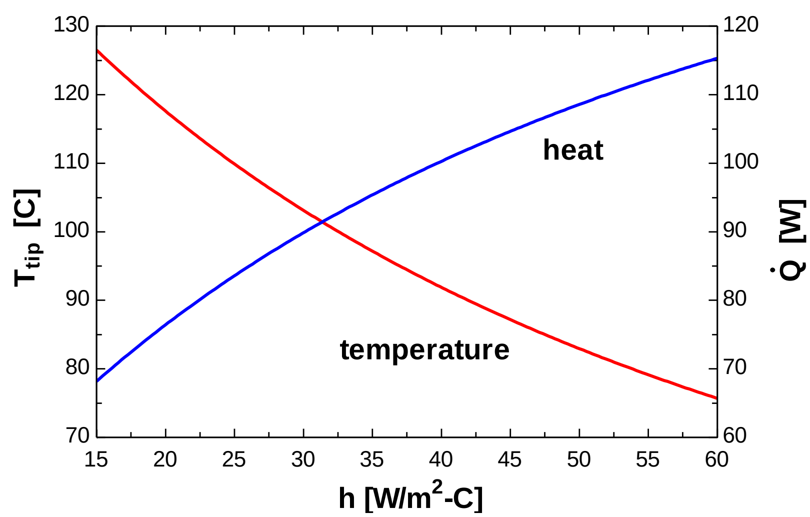
```
"GIVEN"
t_pipe=0.004 [m]
k=52 [W/m-C]
epsilon=0.8
D_o_pipe=0.10 [m]
t_flange=0.01 [m]
D_o_flange=0.20 [m]
T_steam=200 [C]
h_i=180 [W/m^2-C]
T_infinity=8 [C]
h=25 [W/m^2-C]
T_surr=290 [K]
DELTAx=0.01 [m]
sigma=5.67E-8 [W/m^2-K^4] “Stefan-Boltzmann constant"

"ANALYSIS"
"(b)"
DELTAx_1=t_pipe "the distance between nodes 0 and 1"
DELTAx_2=t_flange "nodal spacing along the flange"
L=(D_o_flange-D_o_pipe)/2
M=L/DELTAx_2+2 "Number of nodes"
t=2*t_flange "total thixkness of the flange"
"The values of radii at the nodes and between the nodes /-(the midpoints) are"
r_0=0.046 "[m]"
r_1=0.05 "[m]"
r_2=0.06 "[m]"
r_3=0.07 "[m]"
r_4=0.08 "[m]"
r_5=0.09 "[m]"
r_6=0.10 "[m]"
r_01=0.048 "[m]"
r_12=0.055 "[m]"
r_23=0.065 "[m]"
r_34=0.075 "[m]"
r_45=0.085 "[m]"
r_56=0.095 "[m]"
"Using the finite difference method, the five equations for the unknown temperatures at 7 nodes are determined to
be"
h_i*(2*pi*t*r_0)*(T_steam-T_0)+k*(2*pi*t*r_01)*(T_1-T_0)/DELTAx_1=0 "node 0"
k*(2*pi*t*r_01)*(T_0-T_1)/DELTAx_1+k*(2*pi*t*r_12)*(T_2-
T_1)/DELTAx_2+2*2*pi*t*(r_1+r_12)/2*(DELTAx_2/2)*(h*(T_infinity-T_1)+epsilon*sigma*(T_surr^4-
(T_1+273)^4))=0 "node 1"
k*(2*pi*t*r_12)*(T_1-T_2)/DELTAx_2+k*(2*pi*t*r_23)*(T_3-T_2)/DELTAx_2+2*2*pi*t*r_2*DELTAx_2*(h*(T_infinity-
T_2)+epsilon*sigma*(T_surr^4-(T_2+273)^4))=0 "node 2"
k*(2*pi*t*r_23)*(T_2-T_3)/DELTAx_2+k*(2*pi*t*r_34)*(T_4-T_3)/DELTAx_2+2*2*pi*t*r_3*DELTAx_2*(h*(T_infinity-
T_3)+epsilon*sigma*(T_surr^4-(T_3+273)^4))=0 "node 3"
k*(2*pi*t*r_34)*(T_3-T_4)/DELTAx_2+k*(2*pi*t*r_45)*(T_5-T_4)/DELTAx_2+2*2*pi*t*r_4*DELTAx_2*(h*(T_infinity-
T_4)+epsilon*sigma*(T_surr^4-(T_4+273)^4))=0 "node 4"
k*(2*pi*t*r_45)*(T_4-T_5)/DELTAx_2+k*(2*pi*t*r_56)*(T_6-T_5)/DELTAx_2+2*2*pi*t*r_5*DELTAx_2*(h*(T_infinity-
T_5)+epsilon*sigma*(T_surr^4-(T_5+273)^4))=0 "node 5"
k*(2*pi*t*r_56)*(T_5-T_6)/DELTAx_2+2*(2*pi*t*(r_56+r_6)/2*(DELTAx_2/2)+2*pi*t*r_6)*(h*(T_infinity-
T_6)+epsilon*sigma*(T_surr^4-(T_6+273)^4))=0 "node 6"
T_tip=T_6
```

"(c)"
Q_dot=Q_dot_1+Q_dot_2+Q_dot_3+Q_dot_4+Q_dot_5+Q_dot_6 "where"
Q_dot_1=h*2*2*pi*t*(r_1+r_12)/2*DELTAx_2/2*(T_1-T_infinity)+epsilon*sigma*2*2*pi*t*(r_1+r_12)/2*DELTAx_2/2*((T_1+273)^4-T_surr^4)
Q_dot_2=h*2*2*pi*t*r_2*DELTAx_2*(T_2-T_infinity)+epsilon*sigma*2*2*pi*t*r_2*DELTAx_2*((T_2+273)^4-T_surr^4)
Q_dot_3=h*2*2*pi*t*r_3*DELTAx_2*(T_3-T_infinity)+epsilon*sigma*2*2*pi*t*r_3*DELTAx_2*((T_3+273)^4-T_surr^4)
Q_dot_4=h*2*2*pi*t*r_4*DELTAx_2*(T_4-T_infinity)+epsilon*sigma*2*2*pi*t*r_4*DELTAx_2*((T_4+273)^4-T_surr^4)
Q_dot_5=h*2*2*pi*t*r_5*DELTAx_2*(T_5-T_infinity)+epsilon*sigma*2*2*pi*t*r_5*DELTAx_2*((T_5+273)^4-T_surr^4)
Q_dot_6=h*2*(2*pi*t*(r_56+r_6)/2*(DELTAx_2/2)+2*pi*t*r_6)*(T_6-T_infinity)+epsilon*sigma*2*(2*pi*t*(r_56+r_6)/2*(DELTAx_2/2)+2*pi*t*r_6)*((T_6+273)^4-T_surr^4)

T_{steam} [C]	T_{tip} [C]	$\dot{Q}$ [W]
150	84.42	60.83
160	89.57	65.33
170	94.69	69.85
180	99.78	74.4
190	104.8	78.98
200	109.9	83.58
210	114.9	88.21
220	119.9	92.87
230	124.8	97.55
240	129.7	102.3
250	134.6	107
260	139.5	111.8
270	144.3	116.6
280	149.1	121.4
290	153.9	126.2
300	158.7	131.1



h [W/m^2.C]	T_{tip} [C]	$\dot{Q}$ [W]
15	126.5	68.18
20	117.6	76.42
25	109.9	83.58
30	103.1	89.85
35	97.17	95.38
40	91.89	100.3
45	87.17	104.7
50	82.95	108.6
55	79.14	112.1
60	75.69	115.3



5-51 Using EES, the solutions of the systems of algebraic equations are determined to be as follows:

"(a)"

3*x_1-x_2+3*x_3=0

-x_1+2*x_2+x_3=3

2*x_1-x_2-x_3=2

***Solution*:** $x_1 = 2$, $x_2 = 3$, $x_3 = -1$

"(b)"

4*x_1-2*x_2^2+0.5*x_3=-2

x_1^3-x_2+-x_3=11.964

x_1+x_2+x_3=3

***Solution*:** $x_1 = 2.33$, $x_2 = 2.29$, $x_3 = -1.62$

5-52 Using EES, the solutions of the systems of algebraic equations are determined to be as follows:

"(a)"

3*x_1+2*x_2-x_3+x_4=6

x_1+2*x_2-x_4=-3

-2*x_1+x_2+3*x_3+x_4=2

3*x_2+x_3-4*x_4=-6

***Solution*:** $x_1 = 13$, $x_2 = -9$, $x_3 = 13$, $x_4 = -2$

"(b)"

3*x_1+x_2^2+2*x_3=8

-x_1^2+3*x_2+2*x_3=-6.293

2*x_1-x_2^4+4*x_3=-12

***Solution*:** $x_1 = 2.825$, $x_2 = 1.791$, $x_3 = -1.841$

5-53 Using EES, the solutions of the systems of algebraic equations are determined to be as follows:

```
"(a)"
4*x_1-x_2+2*x_3+x_4=-6
x_1+3*x_2-x_3+4*x_4=-1
-x_1+2*x_2+5*x_4=5
2*x_2-4*x_3-3*x_4=-5
```

***Solution*:** $x_1 = -2$, $x_2 = -1$, $x_3 = 0$, $x_4 = 1$

```
"(b)"
2*x_1+x_2^4-2*x_3+x_4=1
x_1^2+4*x_2+2*x_3^2-2*x_4=-3
-x_1+x_2^4+5*x_3=10
3*x_1-x_3^2+8*x_4=15
```

***Solution*:** $x_1 = 0.263$, $x_2 = -1.15$, $x_3 = 1.70$, $x_4 = 2.14$

Two-Dimensional Steady Heat Conduction

5-54C A region that cannot be filled with simple volume elements such as strips for a plane wall, and rectangular elements for two-dimensional conduction is said to have *irregular boundaries*. A practical way of dealing with such geometries in the finite difference method is to replace the elements bordering the irregular geometry by a series of simple volume elements.

5-55C For a medium in which the finite difference formulation of a general interior node is given in its simplest form as $T_{\text{node}} = (T_{\text{left}} + T_{\text{top}} + T_{\text{right}} + T_{\text{bottom}})/4$:

(*a*) Heat transfer is steady, (*b*) heat transfer is two-dimensional, (*c*) there is no heat generation in the medium, (*d*) the nodal spacing is constant, and (*e*) the thermal conductivity of the medium is constant.

5-56C For a medium in which the finite difference formulation of a general interior node is given in its simplest form as

$$T_{\text{left}} + T_{\text{top}} + T_{\text{right}} + T_{\text{bottom}} - 4T_{\text{node}} + \frac{\dot{e}_{\text{node}} l^2}{k} = 0:$$

(*a*) Heat transfer is steady, (*b*) heat transfer is two-dimensional, (*c*) there is heat generation in the medium, (*d*) the nodal spacing is constant, and (*e*) the thermal conductivity of the medium is constant.

5-57 Starting with an energy balance on a volume element, the steady two-dimensional finite difference equation for a general interior node in rectangular coordinates for $T(x, y)$ for the case of variable thermal conductivity and uniform heat generation is to be obtained.

Analysis We consider a *volume element* of size $\Delta x \times \Delta y \times 1$ centered about a general interior node (m, n) in a region in which heat is generated at a constant rate of $\dot{e}$ and the thermal conductivity k is variable (see Fig. 5-24 in the text). Assuming the direction of heat conduction to be *towards* the node under consideration at all surfaces, the energy balance on the volume element can be expressed as

$$\dot{Q}_{\text{cond,left}} + \dot{Q}_{\text{cond,top}} + \dot{Q}_{\text{cond,right}} + \dot{Q}_{\text{cond,bottom}} + \dot{G}_{\text{element}} = \frac{\Delta E_{\text{element}}}{\Delta t} = 0$$

for the *steady* case. Again assuming the temperatures between the adjacent nodes to vary linearly and noting that the heat transfer area is $\Delta y \times 1$ in the x direction and $\Delta x \times 1$ in the y direction, the energy balance relation above becomes

$$k_{m,n}(\Delta y \times 1)\frac{T_{m-1,n} - T_{m,n}}{\Delta x} + k_{m,n}(\Delta x \times 1)\frac{T_{m,n+1} - T_{m,n}}{\Delta y} + k_{m,n}(\Delta y \times 1)\frac{T_{m+1,n} - T_{m,n}}{\Delta x}$$
$$+ k_{m,n}(\Delta x \times 1)\frac{T_{m,n-1} - T_{m,n}}{\Delta y} + \dot{e}_0(\Delta x \times \Delta y \times 1) = 0$$

Dividing each term by $\Delta x \times \Delta y \times 1$ and simplifying gives

$$\frac{T_{m-1,n} - 2T_{m,n} + T_{m+1,n}}{\Delta x^2} + \frac{T_{m,n-1} - 2T_{m,n} + T_{m,n+1}}{\Delta y^2} + \frac{\dot{e}_0}{k_{m,n}} = 0$$

For a square mesh with $\Delta x = \Delta y = l$, and the relation above simplifies to

$$T_{m-1,n} + T_{m+1,n} + T_{m,n-1} + T_{m,n+1} - 4T_{m,n} + \frac{\dot{e}_0 l^2}{k_{m,n}} = 0$$

It can also be expressed in the following easy-to-remember form

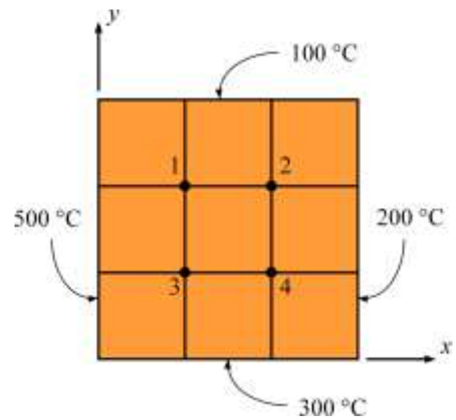
$$T_{\text{left}} + T_{\text{top}} + T_{\text{right}} + T_{\text{bottom}} - 4T_{\text{node}} + \frac{\dot{e}_0 l^2}{k_{\text{node}}} = 0$$

5-58 A square cross section is undergoing a steady two-dimensional heat transfer. The finite difference equations and the nodal temperatures are to be determined.

Assumptions **1** Steady heat conduction is two-dimensional. **2** Thermal properties are constant. **3** There is no heat generation in the body.

Analysis (*a*) There are 4 unknown nodal temperatures, thus we need to have 4 equations to determine them uniquely. For nodes 1 to 4, we can use the general finite difference relation expressed as

$$\frac{T_{m-1,n}-2T_{m,n}+T_{m+1,n}}{\Delta x^2}+\frac{T_{m,n-1}-2T_{m,n}+T_{m,n+1}}{\Delta y^2}=0$$



Since $\Delta x=\Delta y$, we have

$$T_{m,n}=0.25(T_{m,n+1}+T_{m+1,n}+T_{m,n-1}+T_{m-1,n})$$

or $$T_{\text{node}}=0.25(T_{\text{top}}+T_{\text{right}}+T_{\text{bottom}}+T_{\text{left}})$$

Then

Node 1: $T_1=0.25(100+T_2+T_3+500)$

Node 2: $T_2=0.25(100+200+T_4+T_1)$

Node 3: $T_3=0.25(T_1+T_4+300+500)$

Node 4: $T_4=0.25(T_2+200+300+T_3)$

(*b*) By letting the initial guesses as T_1 = 300°C, T_2 = 150°C, T_3 = 400°C, and T_4 = 250°C the results obtained from successive iterations are listed in the following table:

Iteration	Nodal temperature,°C			
	T_1	T_2	T_3	T_4
1	287.5	209.4	334.4	260.9
2	285.9	211.7	336.7	262.1
3	287.1	212.3	337.3	262.4
4	287.4	212.5	337.5	262.5
5	287.5	212.5	337.5	262.5
6	287.5	212.5	337.5	262.5

Hence, the converged nodal temperatures are

T_1 = **287.5°C**, T_2 = **212.5°C**, T_3 = **337.5°C**, T_4 = **262.5°C**

Discussion The finite difference equations can also be calculated using the EES. Copy the following lines and paste on a blank EES screen to solve the above equations:

```
T_1=0.25*(100+T_2+T_3+500)
T_2=0.25*(100+200+T_4+T_1)
T_3=0.25*(T_1+T_4+300+500)
T_4=0.25*(T_2+200+300+T_3)
```

Solving by EES software, we get the same results:

T_1 = **287.5°C**, T_2 = **212.5°C**, T_3 = **337.5°C**, T_4 = **262.5°C**

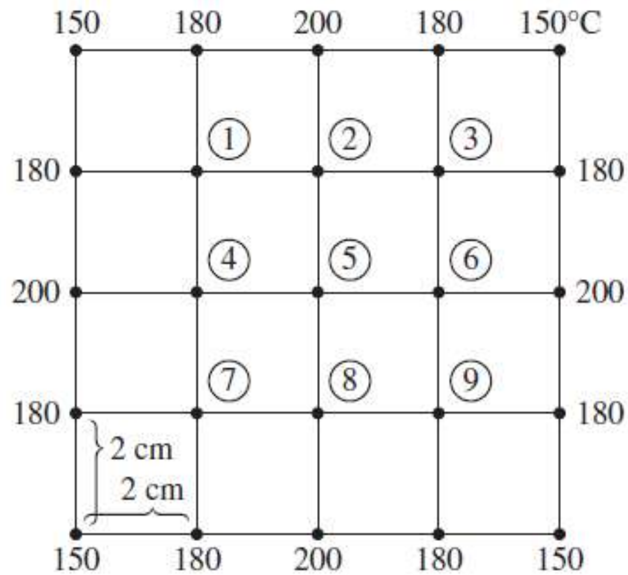
5-59 A long solid body is subjected to steady two-dimensional heat transfer. The unknown nodal temperatures are to be determined.

Assumptions **1** Heat transfer through the body is given to be steady and two-dimensional. **2** There is no heat generation in the body.

Properties The thermal conductivity is given to be k = 45 W/m·°C.

Analysis The nodal spacing is given to be $\Delta x = \Delta x = l$ = 0.02 m, and the general finite difference form of an interior node for steady two-dimensional heat conduction for the case of no heat generation is expressed as

$$T_{\text{left}} + T_{\text{top}} + T_{\text{right}} + T_{\text{bottom}} - 4T_{\text{node}} = 0 \;\rightarrow\; T_{\text{node}} = (T_{\text{left}} + T_{\text{top}} + T_{\text{right}} + T_{\text{bottom}})/4$$



There is symmetry about the horizontal, vertical, and diagonal lines passing through the midpoint, and thus we need to consider only 1/8$^{\text{th}}$ of the region. Then,

$$T_1 = T_3 = T_7 = T_9$$
$$T_2 = T_4 = T_6 = T_8$$

Therefore, there are there are only 3 unknown nodal temperatures, T_1, T_2, and T_5, and thus we need only 3 equations to determine them uniquely. Also, we can replace the symmetry lines by insulation and utilize the mirror-image concept when writing the finite difference equations for the interior nodes.

Node 1 (interior) : $T_1 = (180 + 180 + 2T_2)/4$

Node 2 (interior) : $T_2 = (200 + T_5 + 2T_1)/4$

Node 5 (interior) : $T_5 = 4T_2/4 = T_2$

Solving the equations above simultaneously gives

$$T_1 = T_3 = T_7 = T_9 = \mathbf{185°C}$$
$$T_2 = T_4 = T_5 = T_6 = T_8 = \mathbf{190°C}$$

Discussion Note that taking advantage of symmetry simplified the problem greatly.

5-60 A long solid body is subjected to steady two-dimensional heat transfer. The unknown nodal temperatures are to be determined.

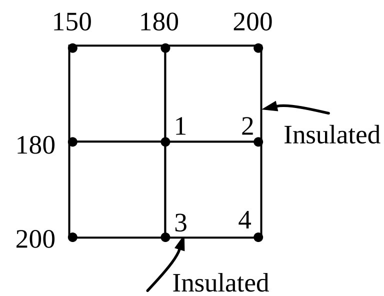
Assumptions **1** Heat transfer through the body is given to be steady and two-dimensional. **2** There is no heat generation in the body.

Properties The thermal conductivity is given to be k = 20 W/m·°C.

Analysis The nodal spacing is given to be $\Delta x = \Delta x = l$ = 0.01 m, and the general finite difference form of an interior node for steady two-dimensional heat conduction for the case of no heat generation is expressed as

$$T_{\text{left}} + T_{\text{top}} + T_{\text{right}} + T_{\text{bottom}} - 4T_{\text{node}} = 0 \;\rightarrow\; T_{\text{node}} = (T_{\text{left}} + T_{\text{top}} + T_{\text{right}} + T_{\text{bottom}})/4$$

(a) There is symmetry about the insulated surfaces as well as about the diagonal line. Therefore $T_3 = T_2$, and T_1, T_2, and T_4 are the only 3 unknown nodal temperatures. Thus we need only 3 equations to determine them uniquely. Also, we can replace the symmetry lines by insulation and utilize the mirror-image concept when writing the finite difference equations for the interior nodes.



Node 1 (interior) : $\quad T_1 = (180 + 180 + T_2 + T_3)/4$

Node 2 (interior) : $\quad T_2 = (200 + T_4 + 2T_1)/4$

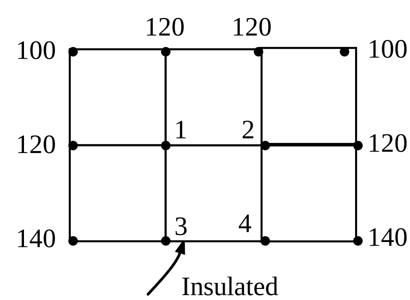
Node 4 (interior) : $\quad T_4 = (2T_2 + 2T_3)/4$

Also, $\quad T_3 = T_2$

Solving the equations above simultaneously gives

$$T_2 = T_3 = T_4 = \mathbf{190°C}$$
$$T_1 = \mathbf{185°C}$$

(b) There is symmetry about the insulated surface as well as the diagonal line. Replacing the symmetry lines by insulation, and utilizing the mirror-image concept, the finite difference equations for the interior nodes can be written as



Node 1 (interior) : $\quad T_1 = (120 + 120 + T_2 + T_3)/4$

Node 2 (interior) : $\quad T_2 = (120 + 120 + T_4 + T_1)/4$

Node 3 (interior) : $\quad T_3 = (140 + 2T1 + T_4)/4 = T_2$

Node 4 (interior) : $\quad T_4 = (2T_2 + 140 + 2T_3)/4$

Solving the equations above simultaneously gives

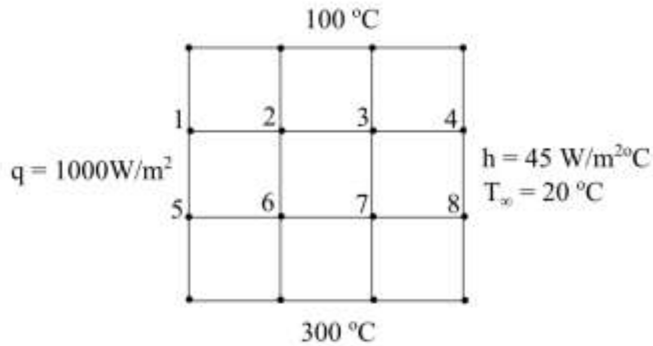
$$T_1 = T_2 = \mathbf{122.9°C}$$
$$T_3 = T_4 = \mathbf{128.6°C}$$

Discussion Note that taking advantage of symmetry simplified the problem greatly.

5-61 A square cross section geometry is subjected to four different boundary conditions. The temperature distribution within the geometry is to be determined using Gauss-Seidel iteration method.

Assumptions **1** Steady heat conduction is two-dimensional without internal heat generation. **2** Thermal conductivity is constant.

Properties Thermal conductivity is given as k=20 W/m·K.



Analysis: (a) There are 4 internal nodes (node 2, 2, 6 and 7) and 4 boundary nodes (node 1, 4, 5 and 8). Thus we need to have 8 equations for 8 unknown temperatures. For internal nodes we can use the general form of the finite difference equation expressed as

$$\frac{T_{m-1,n}-2T_{m,n}+T_{m+1,n}}{\Delta x^2}+\frac{T_{m,n-1}-2T_{m,n}+T_{m,n+1}}{\Delta y^2}=0$$

For $\Delta x=\Delta y$, above equation simplifies to

$$T_{m,n}=0.25(T_{m,n+1}+T_{m+1,n}+T_{m,n-1}+T_{m-1,n})$$

The finite difference equation of the boundary nodes can be found by energy balance at the boundary node control volume assuming all heat transfer is to the node volume.

Node 1: (Left boundary node)

$$\dot{q}\Delta y+k\Delta y\frac{(T_2-T_1)}{\Delta x}+k\frac{\Delta x}{2}\frac{(100-T_1)}{\Delta y}+k\frac{\Delta x}{2}\frac{(T_5-T_1)}{\Delta y}=0 \Rightarrow T_1=25.25+0.5T_1+0.25T_5$$

Node 2: (Internal node)

$$T_2=0.25(T_1+100+T_3+T_6)$$

Node 3: (Internal node)

$$T_3=0.25(T_2+100+T_4+T_7)$$

Node 4: (Right boundary node)

$$k\frac{\Delta x}{2}\frac{(100-T_4)}{\Delta y}+k\frac{\Delta x}{2}\frac{(T_8-T_4)}{\Delta y}+k\Delta y\frac{(T_3-T_4)}{\Delta x}+h\Delta y(T_\infty-T_4)=0 \Rightarrow T_4=0.4944(50.45+0.5T_8+T_3)$$

Node 5: (Left boundary node)

$$\dot{q}\Delta y+k\Delta y\frac{(T_6-T_5)}{\Delta x}+k\frac{\Delta x}{2}\frac{(T_1-T_5)}{\Delta y}+k\frac{\Delta x}{2}\frac{(300-T_1)}{\Delta y}=0 \Rightarrow T_5=0.5(0.5T_1+150.5+T_6)$$

Node 6: (Internal node)

$$T_6=0.25(T_5+T_2+T_7+300)$$

Node 7: (Internal node

$$T_7=0.25(T_6+T_3+T_8+300)$$

Node 8: (Right boundary node)

$$k\frac{\Delta x}{2}\frac{(300-T_8)}{\Delta y}+k\frac{\Delta x}{2}\frac{(T_4-T_8)}{\Delta y}+k\Delta y\frac{(T_7-T_8)}{\Delta x}+h\Delta y(T_\infty-T_8)=0 \Rightarrow T_8=0.4944(150.45+0.5T_4+T_7)$$

(b) By letting the initial guess as 200°C at each node, the temperature distribution obtained using Gauss-Seidel iteration method is as follows

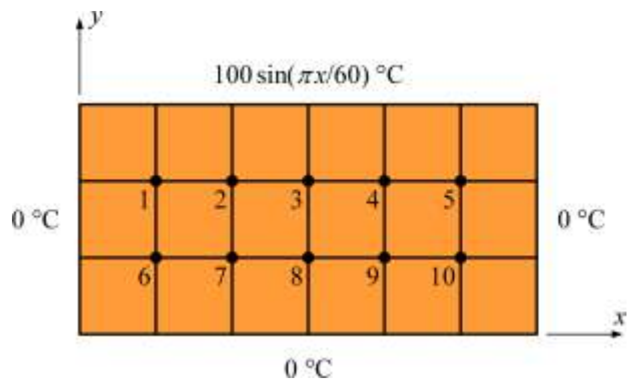
	Nodal temperature, °C							
Iteration	T_1	T_2	T_3	T_4	T_5	T_6	T_7	T_8
0	200.0	200.0	200.0	200.0	200.0	200.0	200.0	200.0
1	175.3	168.8	167.2	157.0	219.1	222.0	222.3	223.1
2	167.6	164.2	160.9	159.6	228.1	228.7	228.2	226.6
3	166.1	163.9	162.9	161.5	231.1	230.8	230.1	228.1
4	166.1	165.0	164.1	162.5	232.2	231.8	231.0	228.8
5	166.3	165.6	164.8	162.9	232.7	232.3	231.5	229.1
6	166.6	165.9	165.1	163.2	233.1	232.6	231.7	229.3
7	166.8	166.1	165.3	163.3	233.3	232.8	231.8	229.4
8	167.0	166.2	165.3	163.4	233.4	232.9	231.9	229.4
9	167.1	166.3	165.4	163.4	233.5	232.9	231.9	229.5
10	167.2	166.4	165.4	163.5	233.5	233.0	232.0	229.5
11	167.2	166.4	165.5	163.5	233.5	233.0	232.0	229.5

5-62 A rectangular cross section is undergoing a steady two-dimensional heat transfer. The finite difference equations and the nodal temperatures are to be determined.

Assumptions **1** Steady heat conduction is two-dimensional. **2** Thermal properties are constant. **3** There is no heat generation in the body.

Analysis (*a*) There are 10 unknown nodal temperatures, thus we need to have 10 equations to determine them uniquely. For nodes 1 to 10, we can use the general finite difference relation expressed as

$$\frac{T_{m-1,n} - 2T_{m,n} + T_{m+1,n}}{\Delta x^2} + \frac{T_{m,n-1} - 2T_{m,n} + T_{m,n+1}}{\Delta y^2} = 0$$



Since $\Delta x = \Delta y$, we have

$$T_{m,n} = 0.25(T_{m,n+1} + T_{m-1,n} + T_{m,n-1} + T_{m+1,n})$$

or $$T_{\text{node}} = 0.25(T_{\text{top}} + T_{\text{left}} + T_{\text{bottom}} + T_{\text{right}})$$

Then

Node 1: $T_1 = 0.25[100\sin(\pi/6) + 0 + T_6 + T_2]$

Node 2: $T_2 = 0.25[100\sin(2\pi/6) + T_1 + T_7 + T_3]$

Node 3: $T_3 = 0.25[100\sin(3\pi/6) + T_2 + T_8 + T_4]$

Node 4: $T_4 = 0.25[100\sin(4\pi/6) + T_3 + T_9 + T_5]$

Node 5: $T_5 = 0.25[100\sin(5\pi/6) + T_4 + T_{10} + 0]$

Node 6: $T_6 = 0.25[T_1 + 0 + 0 + T_7]$

Node 7: $T_7 = 0.25[T_2 + T_6 + 0 + T_8]$

Node 8: $T_8 = 0.25[T_3 + T_7 + 0 + T_9]$

Node 9: $T_9 = 0.25[T_4 + T_8 + 0 + T_{10}]$

Node 10: $T_{10} = 0.25[T_5 + T_9 + 0 + 0]$

(*b*) The nodal temperatures under steady conditions are determined by solving the 10 equations above simultaneously with an equation solver. Copy the following lines and paste on a blank EES screen to solve the above equations:

```
T_1=0.25*(100*sin(pi/6)+0+T_6+T_2)
T_2=0.25*(100*sin(2*pi/6)+T_1+T_7+T_3)
T_3=0.25*(100*sin(3*pi/6)+T_2+T_8+T_4)
T_4=0.25*(100*sin(4*pi/6)+T_3+T_9+T_5)
T_5=0.25*(100*sin(5*pi/6)+T_4+T_10+0)
T_6=0.25*(T_1+0+0+T_7)
T_7=0.25*(T_2+T_6+0+T_8)
T_8=0.25*(T_3+T_7+0+T_9)
T_9=0.25*(T_4+T_8+0+T_10)
T_10=0.25*(T_5+T_9+0+0)
```

Solving by EES software, we get

T_1 = **27.4°C**, T_2 = **47.4°C**, T_3 = **54.7°C**, T_4 = **47.4°C**, T_5 = **27.4°C**

T_6 = **12.1°C**, T_7 = **20.9°C**, T_8 = **24.1°C**, T_9 = **20.9°C**, T_{10} = **12.1°C**

Discussion The numerical solution can be verified using the following analytical solution:

$$T(x,y) = \frac{100\sinh(\pi\ y/60)\sin(\pi\ x/60)}{\sinh(30\pi/60)}$$

For example, at x = 30 cm and y = 20 cm, we have

$$T(30\text{ cm}, 20\text{ cm}) = \frac{100\sinh(20\pi/60)\sin(30\pi/60)}{\sinh(30\pi/60)} = 54.3\,°\text{C}$$

When compared with the numerical solution, T_3 = 54.7°C, the difference is within 0.8%.

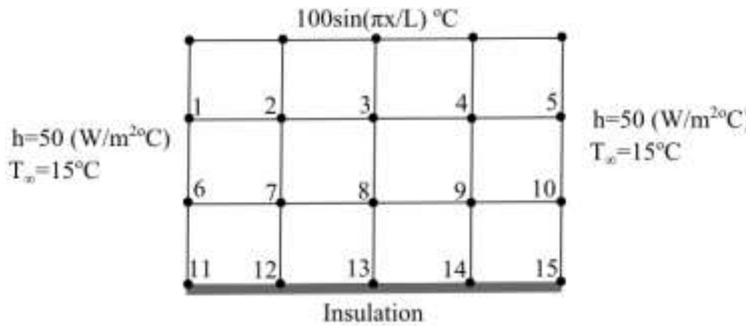
5-63 A rectangular metal block is subjected to specified boundary conditions. The finite difference equations and the nodal temperatures are to be determined.

Assumptions **1**Two-dimensional steady state heat conduction. **2** No internal heat generation. **3** Thermal conductivity is constant.

Properties Thermal conductivity of the metal block is given as k = 35 W/m·K.

Analysis (a) There are 15 unknown temperatures while the temperatures on the top surface of the block are to be determined from the given sinusoidal temperature distribution. Given that $\Delta x = \Delta y = 25cm$, the number of nodes in x and y directions are 5 and 4, respectively. For internal nodes i.e., 2, 3, 4, 7, 8 and 9 we can use the general form of the finite difference equations without heat generation expressed as

$$\frac{T_{m-1,n} - 2T_{m,n} + T_{m+1,n}}{\Delta x^2} + \frac{T_{m,n-1} - 2T_{m,n} + T_{m,n+1}}{\Delta y^2} = 0$$



Since $\Delta x = \Delta y$, we have

$$T_{m,n} = 0.25(T_{m,n+1} + T_{m-1,n} + T_{m,n-1} + T_{m+1,n})$$

For the boundary (external nodes) the finite difference formulation is obtained using energy balance and considering all heat transfer towards these nodes.

Thus the finite difference equations at each node are expressed as follows

Node 1: $$h\Delta y(T_\infty - T_1) + k\Delta y\frac{(T_2 - T_1)}{\Delta x} + k\frac{\Delta x}{2}\frac{(T_6 - T_1)}{\Delta y} + k\frac{\Delta x}{2}\frac{(100\sin(0\pi) - T_1)}{\Delta y} = 0$$

Node 2: $$T_1 + T_7 + T_3 + 100\sin(\pi/4) - 4T_2 = 0$$

Node 3: $$T_2 + T_8 + T_4 + 100\sin(\pi/2) - 4T_3 = 0$$

Node 4: $$T_3 + T_9 + T_5 + 100\sin(3\pi/4) - 4T_4 = 0$$

Node 5: $$h\Delta y(T_\infty - T_5) + k\Delta y\frac{(T_4 - T_5)}{\Delta x} + k\frac{\Delta x}{2}\frac{(100\sin(\pi) - T_5)}{\Delta y} + k\frac{\Delta x}{2}\frac{(T_{10} - T_5)}{\Delta y} = 0$$

Node 6: $$h\Delta y(T_\infty - T_6) + k\Delta y\frac{(T_7 - T_6)}{\Delta x} + k\frac{\Delta x}{2}\frac{(T_1 - T_6)}{\Delta y} + k\frac{\Delta x}{2}\frac{(T_{11} - T_6)}{\Delta y} = 0$$

Node 7: $$T_6 + T_{12} + T_8 + T_2 - 4T_7 = 0$$

Node 8: $$T_7 + T_{13} + T_9 + T_3 - 4T_8 = 0$$

Node 9: $$T_8 + T_{14} + T_{10} + T_4 - 4T_9 = 0$$

Node 10: $$h\Delta y(T_\infty - T_{10}) + k\Delta y\frac{(T_9 - T_{10})}{\Delta x} + k\frac{\Delta x}{2}\frac{(T_5 - T_{10})}{\Delta y} + k\frac{\Delta x}{2}\frac{(T_{15} - T_{10})}{\Delta y} = 0$$

Node 11: $$h\frac{\Delta y}{2}(T_\infty - T_{11}) + \dot{q}\frac{\Delta x}{2} + k\frac{\Delta y}{2}\frac{(T_{12} - T_{11})}{\Delta x} + k\frac{\Delta x}{2}\frac{(T_6 - T_{11})}{\Delta y} = 0$$

Node 12: $$k\frac{\Delta y}{2}\frac{(T_{11} - T_{12})}{\Delta x} + k\frac{\Delta y}{2}\frac{(T_{13} - T_{12})}{\Delta x} + k\Delta x\frac{(T_7 - T_{12})}{\Delta y} + \dot{q}\Delta x = 0$$

Node 13: $$k\frac{\Delta y}{2}\frac{(T_{12} - T_{13})}{\Delta x} + k\frac{\Delta y}{2}\frac{(T_{14} - T_{13})}{\Delta x} + k\Delta x\frac{(T_8 - T_{13})}{\Delta y} + \dot{q}\Delta x = 0$$

Node 14: $$k\frac{\Delta y}{2}\frac{(T_{13} - T_{14})}{\Delta x} + k\frac{\Delta y}{2}\frac{(T_{15} - T_{14})}{\Delta x} + k\Delta x\frac{(T_9 - T_{14})}{\Delta y} + \dot{q}\Delta x = 0$$

Node 15: $$h\frac{\Delta y}{2}(T_\infty - T_{15}) + \dot{q}\frac{\Delta x}{2} + k\frac{\Delta y}{2}\frac{(T_{14} - T_{15})}{\Delta x} + k\frac{\Delta x}{2}\frac{(T_{10} - T_{15})}{\Delta y} = 0$$

(b) The nodal temperatures at these different nodes can be determined by solving the equations simultaneously in equation solver.

Solving these equations simultaneously in EES or any other software gives

T_1 = **32.91°C**, T_2 = **53.77°C,** T_3 = **64.77°C,** T_4 = **53.77°C,** T_5 = **32.91°C,**

T_6 = **36.87°C,** T_7 = **46.69°C,** T_8 = **51.55°C,** T_9 = **46.69°C,** T_{10} = **36.87°C,**

T_{11} = **36.82°C,** T_{12} = **44.56°C,** T_{13} = **48.06°C,** T_{14} = **44.56°C,** T_{15} = **36.82°C.**

Discussion Thermal symmetry is observed about the centerline due to sinusoidal temperature distribution on the top surface and similar convective environments on both sides of the metal block. Thermal symmetry may not exist in case of the change in environment temperature or heat transfer coefficient on either side.

5-64 A rectangular cross section is subjected to convection on the top surface and constant temperature boundary condition on the other three surfaces. The temperatures at nodes 1, 2, and 3 are to be determined using Gauss-Seidel iteration method.

Assumptions **1**Steady heat conduction is two-dimensional without internal heat generation. **2** Thermal conductivity is constant.

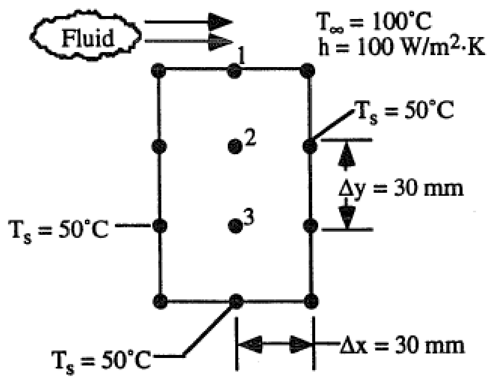
Properties Thermal conductivity is given as k = 1 W/m·K.

Analysis:

Node 1: Follow the development of the equation for node 7 of example problem 5-3 in the text with $\Delta x = \Delta y$

$$h\Delta x(T_\infty - T_1) + k\,\frac{\Delta x}{2}\,\frac{50 - T_1}{\Delta x} + k\Delta x\,\frac{T_2 - T_1}{\Delta x} + k\,\frac{\Delta x}{2}\,\frac{50 - T_1}{\Delta x} = 0$$

$$\frac{h\Delta x}{k}(T_\infty - T_1) + 50 - T_1 + T_2 - T_1 = 0$$



where

$$\frac{h\Delta x}{k} = \frac{\left(100\frac{W}{m^2}\cdot K\right)(0.03\ m)}{1\ \mathrm{W/m\cdot K}} = 3 \quad and\ T_\infty = 100\,℃$$

The eq. for Node 1 reduces to $\rightarrow$ $3(100 - T_1) + 50 - 2T_1 + T_2 = 0$ $\rightarrow$ $T_1 = 0.2\ T_2 + 70$

Node2: This is an interior node, use Eq. 5-35 $\rightarrow$ $50 + T_1 + 50 + T_3 - 4T_2 = 0$ $\rightarrow T_2 = 0.25T_2 + 0.25T_3 + 25$

Node 3: This is an interior node, use Eq. 5-35 $\rightarrow$ $50 + T_2 + 50 + 50 - 4T_3 = 0$ $\rightarrow T_3 = 0.25T_2 + 37.5$

	Nodal temperature, °C			
Iteration	T_1	T_2	T_3	
Initial Guess	0	0	0	
1	70.00	42.50	48.13	
2	78.50	56.66	51.67	
3	81.33	58.25	52.06	
4	**81.65**	**58.43**	**52.11**	Final answer ε < 0.35 for all temps.

5-65 A long solid body is subjected to steady two-dimensional heat transfer. The unknown nodal temperatures and the rate of heat loss from the top surface are to be determined.

Assumptions **1** Heat transfer through the body is given to be steady and two-dimensional. **2** Heat is generated uniformly in the body.

Properties The thermal conductivity is given to be k = 180 W/m·°C.

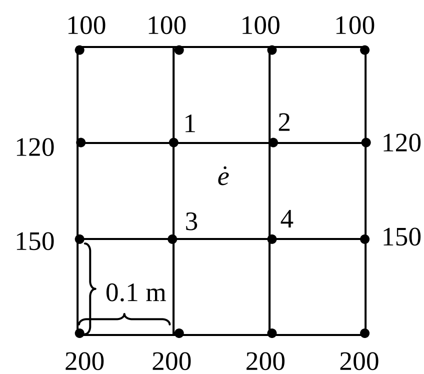
Analysis (*a*) The nodal spacing is given to be Δx=Δx=l=0.1 m, and the general finite difference form of an interior node equation for steady two-dimensional heat conduction for the case of constant heat generation is expressed as

$$T_{\text{left}} + T_{\text{top}} + T_{\text{right}} + T_{\text{bottom}} - 4T_{\text{node}} + \frac{\dot{e}_{\text{node}} l^2}{k} = 0$$

There is symmetry about a vertical line passing through the middle of the region, and thus we need to consider only half of the region. Then,

$$T_1 = T_2 \text{ and } T_3 = T_4$$

Therefore, there are there are only 2 unknown nodal temperatures, T_1 and T_3, and thus we need only 2 equations to determine them uniquely. Also, we can replace the symmetry lines by insulation and utilize the mirror-image concept when writing the finite difference equations for the interior nodes.



$$\text{Node 1 (interior)}: \quad 100 + 120 + T_2 + T_3 - 4T_1 + \frac{\dot{e}l^2}{k} = 0$$

$$\text{Node 3 (interior)}: \quad 150 + 200 + T_1 + T_4 - 4T_3 + \frac{\dot{e}l^2}{k} = 0$$

Noting that $T_1 = T_2$ and $T_3 = T_4$ and substituting,

$$220 + T_3 - 3T_1 + \frac{(1\times10^7 \text{ W/m}^3)(0.1\text{ m})^2}{150 \text{ W/m}\cdot{}^\circ\text{C}} = 0$$

$$350 + T_1 - 3T_3 + \frac{(1\times10^7 \text{ W/m}^3)(0.1\text{ m})^2}{150 \text{ W/m}\cdot{}^\circ\text{C}} = 0$$

The solution of the above system is

$$T_1 = T_2 = \mathbf{404.0^\circ C}$$
$$T_3 = T_4 = \mathbf{436.5^\circ C}$$

(*b*) The total rate of heat transfer from the top surface $\dot{Q}_{\text{top}}$ can be determined from an energy balance on a volume element at the top surface whose height is l/2, length 0.3 m, and depth 1 m:

$$\dot{Q}_{\text{top}} + \dot{e}_0(0.3\times1\times l/2) + \left(2k\frac{l\times1}{2}\frac{120-100}{l} + 2kl\times1\frac{T_1-100}{l}\right) = 0$$

$$\dot{Q}_{\text{top}} = -(1\times10^7 \text{ W/m}^3)(0.3\times0.1/2)\text{m}^3 - 2(150 \text{ W/m}\cdot{}^\circ\text{C})\left(\frac{1\text{ m}}{2}(120-100)^\circ\text{C} + (1\text{ m})(404.0 - 100)^\circ\text{C}\right)$$
$$= \mathbf{263{,}050\ W} \qquad \text{(per } m \text{ depth)}$$

5-66 Prob. 5-65 is reconsidered. The unknown nodal temperatures and the rate of heat loss from the top surface are to be determined.

Analysis The problem is solved using EES, and the solution is given below.

```
"GIVEN"
e_gen=1e7 [W/m^3] "heat generation"
k=180 [W/m-K] "thermal conductivity"
L=0.10 [m] "mesh size"

"ANALYSIS"
"(a) Using the finite difference method, the nodal temperatures can be determined"
100+120+T_2+T_3-4*T_1+e_gen*L^2/k=0 "for node 1"
T_2=T_1 "for node 2"
150+200+T_1+T_4-4*T_3+e_gen*L^2/k=0 "for node 3"
T_4=T_3 "for node 4"

"(b) The rate of heat loss from the top surface is calculated using"
Q_dot=e_gen*(0.3*1*L/2)+(2*k*L/2*(120-100)/L+2*k*L*(T_1-100)/L)
```

(*a*) The nodal temperatures are determined to be

$T_1 = T_2 = \mathbf{404.0°C}$ and $T_3 = T_4 = \mathbf{436.5°C}$

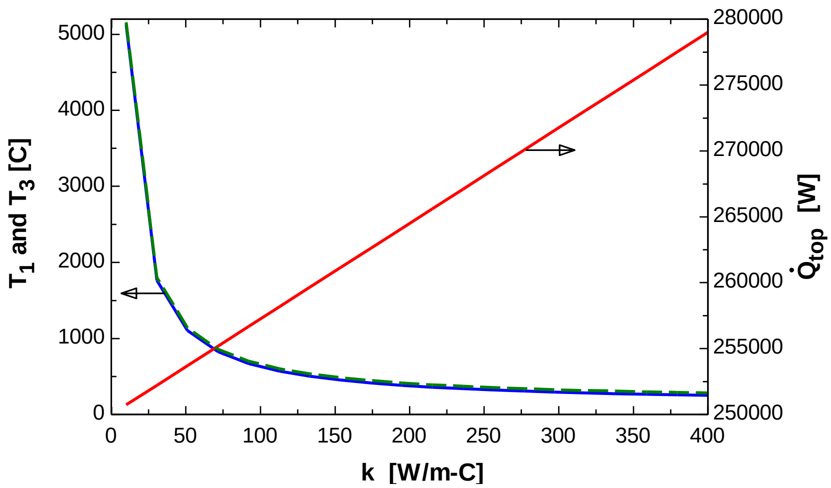
(*b*) The rate of heat loss from the top surface is $\dot{Q} = \mathbf{263{,}050\ W}$.

5-67 Prob. 5-65 is reconsidered. The effects of the thermal conductivity and the heat generation rate on the temperatures at nodes 1 and 3, and the rate of heat loss from the top surface are to be investigated.

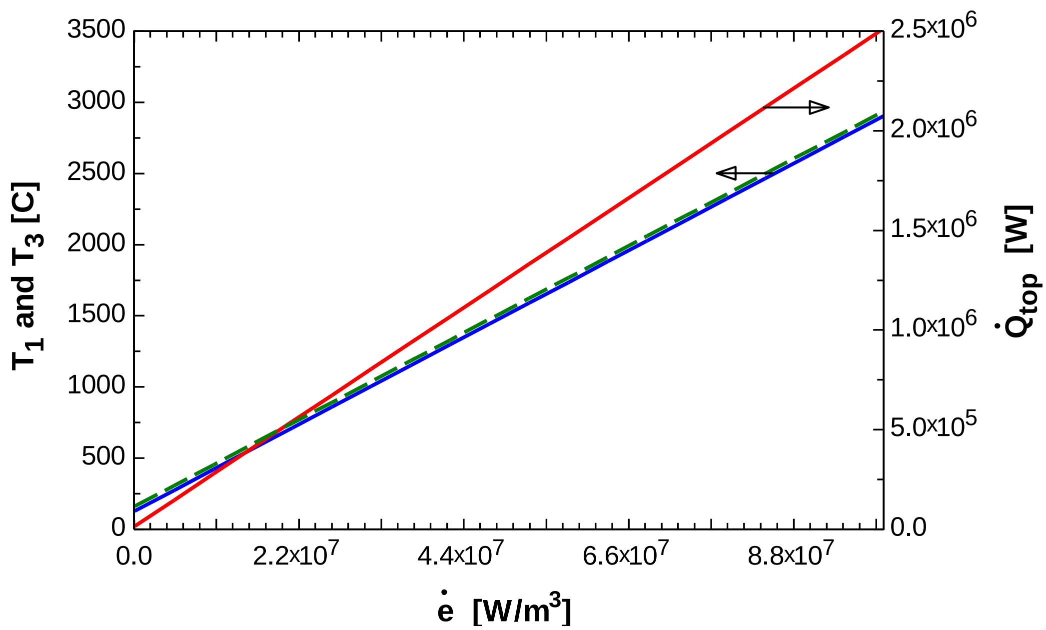
Analysis The problem is solved using EES, and the solution is given below.

```
"GIVEN"
k=180 [W/m-C]
e_dot=1E7 [W/m^3]
DELTAx=0.10 [m]
DELTAy=0.10 [m]
d=1 [m] "depth"
"Temperatures at the selected nodes are also given in the figure"

"ANALYSIS"
"(a)"
l=DELTAx
T_1=T_2 "due to symmetry"
T_3=T_4 "due to symmetry"
"Using the finite difference method, the two equations for the two unknown temperatures are determined to be"
100+120+T_2+T_3-4*T_1+(e_dot*l^2)/k=0
150+200+T_1+T_4-4*T_3+(e_dot*l^2)/k=0
"(b)"
"The rate of heat loss from the top surface can be determined from an energy balance on a volume element whose height is l/2, length 3*l, and depth d=1 m"
-Q_dot_top+e_dot*(3*l*d*l/2)+2*(k*(l*d)/2*(120-100)/l+k*l*d*(T_1-100)/l)=0
```

k [W/m.C]	T_1 [C]	T_3 [C]	$\dot{Q}_{top}$ [W]
10	5126	5159	250725
30.53	1764	1797	252213
51.05	1106	1138	253701
71.58	824.8	857.3	255189
92.11	669.1	701.6	256678
112.6	570.2	602.7	258166
133.2	501.7	534.2	259654
153.7	451.6	484.1	261142
174.2	413.3	445.8	262630
194.7	383	415.5	264118
215.3	358.5	391	265607
235.8	338.3	370.8	267095
256.3	321.3	353.8	268583
276.8	306.9	339.4	270071
297.4	294.4	326.9	271559
317.9	283.5	316	273047
338.4	274	306.5	274536
358.9	265.5	298	276024
379.5	258	290.5	277512
400	251.2	283.7	279000



$\dot{e}$ [W/m^3]	T_1 [C]	T_3 [C]	$\dot{Q}_{top}$ [W]
100000	129	161.5	15550
5.358E+06	275.1	307.6	146997
1.061E+07	421.1	453.6	278445
1.587E+07	567.2	599.7	409892
2.113E+07	713.2	745.7	541339
2.639E+07	859.3	891.8	672787
3.165E+07	1005	1038	804234
3.691E+07	1151	1184	935682
4.216E+07	1297	1330	1.067E+06
4.742E+07	1444	1476	1.199E+06
5.268E+07	1590	1622	1.330E+06
5.794E+07	1736	1768	1.461E+06
6.319E+07	1882	1914	1.593E+06
6.845E+07	2028	2060	1.724E+06
7.371E+07	2174	2206	1.856E+06
7.897E+07	2320	2352	1.987E+06
8.423E+07	2466	2498	2.119E+06
8.948E+07	2612	2644	2.250E+06
9.474E+07	2758	2790	2.382E+06
1.000E+08	2904	2937	2.513E+06

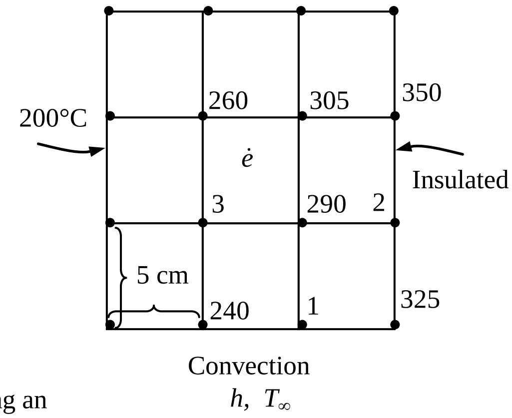


5-68 A long solid body is subjected to steady two-dimensional heat transfer. The unknown nodal temperatures and the rate of heat loss from the bottom surface through a 1-m long section are to be determined.

Assumptions **1** Heat transfer through the body is given to be steady and two-dimensional. **2** Heat is generated uniformly in the body. **3** Radiation heat transfer is negligible.

Properties The thermal conductivity is given to be k = 45 W/m·°C.

Analysis The nodal spacing is given to be $\Delta x = \Delta y = l$ = 0.05 m, and the general finite difference form of an interior node for steady two-dimensional heat conduction is expressed as



$$T_{\text{left}} + T_{\text{top}} + T_{\text{right}} + T_{\text{bottom}} - 4T_{\text{node}} + \frac{\dot{e}_{\text{node}} l^2}{k} = 0$$

where

$$\frac{\dot{e}_{\text{node}} l^2}{k} = \frac{\dot{e}_0 l^2}{k} = \frac{(4\times10^6 \text{ W/m}^3)(0.05\,\text{m})^2}{45 \text{ W/m}\cdot\text{°C}} = 222.2\text{°C}$$

The finite difference equations for boundary nodes are obtained by applying an energy balance on the volume elements and taking the direction of all heat transfers to be towards the node under consideration:

$$\text{Node 1 (convection)}: \quad k\frac{l}{2}\frac{240 - T_1}{l} + kl\frac{290 - T_1}{l} + k\frac{l}{2}\frac{325 - T_1}{l} + hl(T_\infty - T_1) + \frac{\dot{e}_0 l^2}{2} = 0$$

$$\text{Node 2 (interior)}: \quad 350 + 290 + 325 + 290 - 4T_2 + \frac{\dot{e}_0 l^2}{k} = 0$$

$$\text{Node 3 (interior)}: \quad 260 + 290 + 240 + 200 - 4T_3 + \frac{\dot{e}_0 l^2}{k} = 0$$

where

$$k = 45 \text{ W/m.°C}, \quad h = 50 \text{ W/m}^2\text{.°C}, \quad \dot{e} = 4\times10^6 \text{ W/m}^3, \quad T_\infty = 20\text{°C}$$

Substituting,

T_1 **= 333.1°C**, T_2 **= 369.3°C**, T_3 **= 303.1°C**,

(b) The rate of heat loss from the bottom surface through a 1-m long section is

$$\begin{aligned}\dot{Q} &= \sum_m \dot{Q}_{\text{element},\,m} = \sum_m hA_{\text{surface},m}(T_m - T_\infty) \\ &= h(l/2)(200 - T_\infty) + hl(240 - T_\infty) + hl(T_1 - T_\infty) + h(l/2)(325 - T_\infty) \\ &= (50 \text{ W/m}^2\cdot\text{°C})(0.05\,\text{m}\times1\,\text{m})[(200-20)/2 + (240-20) + (333.1-20) + (325-20)/2]\text{°C} \\ &= \mathbf{1939\ \ W}\end{aligned}$$

5-69 Prob. 5-68 is reconsidered. The unknown nodal temperatures and the rate of heat loss from the bottom surface through a 1-m long section are to be determined.

Analysis The problem is solved using EES, and the solution is given below.

```
"GIVEN"
e_gen=4e6 [W/m^3] "heat generation"
h=50 [W/m^2-K] "convection coefficient"
k=45 [W/m-K] "thermal conductivity"
L=0.05 [m] "mesh size"
T_inf=20 [C] "ambient temperature"

"ANALYSIS"
"(a) Using the finite difference method, the 3 equations for the 3 nodal temperatures can be determined"
k*L/2*(240-T_1)/L+k*L*(290-T_1)/L+k*L/2*(325-T_1)/L+h*L*(T_inf-T_1)+e_gen*L^2/2=0 "for node 1"
350+290+325+290-4*T_2+e_gen*L^2/k=0 "for node 2"
260+290+240+200-4*T_3+e_gen*L^2/k=0 "for node 3"

"(b) The rate of heat loss from the bottom surface is calculated by summing the heat loss from each node"
Q_dot=h*L/2*(200-T_inf)+h*L*(240-T_inf)+h*L*(T_1-T_inf)+h*L/2*(325-T_inf)
```

(*a*) The nodal temperatures are determined to be

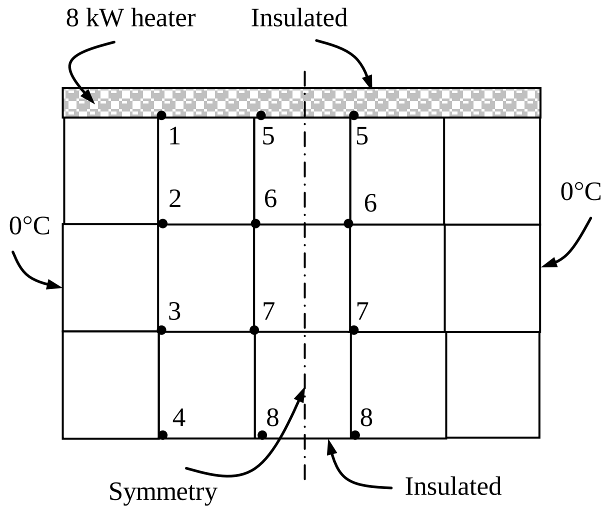
T_1 **= 333.1°C**, T_2 **= 369.3°C**, T_3 **= 303.1°C**,

(*b*) The rate of heat loss from the bottom surface is $\dot{Q} = \mathbf{1939\ W}$.

5-70 A rectangular block is subjected to uniform heat flux at the top, and iced water at 0°C at the sides. The steady finite difference formulation of the problem is to be obtained, and the unknown nodal temperatures as well as the rate of heat transfer to the iced water are to be determined.

Assumptions **1** Heat transfer through the body is given to be steady and two-dimensional. **2** There is no heat generation within the block. **3** The heat transfer coefficient is very high so that the temperatures on both sides of the block can be taken to be 0°C. **4** Heat transfer through the bottom surface is negligible.

Properties The thermal conductivity is given to be k = 23 W/m·°C.



Analysis The nodal spacing is given to be Δx=Δx=l=0.1 m, and the general finite difference form of an interior node equation for steady 2-D heat conduction is expressed as

$$T_{\text{left}} + T_{\text{top}} + T_{\text{right}} + T_{\text{bottom}} - 4T_{\text{node}} + \frac{\dot{e}_{\text{node}} l^2}{k} = 0$$

$$T_{\text{left}} + T_{\text{top}} + T_{\text{right}} + T_{\text{bottom}} - 4T_{\text{node}} = 0$$

There is symmetry about a vertical line passing through the middle of the region, and we need to consider only half of the region. Note that all side surfaces are at T_0 = 0°C, and there are 8 nodes with unknown temperatures. Replacing the symmetry lines by insulation and utilizing the mirror-image concept, the finite difference equations are obtained to be as follows:

Node 1 (heat flux): $\dot{q}_0 l + k\frac{l}{2}\frac{T_0 - T_1}{l} + k\frac{l}{2}\frac{T_5 - T_1}{l} + kl\frac{T_2 - T_1}{l} = 0$

Node 2 (interior): $T_0 + T_1 + T_3 + T_6 - 4T_2 = 0$

Node 3 (interior): $T_0 + T_2 + T_4 + T_7 - 4T_3 = 0$

Node 4 (insulation): $T_0 + 2T_3 + T_8 - 4T_4 = 0$

Node 5 (heat flux): $\dot{q}_0 l + k\frac{l}{2}\frac{T_1 - T_5}{l} + kl\frac{T_6 - T_5}{l} + 0 = 0$

Node 6 (interior): $T_2 + T_5 + T_6 + T_7 - 4T_6 = 0$

Node 7 (interior): $T_3 + T_6 + T_7 + T_8 - 4T_7 = 0$

Node 8 (insulation): $T_4 + 2T_7 + T_8 - 4T_8 = 0$

where

l = 0.1 m, k = 23 W/m·°C, T_0 =0°C, and $\dot{q}_0 = \dot{Q}_0 / A = (8000\text{ W})/(5\times 0.5\text{ m}^2) = 3200\text{ W/m}^2$

This system of 8 equations with 8 unknowns constitutes the finite difference formulation of the problem.

(b) The 8 nodal temperatures under steady conditions are determined by solving the 8 equations above simultaneously with an equation solver to be

T_1 = **18.2°C**, T_2 = **9.9°C**, T_3 = **6.2°C**, T_4 = **5.2°C**, T_5 = **25.4°C**, T_6 = **15.0°C**, T_7 = **9.9°C**, T_8 = **8.3°C**

(c) The rate of heat transfer from the block to the iced water is 8 kW since all the heat supplied to the block from the top must be equal to the heat transferred from the block. Therefore, $\dot{Q} = \mathbf{8\,kW}$.

Discussion The rate of heat transfer can also be determined by calculating the heat loss from the side surfaces using the heat conduction relation.

5-71 A square cross section with uniform heat generation is undergoing a steady two-dimensional heat transfer. The finite difference equations and the nodal temperatures are to be determined.

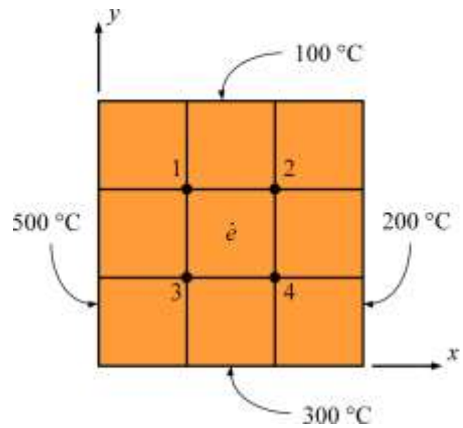
Assumptions **1** Steady heat conduction is two-dimensional. **2** Thermal properties are constant. **3** The heat generation in the body is uniform.

Properties The conductivity is given to be $k = 25$ W/m·K .

Analysis (*a*) There are 4 unknown nodal temperatures, thus we need to have 4 equations to determine them uniquely. For nodes 1 to 4, we can use the general finite difference relation expressed as

$$\frac{T_{m-1,n} - 2T_{m,n} + T_{m+1,n}}{\Delta x^2} + \frac{T_{m,n-1} - 2T_{m,n} + T_{m,n+1}}{\Delta y^2} + \frac{\dot{e}_{m,n}}{k} = 0$$

$$T_{m,n} = 0.25(T_{m-1,n} + T_{m+1,n} + T_{m,n-1} + T_{m,n+1} + \dot{e}_{m,n}\Delta x^2 / k)$$



Since $\Delta x = \Delta y$, we have

$$T_{m,n} = 0.25(T_{m,n+1} + T_{m+1,n} + T_{m,n-1} + T_{m-1,n} + \dot{e}_{m,n}\Delta x^2 / k)$$

or $$T_{\text{node}} = 0.25(T_{\text{top}} + T_{\text{right}} + T_{\text{bottom}} + T_{\text{left}} + \dot{e}_{\text{node}}\Delta x^2 / k)$$

Then

Node 1: $T_1 = 0.25(100 + T_2 + T_3 + 500 + \dot{e}_{\text{node}}\Delta x^2 / k)$

Node 2: $T_2 = 0.25(100 + 200 + T_4 + T_1 + \dot{e}_{\text{node}}\Delta x^2 / k)$

Node 3: $T_3 = 0.25(T_1 + T_4 + 300 + 500 + \dot{e}_{\text{node}}\Delta x^2 / k)$

Node 4: $T_4 = 0.25(T_2 + 200 + 300 + T_3 + \dot{e}_{\text{node}}\Delta x^2 / k)$

where $\dot{e}_{\text{node}}\Delta x^2 / k = 20\,°\text{C}$.

(*b*) By letting the initial guesses as T_1 = 300°C, T_2 = 150°C, T_3 = 400°C, and T_4 = 250°C the results obtained from successive iterations are listed in the following table:

Iteration	Nodal temperature,°C			
	T_1	T_2	T_3	T_4
1	292.5	215.6	340.6	269.1
2	294.1	220.8	345.8	271.6
3	296.6	222.1	347.1	272.3
4	297.3	222.4	347.4	272.4
5	297.4	222.5	347.5	272.5
6	297.5	222.5	347.5	272.5
7	297.5	222.5	347.5	272.5

Hence, the converged nodal temperatures are

T_1 = **297.5°C**, T_2 = **222.5°C**, T_3 = **347.5°C**, T_4 = **272.5°C**

Discussion The finite difference equations can also be calculated using the EES. Copy the following lines and paste on a blank EES screen to solve the above equations:

```
T_1=0.25*(100+T_2+T_3+500+20)
T_2=0.25*(100+200+T_4+T_1+20)
T_3=0.25*(T_1+T_4+300+500+20)
T_4=0.25*(T_2+200+300+T_3+20)
```

Solving by EES software, we get the same results:

T_1 = **297.5°C**, T_2 = **222.5°C**, T_3 = **347.5°C**, T_4 = **272.5°C**

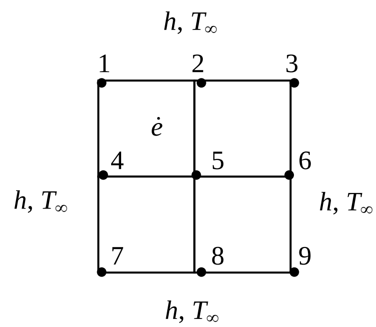
5-72E A long solid bar is subjected to steady two-dimensional heat transfer. The unknown nodal temperatures and the rate of heat loss from the bar through a 1-ft long section are to be determined.

Assumptions **1** Heat transfer through the body is given to be steady and two-dimensional. **2** Heat is generated uniformly in the body. **3** The heat transfer coefficient also includes the radiation effects.

Properties The thermal conductivity is given to be k = 16 Btu/h.ft·°C.

Analysis The nodal spacing is given to be $\Delta x=\Delta y=l$=0.25 ft, and the general finite difference form of an interior node for steady two-dimensional heat conduction is expressed as

$$T_{\text{left}}+T_{\text{top}}+T_{\text{right}}+T_{\text{bottom}}-4T_{\text{node}}+\frac{\dot{e}_{\text{node}}l^2}{k}=0$$



(a) There is symmetry about the vertical, horizontal, and diagonal lines passing through the center. Therefore, $T_1=T_3=T_7=T_9$ and $T_2=T_4=T_6=T_8$, and T_1, T_2, and T_5 are the only 3 unknown nodal temperatures, and thus we need only 3 equations to determine them uniquely. Also, we can replace the symmetry lines by insulation and utilize the mirror-image concept for the interior nodes.

The finite difference equations for boundary nodes are obtained by applying an energy balance on the volume elements and taking the direction of all heat transfers to be towards the node under consideration:

$$\text{Node 1 (convection)}: \quad 2k\frac{l}{2}\frac{T_2-T_1}{l}+2h\frac{l}{2}(T_\infty-T_1)+\frac{\dot{e}_0 l^2}{4}=0$$

$$\text{Node 2 (convection)}: \quad 2k\frac{l}{2}\frac{T_1-T_2}{l}+kl\frac{T_5-T_2}{l}+hl(T_\infty-T_2)+\frac{\dot{e}_0 l^2}{2}=0$$

$$\text{Node 5 (interior)}: \quad 4T_2-4T_5+\frac{\dot{e}_0 l^2}{k}=0$$

where $\dot{e}_0=0.19\times10^5$ Btu/h·ft^3, l = 0.25 ft, k = 16 Btu/h.ft·°F, h =7.9 Btu/h.ft^2·°F, and T_∞ =70°F. The 3 nodal temperatures under steady conditions are determined by solving the 3 equations above simultaneously with an equation solver to be

$T_1=T_3=T_7=T_9$ **= 361.89°F,**

$T_2=T_4=T_6=T_8$ **= 379.37°F,** T_5 = **397.93°F**

(b) The rate of heat loss from the bar through a 1-ft long section is determined from an energy balance on one-eight section of the bar, and multiplying the result by 8:

$$\dot{Q}=8\times\dot{Q}_{\text{one-eight section,conv}}=8\times\left[h\frac{l}{2}(T_1-T_\infty)+h\frac{l}{2}(T_2-T_\infty)\right](1\text{ ft})=8\times h\frac{l}{2}\left[T_1+T_2-2T_\infty\right](1\text{ ft})$$

$$=8(7.9\text{ Btu/h}\cdot\text{ft}^2\cdot°\text{F})(0.25/2\text{ ft})(1\text{ ft})[361.89+379.37-2\times70]°\text{F}$$

$$=\mathbf{4750\ Btu/h}\quad(\text{per ft length})$$

Discussion Under steady conditions, the rate of heat loss from the bar is equal to the rate of heat generation within the bar per unit length, and is determined to be

$$\dot{Q}=\dot{E}_{\text{gen}}=\dot{e}_0 V=(0.19\times10^5\text{ Btu/h.ft}^3)(0.5\text{ ft}\times0.5\text{ ft}\times1\text{ ft})=4750\text{ Btu/h}\ (\text{per ft length})$$

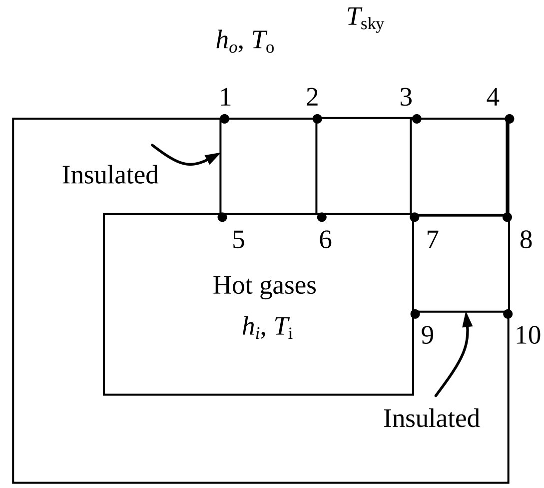
which confirms the results obtained by the finite difference method.

5-73 Heat transfer through a square chimney is considered. The nodal temperatures and the rate of heat loss per unit length are to be determined with the finite difference method.

Assumptions **1** Heat transfer is given to be steady and two-dimensional since the height of the chimney is large relative to its cross-section, and thus heat conduction through the chimney in the axial direction is negligible. It is tempting to simplify the problem further by considering heat transfer in each wall to be one dimensional which would be the case if the walls were thin and thus the corner effects were negligible. This assumption cannot be justified in this case since the walls are very thick and the corner sections constitute a considerable portion of the chimney structure. **2** There is no heat generation in the chimney. **3** Thermal conductivity is constant.

Properties The thermal conductivity and emissivity are given to be k = 1.4 W/m·°C and ε = 0.9.

Analysis (*a*) The most striking aspect of this problem is the apparent symmetry about the horizontal and vertical lines passing through the midpoint of the chimney. Therefore, we need to consider only one-fourth of the geometry in the solution whose nodal network consists of 10 equally spaced nodes. No heat can cross a symmetry line, and thus symmetry lines can be treated as insulated surfaces and thus "mirrors" in the finite-difference formulation. Considering a unit depth and using the energy balance approach for the boundary nodes (again assuming all heat transfer to be into the volume element for convenience), the finite difference formulation is obtained to be



Node 1: $h_o \frac{l}{2}(T_0 - T_1) + k\frac{l}{2}\frac{T_2 - T_1}{l} + k\frac{l}{2}\frac{T_5 - T_1}{l} + \varepsilon\sigma\frac{l}{2}[T_{sky}^4 - (T_1 + 273)^4] = 0$

Node 2: $h_o l(T_0 - T_2) + k\frac{l}{2}\frac{T_1 - T_2}{l} + k\frac{l}{2}\frac{T_3 - T_2}{l} + kl\frac{T_6 - T_2}{l} + \varepsilon\sigma l[T_{sky}^4 - (T_2 + 273)^4] = 0$

Node 3: $h_o l(T_0 - T_3) + k\frac{l}{2}\frac{T_2 - T_3}{l} + k\frac{l}{2}\frac{T_4 - T_3}{l} + kl\frac{T_7 - T_3}{l} + \varepsilon\sigma l[T_{sky}^4 - (T_3 + 273)^4] = 0$

Node 4: $h_o l(T_0 - T_4) + k\frac{l}{2}\frac{T_3 - T_4}{l} + k\frac{l}{2}\frac{T_8 - T_4}{l} + \varepsilon\sigma l[T_{sky}^4 - (T_4 + 273)^4] = 0$

Node 5: $h_i \frac{l}{2}(T_i - T_5) + k\frac{l}{2}\frac{T_6 - T_5}{l} + k\frac{l}{2}\frac{T_1 - T_5}{l} = 0$

Node 6: $h_i l(T_i - T_6) + k\frac{l}{2}\frac{T_5 - T_6}{l} + k\frac{l}{2}\frac{T_7 - T_6}{l} + kl\frac{T_2 - T_6}{l} = 0$

Node 7: $h_i l(T_i - T_7) + k\frac{l}{2}\frac{T_6 - T_7}{l} + k\frac{l}{2}\frac{T_9 - T_7}{l} + kl\frac{T_3 - T_7}{l} + kl\frac{T_8 - T_7}{l} = 0$

Node 8: $h_0 l(T_0 - T_8) + k\frac{l}{2}\frac{T_4 - T_8}{l} + k\frac{l}{2}\frac{T_{10} - T_8}{l} + kl\frac{T_7 - T_8}{l} + \varepsilon\sigma l[T_{sky}^4 - (T_8 + 273)^4] = 0$

Node 9: $h_i \frac{l}{2}(T_i - T_9) + k\frac{l}{2}\frac{T_7 - T_9}{l} + k\frac{l}{2}\frac{T_{10} - T_9}{l} = 0$

Node 10: $h_o \frac{l}{2}(T_0 - T_{10}) + k\frac{l}{2}\frac{T_8 - T_{10}}{l} + k\frac{l}{2}\frac{T_9 - T_{10}}{l} + \varepsilon\sigma\frac{l}{2}[T_{sky}^4 - (T_{10} + 273)^4] = 0$

where l = 0.1 m, k = 1.4 W/m·°C, h_i = 75 W/m^2·°C, T_i =280°C, h_o = 18 W/m^2·°C, T_0 =15°C, T_{surr} =250 K, ε = 0.9, and σ = 5.67×10^{-8} W/m^2.K^4. This system of 10 equations with 10 unknowns constitutes the finite difference formulation of the problem.

(*b*) The 10 nodal temperatures under steady conditions are determined by solving the 10 equations above simultaneously with an equation solver to be

T_1 = **94.5°C**, T_2 = **93.0°C**, T_3 = **82.1°C**, T_4 = **36.1°C**, T_5 = **250.6°C**,

T_6 = **249.2°C**, T_7 = **229.7°C**, T_8 = **82.3°C**, T_9 = **261.5°C**, T_{10} = **94.6°C**

(*c*) The rate of heat loss through a 1-m long section of the chimney is determined from

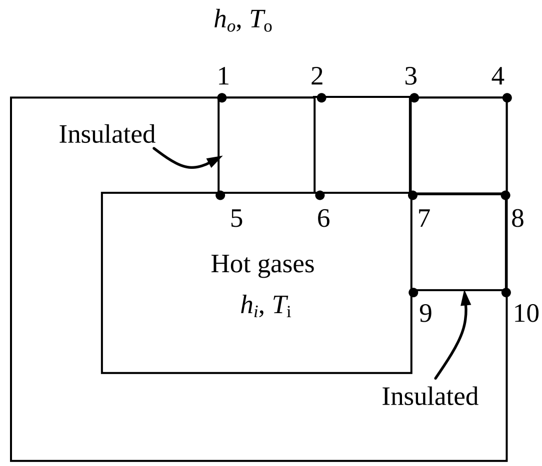
$$\begin{aligned}\dot{Q} &= 4\sum \dot{Q}_{\text{one-fourth of chimney}} = 4\sum \dot{Q}_{\text{element, inner surface}} = 4\sum_m h_i A_{\text{surface},m}(T_i - T_m)\\ &= 4[h_i(l/2)(T_i - T_5) + h_i l(T_i - T_6) + h_i l(T_i - T_7) + h_i(l/2)(T_i - T_9)]\\ &= 4(75\ \text{W/m}^2\cdot{}^\circ\text{C})(0.1\,\text{m}\times 1\,\text{m})[(280-250.6)/2 + (280-249.2) + (280-229.7) + (280-261.5)/2]{}^\circ\text{C}\\ &= \mathbf{3153\ \ W}\end{aligned}$$

Discussion The rate of heat transfer can also be determined by calculating the heat loss from the outer surface by convection and radiation.

5-74 Heat transfer through a square chimney is considered. The nodal temperatures and the rate of heat loss per unit length are to be determined with the finite difference method.

Assumptions **1** Heat transfer is given to be steady and two-dimensional since the height of the chimney is large relative to its cross-section, and thus heat conduction through the chimney in the axial direction is negligible. It is tempting to simplify the problem further by considering heat transfer in each wall to be one dimensional which would be the case if the walls were thin and thus the corner effects were negligible. This assumption cannot be justified in this case since the walls are very thick and the corner sections constitute a considerable portion of the chimney structure. **2** There is no heat generation in the chimney. **3** Thermal conductivity is constant. **4** Radiation heat transfer is negligible.

Properties The thermal conductivity of chimney is given to be k = 1.4 W/m·°C.



Analysis (*a*) The most striking aspect of this problem is the apparent symmetry about the horizontal and vertical lines passing through the midpoint of the chimney. Therefore, we need to consider only one-fourth of the geometry in the solution whose nodal network consists of 10 equally spaced nodes. No heat can cross a symmetry line, and thus symmetry lines can be treated as insulated surfaces and thus "mirrors" in the finite-difference formulation. Considering a unit depth and using the energy balance approach for the boundary nodes (again assuming all heat transfer to be into the volume element for convenience), the finite difference formulation is obtained to be

Node 1: $h_o\frac{l}{2}(T_0-T_1)+k\frac{l}{2}\frac{T_2-T_1}{l}+k\frac{l}{2}\frac{T_5-T_1}{l}=0$

Node 2: $h_o l(T_0-T_2)+k\frac{l}{2}\frac{T_1-T_2}{l}+k\frac{l}{2}\frac{T_3-T_2}{l}+kl\frac{T_6-T_2}{l}=0$

Node 3: $h_o l(T_0-T_3)+k\frac{l}{2}\frac{T_2-T_3}{l}+k\frac{l}{2}\frac{T_4-T_3}{l}+kl\frac{T_7-T_3}{l}=0$

Node 4: $h_o l(T_0-T_4)+k\frac{l}{2}\frac{T_3-T_4}{l}+k\frac{l}{2}\frac{T_8-T_4}{l}=0$

Node 5: $h_i\frac{l}{2}(T_i-T_5)+k\frac{l}{2}\frac{T_6-T_5}{l}+k\frac{l}{2}\frac{T_1-T_5}{l}=0$

Node 6: $h_i l(T_i-T_6)+k\frac{l}{2}\frac{T_5-T_6}{l}+k\frac{l}{2}\frac{T_7-T_6}{l}+kl\frac{T_2-T_6}{l}=0$

Node 7: $h_i l(T_i-T_7)+k\frac{l}{2}\frac{T_6-T_7}{l}+k\frac{l}{2}\frac{T_9-T_7}{l}+kl\frac{T_3-T_7}{l}+kl\frac{T_8-T_7}{l}=0$

Node 8: $h_o l(T_0-T_8)+k\frac{l}{2}\frac{T_4-T_8}{l}+k\frac{l}{2}\frac{T_{10}-T_8}{l}+kl\frac{T_7-T_8}{l}=0$

Node 9: $h_i\frac{l}{2}(T_i-T_9)+k\frac{l}{2}\frac{T_7-T_9}{l}+k\frac{l}{2}\frac{T_{10}-T_9}{l}=0$

Node 10: $h_o\frac{l}{2}(T_0-T_{10})+k\frac{l}{2}\frac{T_8-T_{10}}{l}+k\frac{l}{2}\frac{T_9-T_{10}}{l}=0$

where l = 0.1 m, k = 1.4 W/m·°C, h_i = 75 W/m^2·°C, T_i =280°C, h_o = 18 W/m^2·°C, T_0 =15°C, and σ = 5.67×10^{-8} W/m^2.K^4. This system of 10 equations with 10 unknowns constitutes the finite difference formulation of the problem.

(*b*) The 10 nodal temperatures under steady conditions are determined by solving the 10 equations above simultaneously with an equation solver to be

T_1 = **118.8°C**, T_2 = **116.7°C**, T_3 = **103.4°C**, T_4 = **53.7°C**, T_5 = **254.4°C**,

T_6 = **253.0°C**, T_7 = **235.2°C**, T_8 = **103.5°C**, T_9 = **263.7°C**, T_{10} = **117.6°C**

(*c*) The rate of heat loss through a 1-m long section of the chimney is determined from

$$\dot{Q} = 4\sum \dot{Q}_{\text{one-fourth of chimney}} = 4\sum \dot{Q}_{\text{element, inner surface}} = 4\sum_{m} h_i A_{\text{surface},m}(T_i - T_m)$$

$$= 4[h_i(l/2)(T_i - T_5) + h_i l(T_i - T_6) + h_i l(T_i - T_7) + h_i(l/2)(T_i - T_9)]$$

$$= 4(75\ \text{W/m}^2\cdot{}^\circ\text{C})(0.1\,\text{m}\times 1\,\text{m})[(280 - 254.4)/2 + (280 - 253.0) + (280 - 235.2) + (280 - 263.7)/2]^\circ\text{C}$$

$$= \mathbf{2783\ W}$$

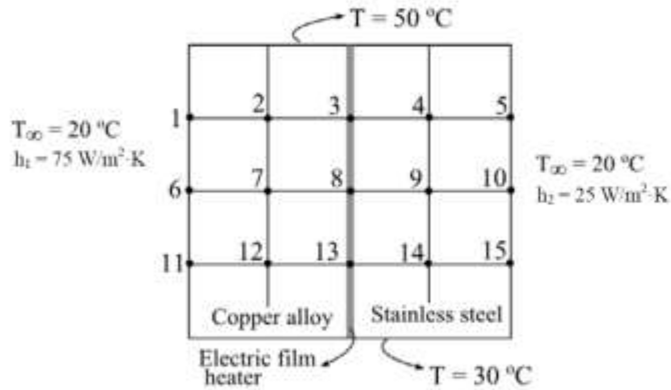
Discussion The rate of heat transfer can also be determined by calculating the heat loss from the outer surface by convection.

5-75 A thin film heater is sandwiched between copper alloy and stainless steel blocks. By developing finite difference formulation at different nodes, the location of maximum temperature is to be determined.

Assumptions **1** Heat transfer is steady state and two-dimensional without heat generation. **2** Thermal conductivities of both copper alloy and stainless steel stay constant.

Properties: The thermal conductivity of the copper alloy and stainless steel is given to be 120 W/m·K and 15 W/m·K, respectively.

Analysis: (a) The finite difference formulation at each node is calculated using energy balance at each node such that all heat transfer is to the node control volume. Nodes 1, 5, 6, 10, 11 and 15 are the boundary nodes subjected to convection boundary condition while nodes 2, 4, 7, 9, 12 and 14 are the internal nodes. The nodes 3, 8 and 13 are the internal node but are subjected to a uniform heat flux by a thin film heater.



Node 1: $$h_1\Delta y(T_\infty - T_1) + k_1\frac{\Delta x}{2}\frac{(50-T_1)}{\Delta y} + k_1\Delta y\frac{(T_2-T_1)}{\Delta x} + k_1\frac{\Delta x}{2}\frac{(T_6-T_1)}{\Delta y} = 0$$

Node 2: $$k_1\Delta y\frac{(T_1-T_2)}{\Delta x} + k_1\Delta x\frac{(50-T_2)}{\Delta y} + k_1\Delta x\frac{(T_7-T_2)}{\Delta y} + k_1\Delta y\frac{(T_3-T_2)}{\Delta x} = 0$$

Node 3: $$k_1\Delta y\frac{(T_2-T_3)}{\Delta x} + (k_1+k_2)\frac{\Delta x}{2}\frac{(50-T_3)}{\Delta y} + (k_1+k_2)\frac{\Delta x}{2}\frac{(T_8-T_3)}{\Delta y} + k_2\Delta y\frac{(T_4-T_3)}{\Delta x} + \dot{q}\Delta y = 0$$

Node 4: $$k_2\Delta y\frac{(T_3-T_4)}{\Delta x} + k_2\Delta x\frac{(50-T_4)}{\Delta y} + k_2\Delta x\frac{(T_9-T_4)}{\Delta y} + k_2\Delta y\frac{(T_5-T_4)}{\Delta x} = 0$$

Node 5: $$h_2\Delta y(T_\infty - T_5) + k_2\frac{\Delta x}{2}\frac{(50-T_5)}{\Delta y} + k_2\Delta y\frac{(T_4-T_5)}{\Delta x} + k_2\frac{\Delta x}{2}\frac{(T_{10}-T_5)}{\Delta y} = 0$$

Node 6: $$h_1\Delta y(T_\infty - T_6) + k_1\frac{\Delta x}{2}\frac{(T_1-T_6)}{\Delta y} + k_1\Delta y\frac{(T_7-T_6)}{\Delta x} + k_1\frac{\Delta x}{2}\frac{(T_{11}-T_6)}{\Delta y} = 0$$

Node 7: $$k_1\Delta y\frac{(T_6-T_7)}{\Delta x} + k_1\Delta x\frac{(T_2-T_7)}{\Delta y} + k_1\Delta x\frac{(T_{12}-T_7)}{\Delta y} + k_1\Delta y\frac{(T_8-T_7)}{\Delta x} = 0$$

Node 8: $$k_1\Delta y\frac{(T_7-T_8)}{\Delta x} + (k_1+k_2)\frac{\Delta x}{2}\frac{(T_3-T_8)}{\Delta y} + (k_1+k_2)\frac{\Delta x}{2}\frac{(T_{13}-T_8)}{\Delta y} + k_2\Delta y\frac{(T_9-T_8)}{\Delta x} + \dot{q}\Delta y = 0$$

Node 9: $$k_2\Delta y\frac{(T_8-T_9)}{\Delta x} + k_2\Delta x\frac{(T_4-T_9)}{\Delta y} + k_2\Delta x\frac{(T_{14}-T_9)}{\Delta y} + k_2\Delta y\frac{(T_{10}-T_9)}{\Delta x} = 0$$

Node 10: $$h_2\Delta y(T_\infty - T_{10}) + k_2\frac{\Delta x}{2}\frac{(T_5-T_{10})}{\Delta y} + k_2\Delta y\frac{(T_9-T_{10})}{\Delta x} + k_2\frac{\Delta x}{2}\frac{(T_{15}-T_{10})}{\Delta y} = 0$$

Node 11: $$h_1\Delta y(T_\infty - T_{11}) + k_1\frac{\Delta x}{2}\frac{(T_6-T_{11})}{\Delta y} + k_1\Delta y\frac{(T_{12}-T_{11})}{\Delta x} + k_1\frac{\Delta x}{2}\frac{(30-T_{11})}{\Delta y} = 0$$

Node 12: $$k_1\Delta y\frac{(T_{11}-T_{12})}{\Delta x} + k_1\Delta x\frac{(T_7-T_{12})}{\Delta y} + k_1\Delta x\frac{(30-T_{12})}{\Delta y} + k_1\Delta y\frac{(T_{13}-T_{12})}{\Delta x} = 0$$

Node 13: $$k_1\Delta y\frac{(T_{12}-T_{13})}{\Delta x} + (k_1+k_2)\frac{\Delta x}{2}\frac{(T_8-T_{13})}{\Delta y} + (k_1+k_2)\frac{\Delta x}{2}\frac{(30-T_{13})}{\Delta y} + k_2\Delta y\frac{(T_{14}-T_{13})}{\Delta x} + \dot{q}\Delta y = 0$$

Node 14: $$k_2 \Delta y \frac{(T_{13} - T_{14})}{\Delta x} + k_2 \Delta x \frac{(T_9 - T_{14})}{\Delta y} + k_2 \Delta x \frac{(30 - T_{14})}{\Delta y} + k_2 \Delta y \frac{(T_{15} - T_{14})}{\Delta x} = 0$$

Node 15: $$h_2 \Delta y (T_\infty - T_{15}) + k_2 \frac{\Delta x}{2} \frac{(T_{10} - T_{15})}{\Delta y} + k_2 \Delta y \frac{(T_{14} - T_{15})}{\Delta x} + k_2 \frac{\Delta x}{2} \frac{(30 - T_{15})}{\Delta y} = 0$$

For a 500 W heater, the heat flux on each slab of 100 mm × 100 mm cross section is 50000 W/m^2.

(b) The temperature at each node is determined by solving the 15 equations for 15 unknown nodal temperatures using EES or any other software. The temperature at different nodes are as follows

T_1 = **49.07°C**, T_2 = **50.71°C**, T_3 = **55.97°C**, T_4 = **50.46°C**, T_5 = **48.38°C,**

T_6 = **45.76°C**, T_7 = **47.8°C**, T_8 = **54°C**, T_9 = **47.49°C**, T_{10} = **44.98°C,**

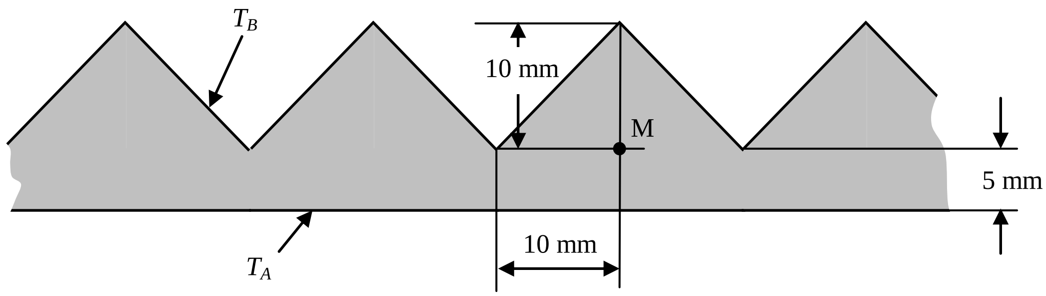
T_{11} = **39.16°C**, T_{12} = **40.74°C**, T_{13} = **45.99°C**, T_{14} = **40.52°C**, T_{15} = **38.62°C.**

The maximum temperature occurs at node 8 (Center node) where the temperature is **54°C**.

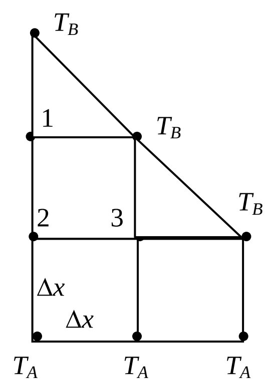
5-76 Two dimensional ridges are machined on the cold side of a heat exchanger. The smallest section of the wall is to be identified. A two-dimensional grid is to be constructed and the unknown temperatures in the grid are to be determined.

Assumptions **1** Heat transfer through the body is given to be steady and two-dimensional. **2** Thermal conductivity is constant. **3** There is no heat generation.

Analysis (*a*) From symmetry, the smallest domain is between the top and the base of one ridge.



(*b*) The unknown temperatures at nodes 1, 2, and 3 are to be determined from finite difference formulations



Node 1:

$$k\frac{T_B-T_1}{\Delta x}\Delta x+k\frac{T_2-T_1}{\Delta x}\frac{\Delta x}{2}+k\frac{T_B-T_1}{\Delta x}\frac{\Delta x}{2}=0$$

$$2T_B-2T_1+T_2-T_1+T_B-T_1=0$$

$$4T_1-T_2=3T_B=3\times 10=30$$

Node 2:

$$k\frac{T_1-T_2}{\Delta x}\frac{\Delta x}{2}+k\frac{T_3-T_2}{\Delta x}\Delta x+k\frac{T_A-T_2}{\Delta x}\frac{\Delta x}{2}=0$$

$$T_1-T_2+2T_3-2T_2+T_A-T_2=0$$

$$-T_1+4T_2-2T_3=T_A=90$$

Node 3:

$$4T_3=T_2+T_A+T_B+T_B$$

$$-T_2+4T_3=2T_B+T_A=2\times 10+90=110$$

The matrix equation is

$$\begin{bmatrix}4 & -1 & 0\\ -1 & 4 & -2\\ 0 & -1 & 4\end{bmatrix}\begin{bmatrix}T_1\\ T_2\\ T_3\end{bmatrix}=\begin{bmatrix}30\\ 90\\ 110\end{bmatrix}$$

(*c*) The temperature T_2 is 46.9°C. Then the temperatures T_1 and T_3 are determined from equations 1 and 3.

$$4T_1-T_2=30$$

$$4T_1-46.9=30\longrightarrow T_1=\mathbf{19.2^\circ C}$$

$$-T_2+4T_3=110$$

$$-46.9+4T_3=110\longrightarrow T_3=\mathbf{39.2^\circ C}$$

5-77 Two long solid bodies are subjected to steady two-dimensional heat transfer. The unknown nodal temperatures are to be determined.

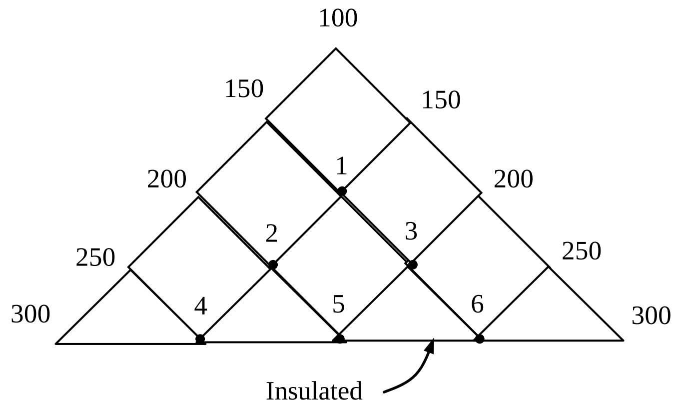
Assumptions **1** Heat transfer through the bodies are given to be steady and two-dimensional. **2** There is no heat generation in the body.

Properties The thermal conductivity is given to be k = 20 W/m·°C.

Analysis The nodal spacing is given to be Δx=Δx=l=0.01m, and the general finite difference form of an interior node for steady two-dimensional heat conduction for the case of no heat generation is expressed as

$$T_{\text{left}} + T_{\text{top}} + T_{\text{right}} + T_{\text{bottom}} - 4T_{\text{node}} + \frac{\dot{e}_{\text{node}} l^2}{k} = 0 \longrightarrow T_{\text{left}} + T_{\text{top}} + T_{\text{right}} + T_{\text{bottom}} - 4T_{\text{node}} = 0$$

(a) There is symmetry about a vertical line passing through the nodes 1 and 3. Therefore, $T_3 = T_2$, $T_6 = T_4$, and T_1, T_2, T_4, and T_5 are the only 4 unknown nodal temperatures, and thus we need only 4 equations to determine them uniquely. Also, we can replace the symmetry lines by insulation and utilize the mirror-image concept when writing the finite difference equations for the interior nodes.



Node 1 (interior) : $150 + 150 + 2T_2 - 4T_1 = 0$

Node 2 (interior) : $200 + T_1 + T_5 + T_4 - 4T_2 = 0$

Node 4 (interior) : $250 + 250 + T_2 + T_2 - 4T_4 = 0$

Node 5 (interior) : $4T_2 - 4T_5 = 0$

Solving the 4 equations above simultaneously gives

T_1 = **175°C**

T_2 = T_3 = **200°C**

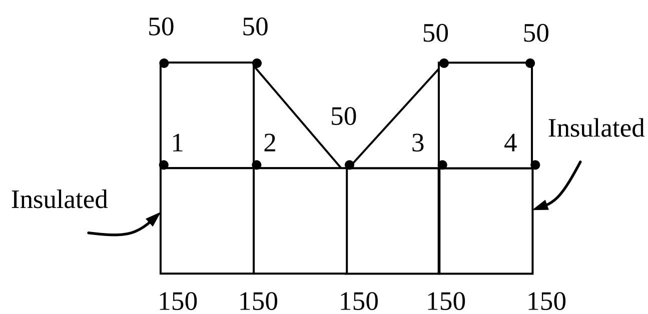
T_4 = T_6 = **225°C**

T_5 = **200°C**

(b) There is symmetry about a vertical line passing through the middle. Therefore, $T_3 = T_2$ and $T_4 = T_1$. Replacing the symmetry lines by insulation and utilizing the mirror-image concept, the finite difference equations for the interior nodes 1 and 2 are determined to be

Node 1 (interior) : $50 + 150 + 2T_2 - 4T_1 = 0$

Node 2 (interior) : $50 + 50 + 150 + T_1 - 4T_2 = 0$



Solving the 2 equations above simultaneously gives

T_1 = T_4 = **92.9°C,** T_2 = T_3 = **85.7°C**

Discussion Note that taking advantage of symmetry simplified the problem greatly.

5-78 The exposed surface of a long concrete damn of triangular cross-section is subjected to solar heat flux and convection and radiation heat transfer. The vertical section of the damn is subjected to convection with water. The temperatures at the top, middle, and bottom of the exposed surface of the damn are to be determined.

Assumptions **1** Heat transfer through the damn is given to be steady and two-dimensional. **2** There is no heat generation within the damn. **3** Heat transfer through the base is negligible. **4** Thermal properties and heat transfer coefficients are constant.

Properties The thermal conductivity and solar absorptivity are given to be k = 0.6 W/m·°C and α_s = 0.7.

Analysis The nodal spacing is given to be Δx=Δx=l=1 m, and all nodes are boundary nodes. Node 5 on the insulated boundary can be treated as an interior node for which $T_{\text{left}} + T_{\text{top}} + T_{\text{right}} + T_{\text{bottom}} - 4T_{\text{node}} = 0$. Using the energy balance approach and taking the direction of all heat transfer to be towards the node, the finite difference equations for the nodes are obtained to be as follows:

Node 1: $$h_i \frac{l}{2}(T_i - T_1) + k\frac{l}{2}\frac{T_2 - T_1}{l} + \frac{l/2}{\sin 45}\left[\alpha_s \dot{q}_s + h_0(T_0 - T_1)\right] = 0$$

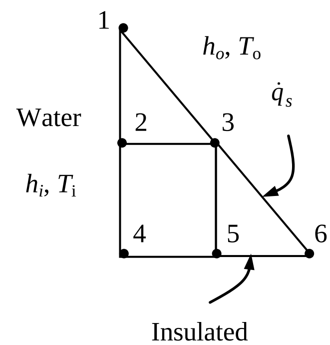
Node 2: $$h_i l(T_i - T_1) + k\frac{l}{2}\frac{T_1 - T_2}{l} + k\frac{l}{2}\frac{T_4 - T_2}{l} + kl\frac{T_3 - T_2}{l} = 0$$

Node 3: $$kl\frac{T_2 - T_3}{l} + kl\frac{T_5 - T_3}{l} + \frac{l}{\sin 45}\left[\alpha_s \dot{q}_s + h_0(T_0 - T_3)\right] = 0$$

Node 4: $$h_i \frac{l}{2}(T_i - T_4) + k\frac{l}{2}\frac{T_2 - T_4}{l} + k\frac{l}{2}\frac{T_5 - T_4}{l} = 0$$

Node 5: $$T_4 + 2T_3 + T_6 - 4T_5 = 0$$

Node 6: $$k\frac{l}{2}\frac{T_5 - T_6}{l} + \frac{l/2}{\sin 45}\left[\alpha_s \dot{q}_s + h_0(T_0 - T_6)\right] = 0$$



where

l = 1 m, k = 0.6 W/m·°C, h_i =150 W/m^2·°C, T_i =15°C, h_o = 30 W/m^2·°C, T_0 =25°C, α_s = 0.7, and $\dot{q}_s = 800$ W/m^2.

The system of 6 equations with 6 unknowns constitutes the finite difference formulation of the problem. The 6 nodal temperatures under steady conditions are determined by solving the 6 equations above simultaneously with an equation solver to be

$T_1 = T_{\text{top}}$ =**21.3**°C, T_2 =15.1°C, $T_3 = T_{\text{middle}}$ =**43.2**°C

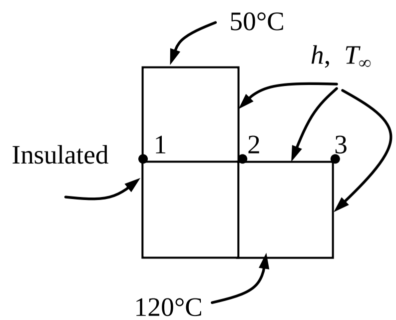
T_4 =15.1°C, T_5 =36.3°C, $T_6 = T_{\text{bottom}}$ =**43.6**°C

Discussion Note that the highest temperature occurs at a location furthest away from the water, as expected.

5-79 The top and bottom surfaces of an L-shaped long solid bar are maintained at specified temperatures while the left surface is insulated and the remaining 3 surfaces are subjected to convection. The finite difference formulation of the problem is to be obtained, and the unknown nodal temperatures are to be determined.

Assumptions **1** Heat transfer through the bar is given to be steady and two-dimensional. **2** There is no heat generation within the bar. **3** Thermal properties and heat transfer coefficients are constant. **4** Radiation heat transfer is negligible.

Properties The thermal conductivity is given to be k = 5 W/m·°C.



Analysis (a) The nodal spacing is given to be Δx=Δx=l=0.1 m, and all nodes are boundary nodes. Node 1 on the insulated boundary can be treated as an interior node for which $T_{\text{left}} + T_{\text{top}} + T_{\text{right}} + T_{\text{bottom}} - 4T_{\text{node}} = 0$. Using the energy balance approach and taking the direction of all heat transfer to be towards the node, the finite difference equations for the nodes are obtained to be as follows:

Node 1: $$50 + 120 + 2T_2 - 4T_1 = 0$$

Node 2: $$hl(T_\infty - T_2) + k\frac{l}{2}\frac{50 - T_2}{l} + k\frac{l}{2}\frac{T_3 - T_2}{l} + kl\frac{T_1 - T_2}{l} + kl\frac{120 - T_2}{l} = 0$$

Node 3: $$hl(T_\infty - T_3) + k\frac{l}{2}\frac{T_2 - T_3}{l} + k\frac{l}{2}\frac{120 - T_3}{l} = 0$$

where

l = 0.1 m, k = 5 W/m·°C, h = 40 W/m^2·°C, and T_∞ =25°C.

This system of 3 equations with 3 unknowns constitute the finite difference formulation of the problem.

(b) The 3 nodal temperatures under steady conditions are determined by solving the 3 equations above simultaneously with an equation solver to be

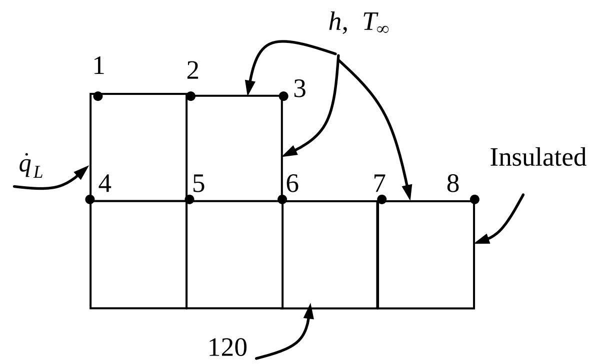
T_1 = **78.8°C,** T_2 = **72.7°C,** T_3 = **64.6°C**

5-80 Heat conduction through a long L-shaped solid bar with specified boundary conditions is considered. The unknown nodal temperatures are to be determined with the finite difference method.

Assumptions **1** Heat transfer through the body is given to be steady and two-dimensional. **2** Thermal conductivity is constant. **3** Heat generation is uniform.

Properties The thermal conductivity is given to be k = 45 W/m·°C.

Analysis (*a*) The nodal spacing is given to be Δx=Δx=l=0.015 m, and the general finite difference form of an interior node for steady two-dimensional heat conduction for the case of constant heat generation is expressed as



$$T_{\text{left}} + T_{\text{top}} + T_{\text{right}} + T_{\text{bottom}} - 4T_{\text{node}} + \frac{\dot{e}_{\text{node}} l^2}{k} = 0$$

We observe that all nodes are boundary nodes except node 5 that is an interior node. Therefore, we will have to rely on energy balances to obtain the finite difference equations. Using energy balances, the finite difference equations for each of the 8 nodes are obtained as follows:

Node 1: $$\dot{q}_L \frac{l}{2} + h\frac{l}{2}(T_\infty - T_1) + k\frac{l}{2}\frac{T_2 - T_1}{l} + k\frac{l}{2}\frac{T_4 - T_1}{l} + \dot{e}_0 \frac{l^2}{4} = 0$$

Node 2: $$hl(T_\infty - T_2) + k\frac{l}{2}\frac{T_1 - T_2}{l} + k\frac{l}{2}\frac{T_3 - T_2}{l} + kl\frac{T_5 - T_2}{l} + \dot{e}_0 \frac{l^2}{2} = 0$$

Node 3: $$hl(T_\infty - T_3) + k\frac{l}{2}\frac{T_2 - T_3}{l} + k\frac{l}{2}\frac{T_6 - T_3}{l} + \dot{e}_0 \frac{l^2}{4} = 0$$

Node 4: $$\dot{q}_L l + k\frac{l}{2}\frac{T_1 - T_4}{l} + k\frac{l}{2}\frac{120 - T_4}{l} + kl\frac{T_5 - T_4}{l} + \dot{e}_0 \frac{l^2}{2} = 0$$

Node 5: $$T_4 + T_2 + T_6 + 120 - 4T_5 + \frac{\dot{e}_0 l^2}{k} = 0$$

Node 6: $$hl(T_\infty - T_6) + k\frac{l}{2}\frac{T_3 - T_6}{l} + kl\frac{T_5 - T_6}{l} + kl\frac{120 - T_6}{l} + k\frac{l}{2}\frac{T_7 - T_6}{l} + \dot{e}_0 \frac{3l^2}{4} = 0$$

Node 7: $$hl(T_\infty - T_7) + k\frac{l}{2}\frac{T_6 - T_7}{l} + k\frac{l}{2}\frac{T_8 - T_7}{l} + kl\frac{120 - T_7}{l} + \dot{e}_0 \frac{l^2}{2} = 0$$

Node 8: $$h\frac{l}{2}(T_\infty - T_8) + k\frac{l}{2}\frac{T_7 - T_8}{l} + k\frac{l}{2}\frac{120 - T_8}{l} + \dot{e}_0 \frac{l^2}{4} = 0$$

where

$$\dot{e}_0 = 5\times 10^6 \text{ W/m}^3,\ \dot{q}_L = 8000 \text{ W/m}^2,\ l = 0.015 \text{ m},\ k = 45 \text{ W/m·°C},\ h = 55 \text{ W/m}^2\text{·°C, and } T_\infty = 30\text{°C}.$$

This system of 8 equations with 8 unknowns is the finite difference formulation of the problem.

(*b*) The 8 nodal temperatures under steady conditions are determined by solving the 8 equations above simultaneously with an equation solver to be

T_1 = **163.6°C**, T_2 = **160.5°C**, T_3 = **156.4°C**, T_4 = **154.0°C**, T_5 = **151.0°C**, T_6 = **144.4°C**,

T_7 = **134.5°C**, T_8 = **132.6°C**

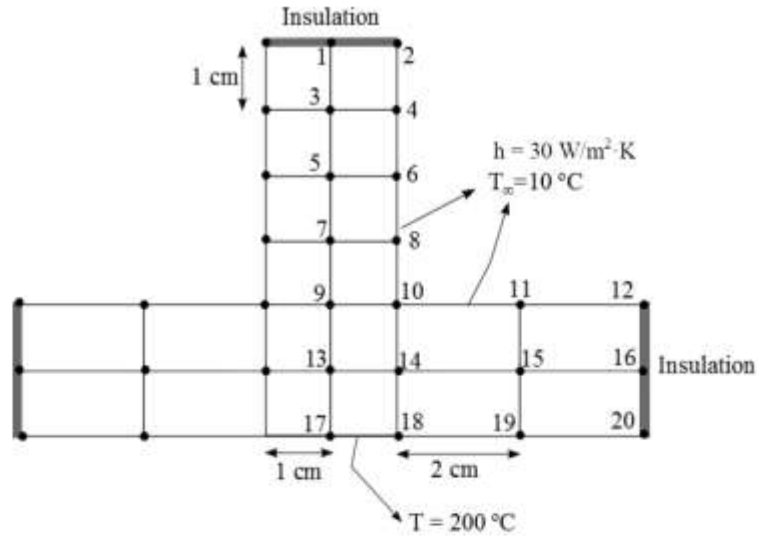
Discussion The accuracy of the solution can be improved by using more nodal points.

5-81 T shaped bar with known thermal properties is subjected to convection environment. Taking the advantage of symmetry develop finite difference formulation and determine the nodal temperatures.

Assumptions **1** Steady state two-dimensional heat conduction. **2** Constant thermal conductivities. 3 No internal heat generation.

Properties: The thermal conductivity of the T shaped bar is given as 28 W/m·K.

Analysis The finite difference formulation at all boundary nodes is done by doing energy balance at each node assuming all heat transfer entering the node.



Node 1: $k\frac{\Delta y}{2}\frac{2(T_2-T_1)}{\Delta x}+2k\Delta x\frac{(T_3-T_1)}{\Delta y}=0$

Node 2: $h\frac{\Delta y}{2}(T_\infty-T_2)+k\frac{\Delta y}{2}\frac{(T_1-T_2)}{\Delta x}+k\frac{\Delta x}{2}\frac{(T_4-T_2)}{\Delta y}=0$

Node 3: $2T_4+T_1+T_5-4T_3=0$

Node 4: $h\Delta y(T_\infty-T_4)+k\Delta y\frac{(T_3-T_4)}{\Delta x}+k\frac{\Delta x}{2}\frac{(T_2-T_4)}{\Delta y}+k\frac{\Delta x}{2}\frac{(T_6-T_4)}{\Delta y}=0$

Node 5: $2T_6+T_3+T_7-4T_5=0$

Node 6: $h\Delta y(T_\infty-T_6)+k\Delta y\frac{(T_5-T_6)}{\Delta x}+k\frac{\Delta x}{2}\frac{(T_4-T_6)}{\Delta y}+k\frac{\Delta x}{2}\frac{(T_8-T_6)}{\Delta y}=0$

Node 7: $2T_8+T_5+T_9-4T_7=0$

Node 8: $h\Delta y(T_\infty-T_8)+k\Delta y\frac{(T_7-T_8)}{\Delta x}+k\frac{\Delta x}{2}\frac{(T_6-T_8)}{\Delta y}+k\frac{\Delta x}{2}\frac{(T_{10}-T_8)}{\Delta y}=0$

Node 9: $2T_{10}+T_7+T_{13}-4T_9=0$

Node 10: $h\left(\frac{2\Delta x}{2}+\frac{\Delta y}{2}\right)(T_\infty-T_{10})+k\Delta y\frac{(T_9-T_{10})}{\Delta x}+k(1.5\Delta x)\frac{(T_{14}-T_{10})}{\Delta y}+k\frac{\Delta y}{2}\frac{(T_{11}-T_{10})}{2\Delta x}+k(1.5\Delta x)\frac{(T_8-T_{10})}{\Delta y}=0$

Node 11: $h(2\Delta x)(T_\infty-T_{11})+k\frac{\Delta y}{2}\frac{(T_{10}-T_{11})}{2\Delta x}+k\frac{\Delta y}{2}\frac{(T_{12}-T_{11})}{2\Delta x}+k(2\Delta x)\frac{(T_{15}-T_{11})}{\Delta y}=0$

Node 12: $h\Delta x(T_\infty-T_{12})+k\frac{\Delta y}{2}\frac{(T_{11}-T_{12})}{2\Delta x}+k\frac{2\Delta x}{2}\frac{(T_{16}-T_{12})}{\Delta y}=0$

Node 13: $2T_{14}+T_9+T_{17}-4T_{13}=0$

Node 14: $k\Delta y\frac{(T_{13}-T_{14})}{\Delta x}+k\Delta y\frac{(T_{15}-T_{14})}{2\Delta x}+k(1.5\Delta x)\frac{(T_{10}-T_{14})}{\Delta y}+k(1.5\Delta x)\frac{(T_{18}-T_{14})}{\Delta y}=0$

This equation for node 14 reduces to

$$T_{13}+0.5T_{15}+1.5T_{10}+1.5T_{18}-4.5T_{14}=0$$

Node 15: $$\frac{(T_{14}+T_{16}-2T_{15})}{(2\Delta x)^2}+\frac{(T_{11}+T_{19}-2T_{15})}{(\Delta y)^2}=0$$

This equation for node 15 reduces to

$$T_{14}+T_{16}+4T_{11}+4T_{19}-10T_{15}=0$$

Node 16: $$k\Delta y\frac{(T_{15}-T_{16})}{2\Delta x}+k\frac{2\Delta x}{2}\frac{(T_{12}-T_{16})}{\Delta y}+k\frac{2\Delta x}{2}\frac{(T_{20}-T_{16})}{\Delta y}=0$$

Solving these 16 equations simultaneously for 16 unknown nodal temperatures using EES or any other software gives

T_1 = **174.8°C**, T_2 = **173.7°C**, T_3 = **175.3°C**, T_4 = **174.4°C**, T_5 = **177.7°C**,

T_6 = **176.8°C**, T_7 = **181.8°C**, T_8 = **181°C**, T_9 = **187.5°C**, T_{10} = **187.2°C**,

T_{11} = **194.1°C**, T_{12} = **195.2°C**, T_{13} = **193.9°C**, T_{14} = **194°C**, T_{15} = **196.8°C**,

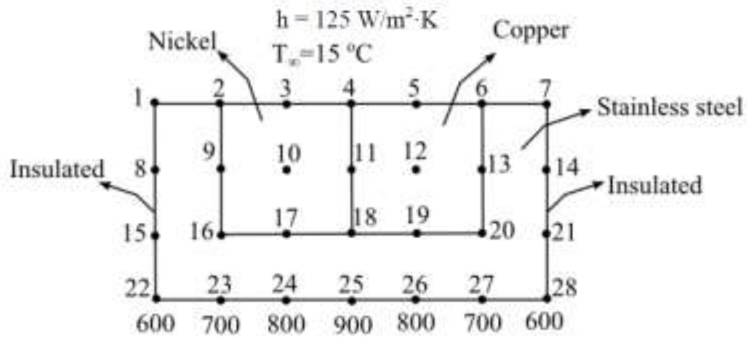
T_{16} = **197.4°C**, T_{17} = **200°C**, T_{18} = **200°C**, T_{19} = **200°C**, T_{20} = **200°C**.

5-82 Nickel and Copper are embedded into stainless steel material during a sintering process. For the given boundary conditions, the finite difference equations and the temperatures at different nodes are to be determined.

Assumptions **1** Two-dimensional steady state heat conduction with no heat generation. **2** Thermal conductivities for each material are constant. **3** Perfect contact at the interface.

Properties: The thermal conductivities of Nickel, Copper, and stainless steel are given as k = 90.7 W/m·K, k = 401 W/m·K, and k = 15.1 W/m·K, respectively.

Analysis The top surface is exposed to the convection while the two sides are insulated. The bottom surface is subjected to a non-uniform temperature. Nodes 9, 10, 11, 12, 13, 16, 17, 18, 19, 20 are the internal nodes. However due to different thermal conductivity of the three materials in use, Eq. (5-35) cannot be used for all internal nodes except node 10 and node 12. The nodes 8, 14, 15 and 21 at the two insulated sides can also be treated as the interior node using the mirror image concept at adiabatic boundary.



For all other nodes, the finite difference formulation can be obtained by doing an energy balance at the node volume and assuming all heat transfers entering the node volume. At node 18, the effect of thermal conductivity of stainless steel, nickel and copper on the heat transfer and hence the nodal temperature must be considered.

For the case of two-dimensional steady state heat conduction without internal heat generation, the finite difference formulations for different nodes are expressed as follows

Node 1: $$h\frac{\Delta x}{2}(T_\infty - T_1) + k_A\frac{\Delta y}{2}\frac{(T_2 - T_1)}{\Delta x} + k_A\frac{\Delta x}{2}\frac{(T_8 - T_1)}{\Delta y} = 0$$

Node 2: $$h\Delta x(T_\infty - T_2) + k_A\frac{\Delta y}{2}\frac{(T_1 - T_2)}{\Delta x} + k_B\frac{\Delta y}{2}\frac{(T_3 - T_2)}{\Delta x} + (k_A + k_B)\frac{\Delta x}{2}\frac{(T_9 - T_2)}{\Delta y} = 0$$

Node 3: $$h\Delta x(T_\infty - T_3) + k_B\frac{\Delta y}{2}\frac{(T_2 - T_3)}{\Delta x} + k_B\frac{\Delta y}{2}\frac{(T_4 - T_3)}{\Delta x} + k_B\Delta x\frac{(T_{10} - T_3)}{\Delta y} = 0$$

Node 4: $$h\Delta x(T_\infty - T_4) + k_B\frac{\Delta y}{2}\frac{(T_3 - T_4)}{\Delta x} + k_C\frac{\Delta y}{2}\frac{(T_5 - T_4)}{\Delta x} + (k_B + k_C)\frac{\Delta x}{2}\frac{(T_{11} - T_4)}{\Delta y} = 0$$

Node 5: $$h\Delta x(T_\infty - T_5) + k_C\frac{\Delta y}{2}\frac{(T_4 - T_5)}{\Delta x} + k_C\frac{\Delta y}{2}\frac{(T_6 - T_5)}{\Delta x} + k_C\Delta x\frac{(T_{12} - T_5)}{\Delta y} = 0$$

Node 6: $$h\Delta x(T_\infty - T_6) + k_A\frac{\Delta y}{2}\frac{(T_7 - T_6)}{\Delta x} + k_C\frac{\Delta y}{2}\frac{(T_5 - T_6)}{\Delta x} + (k_A + k_C)\frac{\Delta x}{2}\frac{(T_{13} - T_6)}{\Delta y} = 0$$

Node 7: $$h\frac{\Delta x}{2}(T_\infty - T_7) + k_A\frac{\Delta y}{2}\frac{(T_6 - T_7)}{\Delta x} + k_A\frac{\Delta x}{2}\frac{(T_{14} - T_7)}{\Delta y} = 0$$

Node 8: $$T_1 + T_{15} + 2T_9 - 4T_8 = 0$$

Node 9: $$k_A\Delta y\frac{(T_8 - T_9)}{\Delta x} + k_B\Delta y\frac{(T_{10} - T_9)}{\Delta x} + (k_A + k_B)\frac{\Delta x}{2}\frac{(T_2 - T_9)}{\Delta y} + (k_A + k_B)\frac{\Delta x}{2}\frac{(T_{16} - T_9)}{\Delta y} = 0$$

Node 10: $$T_3 + T_{17} + T_9 + T_{11} - 4T_{10} = 0$$

Node 11: $$k_B\Delta y\frac{(T_{10} - T_{11})}{\Delta x} + k_C\Delta y\frac{(T_{12} - T_{11})}{\Delta x} + (k_B + k_C)\frac{\Delta x}{2}\frac{(T_4 - T_{11})}{\Delta y} + (k_B + k_C)\frac{\Delta x}{2}\frac{(T_{18} - T_{11})}{\Delta y} = 0$$

Node 12: $$T_{11} + T_5 + T_{19} + T_{13} - 4T_{12} = 0$$

Node 13: $k_A \Delta y \frac{(T_{14} - T_{13})}{\Delta x} + k_C \Delta y \frac{(T_{12} - T_{13})}{\Delta x} + (k_A + k_C) \frac{\Delta x}{2} \frac{(T_6 - T_{13})}{\Delta y} + (k_A + k_C) \frac{\Delta x}{2} \frac{(T_{20} - T_{13})}{\Delta y} = 0$

Node 14: $T_7 + T_{21} + 2T_{13} - 4T_{14} = 0$

Node 15: $T_8 + T_{22} + 2T_{16} - 4T_{15} = 0$

Node 16: $k_A \Delta y \frac{(T_{15} - T_{16})}{\Delta x} + k_A \Delta x \frac{(T_{23} - T_{16})}{\Delta y} + (k_A + k_B) \frac{\Delta x}{2} \frac{(T_9 - T_{16})}{\Delta y} + (k_A + k_B) \frac{\Delta y}{2} \frac{(T_{17} - T_{16})}{\Delta x} = 0$

Node 17: $k_B \Delta x \frac{(T_{10} - T_{17})}{\Delta y} + k_A \Delta x \frac{(T_{24} - T_{17})}{\Delta y} + (k_A + k_B) \frac{\Delta y}{2} \frac{(T_{16} - T_{17})}{\Delta x} + (k_A + k_B) \frac{\Delta y}{2} \frac{(T_{18} - T_{17})}{\Delta x} = 0$

Node 18: $(k_A + k_B) \frac{\Delta y}{2} \frac{(T_{17} - T_{18})}{\Delta x} + (k_A + k_C) \frac{\Delta y}{2} \frac{(T_{19} - T_{18})}{\Delta x} + (k_B + k_C) \frac{\Delta x}{2} \frac{(T_{11} - T_{18})}{\Delta y} + k_A \Delta x \frac{(T_{25} - T_{18})}{\Delta y} = 0$

Node 19: $k_C \Delta x \frac{(T_{12} - T_{19})}{\Delta y} + k_A \Delta x \frac{(T_{26} - T_{19})}{\Delta y} + (k_A + k_C) \frac{\Delta y}{2} \frac{(T_{18} - T_{19})}{\Delta x} + (k_A + k_C) \frac{\Delta y}{2} \frac{(T_{20} - T_{19})}{\Delta x} = 0$

Node 20: $k_A \Delta y \frac{(T_{21} - T_{20})}{\Delta x} + k_A \Delta x \frac{(T_{27} - T_{20})}{\Delta y} + (k_A + k_C) \frac{\Delta x}{2} \frac{(T_{13} - T_{20})}{\Delta y} + (k_A + k_C) \frac{\Delta y}{2} \frac{(T_{19} - T_{20})}{\Delta x} = 0$

Node 21: $T_{14} + T_{28} + 2T_{20} - 4T_{21} = 0$

Solving these 21 equations simultaneously for 21 unknown nodal temperatures using EES or any other software we get,

T_1 = **605.4°C**, T_2 = **641.6°C**, T_3 = **652.9°C**, T_4 = **664.7°C**, T_5 = **665.7°C**, T_6 = **663.5°C**, T_7 = **620°C**,
T_8 = **642.4°C**, T_9 = **659.3°C**, T_{10} = **665.9°C**, T_{11} = **671°C**, T_{12} = **670.3°C**, T_{13} = **668.8°C**, T_{14} = **651.7°C**,
T_{15} = **645.8°C**, T_{16} = **670.5°C**, T_{17} = **680.2°C**, T_{18} = **680.5°C**, T_{19} = **675.7°C**, T_{20} = **672.4°C**, T_{21} = **649.1°C**.

5–83E The top and bottom surfaces of a V-grooved long solid bar are maintained at specified temperatures while the left and right surfaces are insulated. The temperature at the middle of the insulated surface is to be determined.

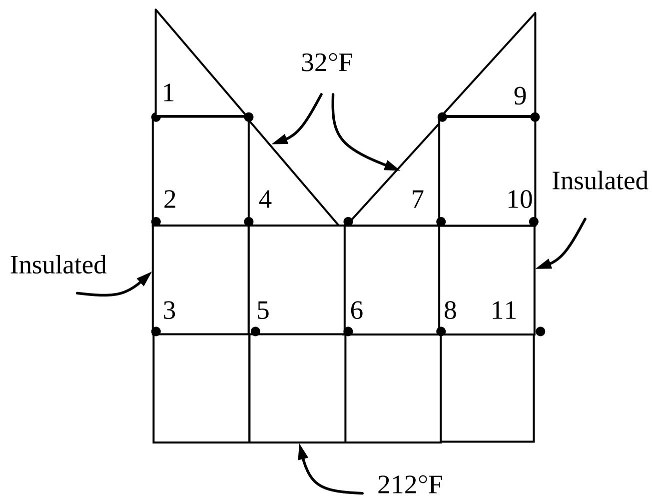
Assumptions 1 Heat transfer through the bar is given to be steady and two-dimensional. **2** There is no heat generation within the bar. **3** Thermal conductivity is constant.

Analysis The nodal spacing is given to be $\Delta x=\Delta y=l=1$ ft, and the general finite difference form of an interior node for steady two-dimensional heat conduction with no heat generation is expressed as

$$T_{\text{left}}+T_{\text{top}}+T_{\text{right}}+T_{\text{bottom}}-4T_{\text{node}}+\frac{\dot{e}_{\text{node}}l^2}{k}=0 \;\rightarrow\; T_{\text{left}}+T_{\text{top}}+T_{\text{right}}+T_{\text{bottom}}-4T_{\text{node}}=0$$

There is symmetry about the vertical plane passing through the center. Therefore, $T_1 = T_9$, $T_2 = T_{10}$, $T_3 = T_{11}$, $T_4 = T_7$, and $T_5 = T_8$. Therefore, there are only 6 unknown nodal temperatures, and thus we need only 6 equations to determine them uniquely. Also, we can replace the symmetry lines by insulation and utilize the mirror-image concept when writing the finite difference equations for the interior nodes.

The finite difference equations for boundary nodes are obtained by applying an energy balance on the volume elements and taking the direction of all heat transfers to be towards the node under consideration:



Node 1: $k\frac{l}{2}\frac{32-T_1}{l}+kl\frac{32-T_1}{l}+k\frac{l}{2}\frac{T_2-T_1}{l}=0$

(Note that k and l cancel out)

Node 2: $T_1+2T_4+T_3-4T_2=0$

Node 3: $T_2+212+2T_5-4T_3=0$

Node 4: $2\times 32+T_2+T_5-4T_4=0$

Node 5: $T_3+212+T_4+T_6-4T_5=0$

Node 6: $32+212+2T_5-4T_6=0$

The 6 nodal temperatures under steady conditions are determined by solving the 6 equations above simultaneously with an equation solver to be

T_1 = 44.7°F, T_2 =82.8°F, T_3 =143.4°F, T_4 = 71.6°F, T_5 =139.4°F, T_6 =130.7°F

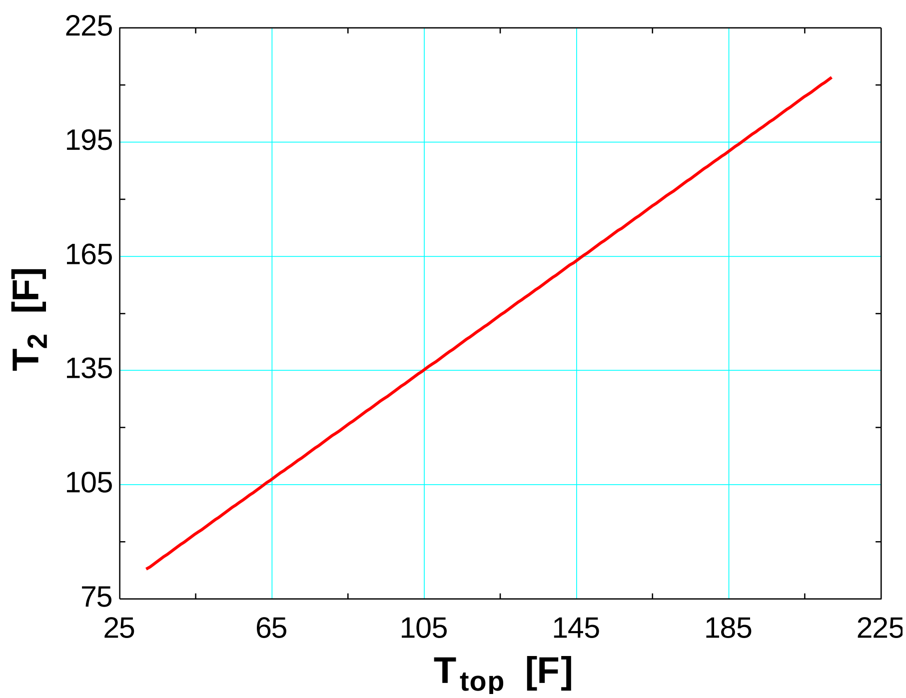
Therefore, the temperature at the middle of the insulated surface will be T_2 = **82.8°F.**

5–84E Prob. 5-83E is reconsidered. The effects of the temperatures at the top and bottom surfaces on the temperature in the middle of the insulated surface are to be investigated.

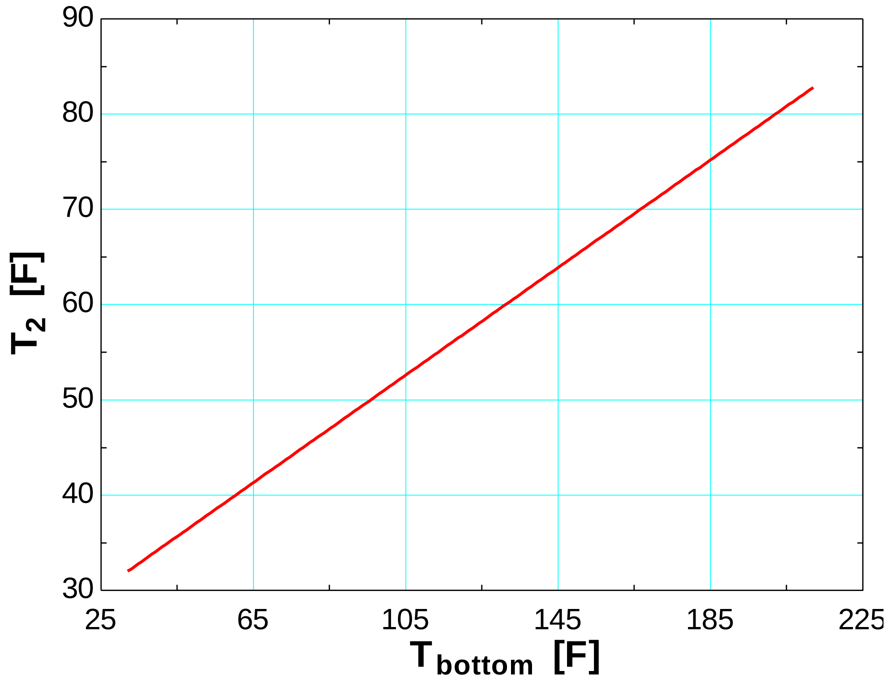
Analysis The problem is solved using EES, and the solution is given below.

```
"GIVEN"
T_top=32 [F]
T_bottom=212 [F]
DELTAx=1 [ft]
DELTAy=1 [ft]

"ANALYSIS"
l=DELTAx
T_1=T_9 "due to symmetry"
T_2=T_10 "due to symmetry"
T_3=T_11 "due to symmetry"
T_4=T_7 "due to symmetry"
T_5=T_8 "due to symmetry"
"Using the finite difference method, the six equations for the six unknown temperatures are determined to be"
"k*l/2*(T_top-T_1)/l+k*l*(T_top-T_1)/l+k*l/2*(T_2-T_1)/l=0 simplifies to for Node 1"
1/2*(T_top-T_1)+(T_top-T_1)+1/2*(T_2-T_1)=0 "Node 1"
T_1+2*T_4+T_3-4*T_2=0 "Node 2"
T_2+T_bottom+2*T_5-4*T_3=0 "Node 3"
2*T_top+T_2+T_5-4*T_4=0 "Node 4"
T_3+T_bottom+T_4+T_6-4*T_5=0 "Node 5"
T_top+T_bottom+2*T_5-4*T_6=0 "Node 6"
```

T_{top} [F]	T_2 [F]
32	82.81
41.47	89.61
50.95	96.41
60.42	103.2
69.89	110
79.37	116.8
88.84	123.6
98.32	130.4
107.8	137.2
117.3	144
126.7	150.8
136.2	157.6
145.7	164.4
155.2	171.2
164.6	178
174.1	184.8
183.6	191.6
193.1	198.4
202.5	205.2
212	212



T_{bottom} [F]	T_2 [F]
32	32
41.47	34.67
50.95	37.35
60.42	40.02
69.89	42.7
79.37	45.37
88.84	48.04
98.32	50.72
107.8	53.39
117.3	56.07
126.7	58.74
136.2	61.41
145.7	64.09
155.2	66.76
164.6	69.44
174.1	72.11
183.6	74.78
193.1	77.46
202.5	80.13
212	82.81



Transient Heat Conduction

5-85C The formulation of a transient heat conduction problem differs from that of a steady heat conduction problem in that the transient problem involves an *additional term* that represents the *change in the energy content* of the medium with time. This additional term $\rho A\Delta x c_p (T_m^{i+1} - T_m^i)/\Delta t$ represent the change in the internal energy content during Δt in the transient finite difference formulation.

5-86C The two basic methods of solution of transient problems based on finite differencing are the *explicit* and the *implicit methods*. The heat transfer terms are expressed at time step *i* in the explicit method, and at the future time step *i* + 1 in the implicit method as

Explicit method: $$\sum_{\text{All sides}} \dot{Q}^i + \dot{E}^{\text{i}}_{\text{gen,element}} = \rho V_{\text{element}} c_p \frac{T_m^{i+1} - T_m^i}{\Delta t}$$

Implicit method: $$\sum_{\text{All sides}} \dot{Q}^{i+1} + \dot{E}^{\text{i+1}}_{\text{gen,element}} = \rho V_{\text{element}} c_p \frac{T_m^{i+1} - T_m^i}{\Delta t}$$

5-87C The explicit finite difference formulation of a general interior node for transient heat conduction in a plane wall is given by $T_{m-1}^i - 2T_m^i + T_{m+1}^i + \dfrac{\dot{e}_m^i \Delta x^2}{k} = \dfrac{T_m^{i+1} - T_m^i}{\tau}$. The finite difference formulation for the steady case is obtained by simply setting $T_m^{i+1} = T_m^i$ and disregarding the time index *i*. It yields

$$T_{m-1} - 2T_m + T_{m+1} + \frac{\dot{e}_m \Delta x^2}{k} = 0$$

5-88C For transient one-dimensional heat conduction in a plane wall with both sides of the wall at specified temperatures, the stability criteria for the explicit method can be expressed in its simplest form as

$$\tau = \frac{\alpha \Delta t}{(\Delta x)^2} \le \frac{1}{2}$$

5-89C For transient one-dimensional heat conduction in a plane wall with specified heat flux on both sides, the stability criteria for the explicit method can be expressed in its simplest form as

$$\tau = \frac{\alpha \Delta t}{(\Delta x)^2} \le \frac{1}{2}$$

which is identical to the one for the interior nodes. This is because the heat flux boundary conditions have no effect on the stability criteria.

5-90C The explicit finite difference formulation of a general interior node for transient two-dimensional heat conduction is given by $T_{\text{node}}^{i+1} = \tau(T_{\text{left}}^{\text{i}} + T_{\text{top}}^{\text{i}} + T_{\text{right}}^{\text{i}} + T_{\text{bottom}}^{\text{i}}) + (1-4\tau)T_{\text{node}}^{\text{i}} + \tau\dfrac{\dot{e}_{\text{node}}^{\text{i}} l^2}{k}$. The finite difference formulation for the steady case is obtained by simply setting $T_m^{i+1} = T_m^i$ and disregarding the time index *i*. It yields

$$T_{\text{left}} + T_{\text{top}} + T_{\text{right}} + T_{\text{bottom}} - 4T_{\text{node}} + \frac{\dot{e}_{\text{node}} l^2}{k} = 0$$

5-91C There is a limitation on the size of the time step Δt in the solution of transient heat conduction problems using the explicit method, but there is no such limitation in the implicit method.

5-92C The general stability criteria for the explicit method of solution of transient heat conduction problems is expressed as follows: *The coefficients of all* T_m^i *in the* T_m^{i+1} *expressions* (*called the primary coefficient*) *in the simplified expressions must be greater than or equal to zero for all nodes m.*

5-93C For transient two-dimensional heat conduction in a rectangular region with insulation or specified temperature boundary conditions, the stability criteria for the explicit method can be expressed in its simplest form as

$$\tau = \frac{\alpha \Delta t}{(\Delta x)^2} \le \frac{1}{4}$$

which is identical to the one for the interior nodes. This is because the insulation or specified temperature boundary conditions have no effect on the stability criteria.

5-94C The implicit method is unconditionally stable and thus any value of time step Δt can be used in the solution of transient heat conduction problems since there is no danger of unstability. However, using a very large value of Δt is equivalent to replacing the time derivative by a very large difference, and thus the solution will not be accurate. Therefore, we should still use the smallest time step practical to minimize the numerical error.

5-95 Starting with an energy balance on a volume element, the two-dimensional transient *explicit* finite difference equation for a general interior node in rectangular coordinates for $T(x, y, t)$ for the case of constant thermal conductivity and no heat generation is to be obtained.

Analysis (See Figure 5-49 in the text). We consider a rectangular region in which heat conduction is significant in the x and y directions, and consider a unit depth of $\Delta z = 1$ in the z direction. There is no heat generation in the medium, and the thermal conductivity k of the medium is constant. Now we divide the x-y plane of the region into a *rectangular mesh* of nodal points which are spaced Δx and Δy apart in the x and y directions, respectively, and consider a general interior node (m, n) whose coordinates are $x = m\Delta x$ and $y = n\Delta y$. Noting that the volume element centered about the general interior node (m, n) involves heat conduction from four sides (right, left, top, and bottom) and expressing them at previous time step i, the transient explicit finite difference formulation for a general interior node can be expressed as

$$k(\Delta y \times 1)\frac{T^i_{m-1,n} - T^i_{m,n}}{\Delta x} + k(\Delta x \times 1)\frac{T^i_{m,n+1} - T^i_{m,n}}{\Delta y} + k(\Delta y \times 1)\frac{T^i_{m+1,n} - T^i_{m,n}}{\Delta x} + k(\Delta x \times 1)\frac{T^i_{m,n-1} - T^i_{m,n}}{\Delta y}$$
$$= \rho(\Delta x \times \Delta y \times 1)c_p \frac{T^{i+1}_{m,n} - T^i_{m,n}}{\Delta t}$$

Taking a square mesh ($\Delta x = \Delta y = l$) and dividing each term by k gives, after simplifying,

$$T^i_{m-1,n} + T^i_{m+1,n} + T^i_{m,n+1} + T^i_{m,n-1} - 4T^i_{m,n} = \frac{T^{i+1}_{m,n} - T^i_{m,n}}{\tau}$$

where $\alpha = k / \rho c_p$ is the thermal diffusivity of the material and $\tau = \alpha \Delta t / l^2$ is the dimensionless mesh Fourier number. It can also be expressed in terms of the temperatures at the neighboring nodes in the following easy-to-remember form:

$$T^i_{\text{left}} + T^i_{\text{top}} + T^i_{\text{right}} + T^i_{\text{bottom}} - 4T^i_{\text{node}} = \frac{T^{i+1}_{\text{node}} - T^i_{\text{node}}}{\tau}$$

Discussion We note that setting $T^{i+1}_{\text{node}} = T^i_{\text{node}}$ gives the steady finite difference formulation.

5-96 Starting with an energy balance on a volume element, the two-dimensional transient *implicit* finite difference equation for a general interior node in rectangular coordinates for $T(x, y, t)$ for the case of constant thermal conductivity and no heat generation is to be obtained.

Analysis (See Figure 5-49 in the text). We consider a rectangular region in which heat conduction is significant in the x and y directions, and consider a unit depth of Δz = 1 in the z direction. There is no heat generation in the medium, and the thermal conductivity k of the medium is constant. Now we divide the x-y plane of the region into a *rectangular mesh* of nodal points which are spaced Δx and Δy apart in the x and y directions, respectively, and consider a general interior node (m, n) whose coordinates are $x = m\Delta x$ and $y = n\Delta y$. Noting that the volume element centered about the general interior node (m, n) involves heat conduction from four sides (right, left, top, and bottom) and expressing them at previous time step i, the transient *implicit* finite difference formulation for a general interior node can be expressed as

$$k(\Delta y \times 1)\frac{T_{m-1,n}^{i+1} - T_{m,n}^{i+1}}{\Delta x} + k(\Delta x \times 1)\frac{T_{m,n+1}^{i+1} - T_{m,n}^{i+1}}{\Delta y} + k(\Delta y \times 1)\frac{T_{m+1,n}^{i+1} - T_{m,n}^{i}}{\Delta x} + k(\Delta x \times 1)\frac{T_{m,n-1}^{i+1} - T_{m,n}^{i+1}}{\Delta y}$$

$$= \rho(\Delta x \times \Delta y \times 1)c_p \frac{T_{m,n}^{i+1} - T_{m,n}^{i}}{\Delta t}$$

Taking a square mesh ($\Delta x = \Delta y = l$) and dividing each term by k gives, after simplifying,

$$T_{m-1,n}^{i+1} + T_{m+1,n}^{i+1} + T_{m,n+1}^{i+1} + T_{m,n-1}^{i+1} - 4T_{m,n}^{i+1} = \frac{T_m^{i+1} - T_m^{i}}{\tau}$$

where $\alpha = k / \rho c_p$ is the thermal diffusivity of the material and $\tau = \alpha \Delta t / l^2$ is the dimensionless mesh Fourier number. It can also be expressed in terms of the temperatures at the neighboring nodes in the following easy-to-remember form:

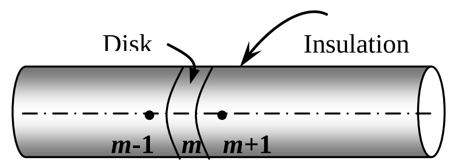
$$T_{\text{left}}^{i+1} + T_{\text{top}}^{i+1} + T_{\text{right}}^{i+1} + T_{\text{bottom}}^{i+1} - 4T_{\text{node}}^{i+1} = \frac{T_{\text{node}}^{i+1} - T_{\text{node}}^{i}}{\tau}$$

Discussion We note that setting $T_{\text{node}}^{i+1} = T_{\text{node}}^{i}$ gives the steady finite difference formulation.

5-97 Starting with an energy balance on a disk volume element, the one-dimensional transient explicit finite difference equation for a general interior node for $T(z,t)$ in a cylinder whose side surface is insulated for the case of constant thermal conductivity with uniform heat generation is to be obtained.

Analysis We consider transient one-dimensional heat conduction in the axial *z* direction in an insulated cylindrical rod of constant cross-sectional area *A* with constant heat generation $\dot{g}_0$ and constant conductivity *k* with a mesh size of Δz in the *z* direction. Noting that the volume element of a general interior node *m* involves heat conduction from two sides and the volume of the element is $V_{\text{element}} = A\Delta z$, the transient explicit finite difference formulation for an interior node can be expressed as

$$kA\frac{T_{m-1}^i - T_m^i}{\Delta x} + kA\frac{T_{m+1}^i - T_m^i}{\Delta x} + \dot{e}_0 A\Delta x = \rho A\Delta x c_p \frac{T_m^{i+1} - T_m^i}{\Delta t}$$



Canceling the surface area *A* and multiplying by *Δx*/*k*, it simplifies to

$$T_{m-1}^i - 2T_m^i + T_{m+1}^i + \frac{\dot{e}_0 \Delta x^2}{k} = \frac{(\Delta x)^2}{\alpha \Delta t}(T_m^{i+1} - T_m^i)$$

where $\alpha = k/\rho c_p$ is the *thermal diffusivity* of the wall material.

Using the definition of the dimensionless *mesh Fourier number* $\tau = \dfrac{\alpha \Delta t}{\Delta x^2}$, the last equation reduces to

$$T_{m-1}^i - 2T_m^i + T_{m+1}^i + \frac{\dot{e}_0 \Delta x^2}{k} = \frac{T_m^{i+1} - T_m^i}{\tau}$$

Discussion We note that setting $T_{\text{m}}^{i+1} = T_{\text{m}}^i$ gives the steady finite difference formulation.

5-98 A plane wall with no heat generation is subjected to specified temperature at the left (node 0) and heat flux at the right boundary (node 6). The explicit transient finite difference formulation of the boundary nodes and the finite difference formulation for the total amount of heat transfer at the left boundary during the first 3 time steps are to be determined.

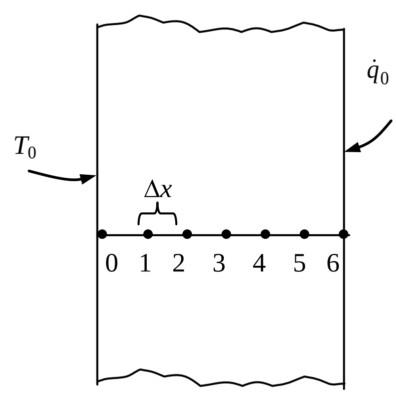
Assumptions **1** Heat transfer through the wall is given to be transient, and the thermal conductivity to be constant. **2** Heat transfer is one-dimensional since the plate is large relative to its thickness. **3** There is no heat generation in the medium.

Analysis Using the energy balance approach and taking the direction of all heat transfers to be towards the node under consideration, the *explicit* finite difference formulations become

Left boundary node: $T_0^i = T_0 = 50°\text{C}$

Right boundary node: $$k\frac{T_5^i - T_6^i}{\Delta x} + \dot{q}_0 = \rho\frac{\Delta x}{2}c_p\frac{T_6^{i+1} - T_6^i}{\Delta t}$$

Heat transfer at left surface: $$\dot{Q}_{\text{left surface}}^i + kA\frac{T_1^i - T_0}{\Delta x} = \rho A\frac{\Delta x}{2}c_p\frac{T_6^{i+1} - T_6^i}{\Delta t}$$



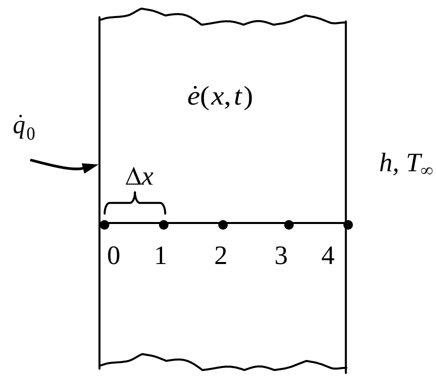
Noting that $Q = \dot{Q}\Delta t = \sum_i \dot{Q}^i \Delta t$, the total amount of heat transfer becomes

$$Q_{\text{left surface}} = \sum_{i=1}^{3}\dot{Q}_{\text{left surface}}^i \Delta t = \sum_{i=1}^{3}\left(kA\frac{T_0 - T_1^i}{\Delta x} + \rho A\frac{\Delta x}{2}c_p\frac{T_0^{i+1} - T_0^i}{\Delta t}\right)\Delta t$$

5-99 A plane wall with variable heat generation and constant thermal conductivity is subjected to uniform heat flux $\dot{q}_0$ at the left (node 0) and convection at the right boundary (node 4). The explicit transient finite difference formulation of the boundary nodes is to be determined.

Assumptions **1** Heat transfer through the wall is given to be transient, and the thermal conductivity to be constant. **2** Heat transfer is one-dimensional since the wall is large relative to its thickness. **3** Radiation heat transfer is negligible.

Analysis Using the energy balance approach and taking the direction of all heat transfers to be towards the node under consideration, the *explicit* finite difference formulations become



Left boundary node:

$$kA\frac{T_1^i - T_0^i}{\Delta x} + \dot{q}_0 A + \dot{e}_0^i (A\Delta x/2) = \rho A \frac{\Delta x}{2} c_p \frac{T_0^{i+1} - T_0^i}{\Delta t}$$

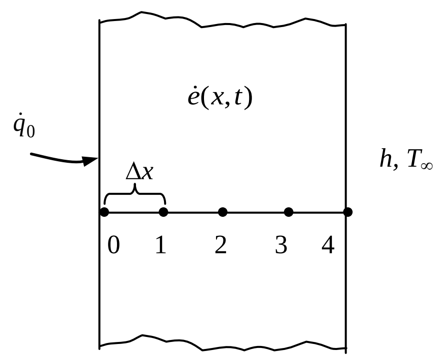
Right boundary node:

$$kA\frac{T_3^i - T_4^i}{\Delta x} + hA(T_\infty^i - T_4^i) + \dot{e}_4^i (A\Delta x/2) = \rho A \frac{\Delta x}{2} c_p \frac{T_4^{i+1} - T_4^i}{\Delta t}$$

5-100 A plane wall with variable heat generation and constant thermal conductivity is subjected to uniform heat flux $\dot{q}_0$ at the left (node 0) and convection at the right boundary (node 4). The implicit transient finite difference formulation of the boundary nodes is to be determined.

Assumptions **1** Heat transfer through the wall is given to be transient, and the thermal conductivity to be constant. **2** Heat transfer is one-dimensional since the wall is large relative to its thickness. **3** Radiation heat transfer is negligible.

Analysis Using the energy balance approach and taking the direction of all heat transfers to be towards the node under consideration, the *implicit* finite difference formulations become



Left boundary node:

$$kA\frac{T_1^{i+1} - T_0^{i+1}}{\Delta x} + \dot{q}_0 A + \dot{e}_0^{i+1} (A\Delta x/2) = \rho A \frac{\Delta x}{2} c_p \frac{T_0^{i+1} - T_0^i}{\Delta t}$$

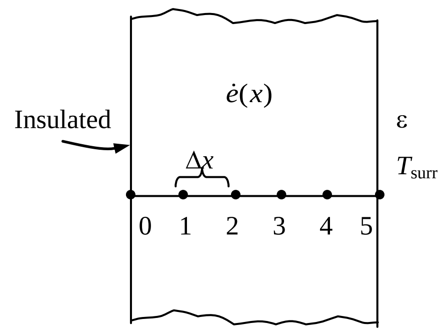
Right boundary node:

$$kA\frac{T_3^{i+1} - T_4^{i+1}}{\Delta x} + hA(T_\infty^{i+1} - T_4^{i+1}) + \dot{e}_4^{i+1} (A\Delta x/2) = \rho A \frac{\Delta x}{2} c_p \frac{T_4^{i+1} - T_4^i}{\Delta t}$$

5-101 A plane wall with variable heat generation and constant thermal conductivity is subjected to insulation at the left (node 0) and radiation at the right boundary (node 5). The explicit transient finite difference formulation of the boundary nodes is to be determined.

Assumptions **1** Heat transfer through the wall is given to be transient and one-dimensional, and the thermal conductivity to be constant. **2** Convection heat transfer is negligible.

Analysis Using the energy balance approach and taking the direction of all heat transfers to be towards the node under consideration, the *explicit* transient finite difference formulations become



Left boundary node:

$$kA\frac{T_1^i - T_0^i}{\Delta x} + \dot{e}_0^i A\frac{\Delta x}{2} = \rho A\frac{\Delta x}{2} c_p \frac{T_0^{i+1} - T_0^i}{\Delta t}$$

Right boundary node:

$$\varepsilon\sigma A[(T_{\text{surr}}^{\text{i}})^4 - (T_5^i)^4] + kA\frac{T_4^i - T_5^i}{\Delta x} + \dot{e}_5^i A\frac{\Delta x}{2} = \rho A\frac{\Delta x}{2} c_p \frac{T_5^{i+1} - T_5^i}{\Delta t}$$

5-102 A plane wall with variable heat generation and constant thermal conductivity is subjected to combined convection, radiation, and heat flux at the left (node 0) and specified temperature at the right boundary (node 4). The explicit finite difference formulation of the left boundary and the finite difference formulation for the total amount of heat transfer at the right boundary are to be determined.

Assumptions **1** Heat transfer through the wall is given to be transient and one-dimensional, and the thermal conductivity to be constant.

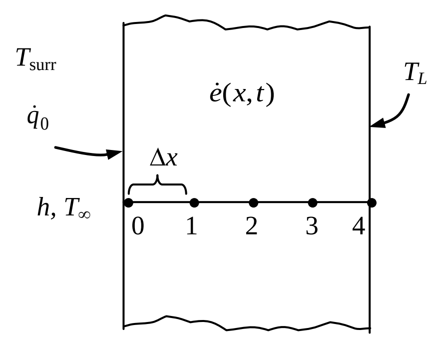
Analysis Using the energy balance approach and taking the direction of all heat transfers to be towards the node under consideration, the *explicit* transient finite difference formulations become

Left boundary node:

$$\dot{q}_0 A + \varepsilon\sigma A[T_{\text{surr}}^4 - (T_0^i)^4] + hA(T_\infty^i - T_0^i) + kA\frac{T_1^i - T_0^i}{\Delta x} + \dot{e}_0^i A\frac{\Delta x}{2} = \rho A\frac{\Delta x}{2} c_p \frac{T_0^{i+1} - T_0^i}{\Delta t}$$

Heat transfer at right surface:

$$\dot{Q}_{\text{right surface}}^i + kA\frac{T_3^i - T_4^i}{\Delta x} + \dot{e}_4^i A\frac{\Delta x}{2} = \rho A\frac{\Delta x}{2} c_p \frac{T_4^{i+1} - T_4^i}{\Delta t}$$



Noting that

$$Q = \dot{Q}\Delta t = \sum_i \dot{Q}^i \Delta t$$

the total amount of heat transfer becomes

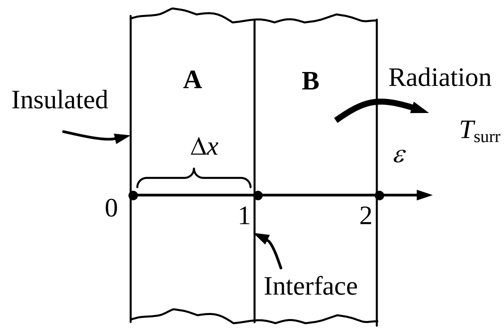
$$Q_{\text{right surface}} = \sum_{i=1}^{20} \dot{Q}_{\text{right surface}}^i \Delta t$$

$$= \sum_{i=1}^{20}\left(kA\frac{T_4^i - T_3^i}{\Delta x} - \dot{e}_4^i A\frac{\Delta x}{2} + \rho A\frac{\Delta x}{2} c_p \frac{T_4^{i+1} - T_4^i}{\Delta t}\right)\Delta t$$

5-103 A composite plane wall consists of two layers A and B in perfect contact at the interface where node 1 is at the interface. The wall is insulated at the left (node 0) and subjected to radiation at the right boundary (node 2). The complete transient explicit finite difference formulation of this problem is to be obtained.

Assumptions **1** Heat transfer through the wall is given to be transient and one-dimensional, and the thermal conductivity to be constant. **2** Convection heat transfer is negligible. **3** There is no heat generation.

Analysis Using the energy balance approach with a unit area A = 1 and taking the direction of all heat transfers to be towards the node under consideration, the finite difference formulations become



Node 0 (at left boundary):

$$k_A \frac{T_1^i - T_0^i}{\Delta x} = \rho_A \frac{\Delta x}{2} c_{p,A} \frac{T_0^{i+1} - T_0^i}{\Delta t}$$

Node 1 (at interface):

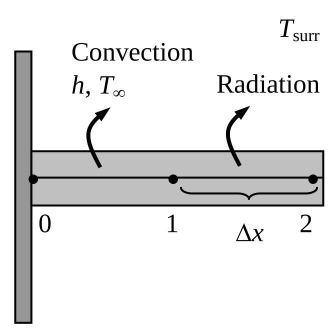
$$k_A \frac{T_0^i - T_1^i}{\Delta x} + k_B \frac{T_2^i - T_1^i}{\Delta x} = \left(\rho_A \frac{\Delta x}{2} c_{p,A} + \rho_B \frac{\Delta x}{2} c_B\right) \frac{T_1^{i+1} - T_1^i}{\Delta t}$$

Node 2 (at right boundary):

$$\varepsilon\sigma[T_{\text{surr}}^4 - (T_2^i)^4] + k_B \frac{T_1^i - T_2^i}{\Delta x} = \rho_B \frac{\Delta x}{2} c_{p,B} \frac{T_2^{i+1} - T_2^i}{\Delta t}$$

5-104 A pin fin with negligible heat transfer from its tip is considered. The complete explicit finite difference formulation for the determination of nodal temperatures is to be obtained.

Assumptions **1** Heat transfer through the pin fin is given to be steady and one-dimensional, and the thermal conductivity to be constant. **2** Convection heat transfer coefficient is constant and uniform. **3** Heat loss from the fin tip is given to be negligible.



Analysis The nodal network consists of 3 nodes, and the base temperature T_0 at node 0 is specified. Therefore, there are two unknowns T_1 and T_2, and we need two equations to determine them. Using the energy balance approach and taking the direction of all heat transfers to be towards the node under consideration, the explicit transient finite difference formulations become

Node 1 (at midpoint):

$$\varepsilon\sigma p\Delta x[T_{\text{surr}}^4 - (T_1^i)^4] + hp\Delta x(T_\infty - T_1^i) + kA\frac{T_2^i - T_1^i}{\Delta x} + kA\frac{T_0^i - T_1^i}{\Delta x} = \rho A\Delta x c_p \frac{T_1^{i+1} - T_1^i}{\Delta t}$$

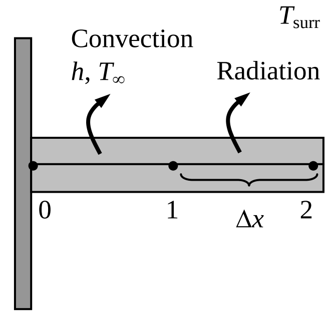
Node 2 (at fin tip):

$$\varepsilon\sigma\left(p\frac{\Delta x}{2}\right)[T_{\text{surr}}^4 - (T_2^i)^4] + h\left(p\frac{\Delta x}{2}\right)(T_\infty - T_2^i) + kA\frac{T_1^i - T_2^i}{\Delta x} = \rho A\frac{\Delta x}{2} c_p \frac{T_2^{i+1} - T_2^i}{\Delta t}$$

where $A = \pi D^2/4$ is the cross-sectional area and $p = \pi D$ is the perimeter of the fin.

5-105 A pin fin with negligible heat transfer from its tip is considered. The complete implicit finite difference formulation for the determination of nodal temperatures is to be obtained.

Assumptions **1** Heat transfer through the pin fin is given to be steady and one-dimensional, and the thermal conductivity to be constant. **2** Convection heat transfer coefficient is constant and uniform. **3** Heat loss from the fin tip is given to be negligible.



Analysis The nodal network consists of 3 nodes, and the base temperature T_0 at node 0 is specified. Therefore, there are two unknowns T_1 and T_2, and we need two equations to determine them. Using the energy balance approach and taking the direction of all heat transfers to be towards the node under consideration, the implicit transient finite difference formulations become

Node 1: $$\varepsilon\sigma p\Delta x[T_{surr}^4 - (T_1^{i+1})^4] + hp\Delta x(T_\infty - T_1^{i+1}) + kA\frac{T_2^{i+1} - T_1^{i+1}}{\Delta x} + kA\frac{T_0^{i+1} - T_1^{i+1}}{\Delta x} = \rho A\Delta x c_p \frac{T_1^{i+1} - T_1^i}{\Delta t}$$

Node 2: $$\varepsilon\sigma\left(p\frac{\Delta x}{2}\right)[T_{surr}^4 - (T_2^{i+1})^4] + h\left(p\frac{\Delta x}{2}\right)(T_\infty - T_2^{i+1}) + kA\frac{T_1^{i+1} - T_2^{i+1}}{\Delta x} = \rho A\frac{\Delta x}{2}c_p\frac{T_2^{i+1} - T_2^i}{\Delta t}$$

where $A = \pi D^2 / 4$ is the cross-sectional area and $p = \pi D$ is the perimeter of the fin.

5-106 A hot brass plate is having its upper surface cooled by impinging jet while its lower surface is insulated. The implicit finite difference equations and the nodal temperatures of the brass plate after 10 seconds of cooling are to be determined.

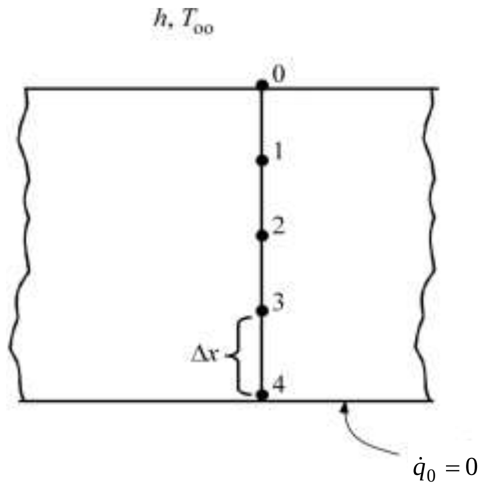
Assumptions **1** Transient heat conduction is one-dimensional. **2** Thermal properties are constant. **3** Convection heat transfer coefficient is uniform. **4** Heat transfer by radiation is negligible. **5** There is no heat generation.

Properties The properties of the brass plate are given as $\rho = 8530$ kg/m^3, $c_p = 380$ J/kg·K, $k = 110$ W/m·K, and $\alpha = 33.9 \times 10^{-6}$ m^2/s.

Analysis The nodal spacing is given to be $\Delta x = 2.5$ cm. Then the number of nodes becomes $M = L/\Delta x + 1 = 10/2.5 + 1 = 5$. This problem involves 5 unknown nodal temperatures, and thus we need to have 5 equations. The finite difference equation for node 0 on the top surface subjected to convection is obtained by applying an energy balance on the half volume element about node 0 and taking the direction of all heat transfers to be towards the node under consideration:

$$h(T_\infty - T_0^{i+1}) + k\frac{T_1^{i+1} - T_0^{i+1}}{\Delta x} = \rho\frac{\Delta x}{2}c_p\frac{T_0^{i+1} - T_0^i}{\Delta t}$$

or $$-\left(1 + 2\tau + 2\frac{h\Delta x}{k}\tau\right)T_0^{i+1} + 2\tau T_1^{i+1} + T_0^i + 2\frac{h\Delta x}{k}\tau T_\infty = 0$$



Node 4 is on insulated boundary, and thus we can treat it as an interior node by using the mirror image concept. Nodes 1, 2, and 3 are interior nodes, and thus for them we can use the general explicit finite difference relation expressed as

$$T_{m-1}^{i+1} - 2T_m^{i+1} + T_{m+1}^{i+1} = \frac{T_m^{i+1} - T_m^i}{\tau}$$

or $$\tau T_{m-1}^{i+1} - (1+2\tau)T_m^{i+1} + \tau T_{m+1}^{i+1} + T_m^i = 0$$

Thus, the explicit finite difference equations are

Node 0: $$-\left(1 + 2\tau + 2\frac{h\Delta x}{k}\tau\right)T_0^{i+1} + 2\tau T_1^{i+1} + T_0^i + 2\frac{h\Delta x}{k}\tau T_\infty = 0$$

Node 1: $$\tau T_0^{i+1} - (1+2\tau)T_1^{i+1} + \tau T_2^{i+1} + T_1^i = 0$$

Node 2: $$\tau T_1^{i+1} - (1+2\tau)T_2^{i+1} + \tau T_3^{i+1} + T_2^i = 0$$

Node 3: $$\tau T_2^{i+1} - (1+2\tau)T_3^{i+1} + \tau T_4^{i+1} + T_3^i = 0$$

Node 4: $$\tau T_3^{i+1} - (1+2\tau)T_4^{i+1} + \tau T_3^{i+1} + T_4^i = 0$$

where $\Delta x = 2.5$ cm, $k = 110$ W/m·K, $h = 220$ W/m^2·K, $T_\infty = 15$°C, $\alpha = 33.9 \times 10^{-6}$ m^2/s, and $h\Delta x/k = 0.05$. For time step of Δt = 10 s. Then the mesh Fourier number becomes

$$\tau = \frac{\alpha\Delta t}{\Delta x^2} = \frac{(33.9\times10^{-6}\text{ m}^2/\text{s})(10\text{ s})}{(0.025\text{ m})^2} = 0.5424 \qquad (\text{for } \Delta t = 10\text{ s})$$

(*b*) The nodal temperatures of the brass plate after 10 seconds of cooling can be determined by solving the 5 equations above simultaneously with an equation solver. Copy the following lines and paste on a blank EES screen to solve the above equations:

```
-(1+2*0.5424+2*0.05*0.5424)*T_0+2*0.5424*T_1+650+2*0.05*0.5424*15=0
0.5424*T_0-(1+2*0.5424)*T_1+0.5424*T_2+650=0
0.5424*T_1-(1+2*0.5424)*T_2+0.5424*T_3+650=0
0.5424*T_2-(1+2*0.5424)*T_3+0.5424*T_4+650=0
0.5424*T_3-(1+2*0.5424)*T_4+0.5424*T_3+650=0
```

Solving by EES software, we get the same results:

T_0 = **631.2°C**, T_1 = **644.7°C**, T_2 = **648.5°C**, T_3 = **649.6°C**, T_4 = **649.8°C**

Discussion Unlike the explicit method, the implicit method does not require any stability criterion, and the solution will converge with large values of time step. However, the large time step tends to give less accurate the results.

5-107 A uranium plate initially at a uniform temperature is subjected to insulation on one side and convection on the other. The transient finite difference formulation of this problem is to be obtained, and the nodal temperatures after 5 min and under steady conditions are to be determined.

Assumptions **1** Heat transfer is one-dimensional since the plate is large relative to its thickness. **2** Thermal conductivity is constant. **3** Radiation heat transfer is negligible.

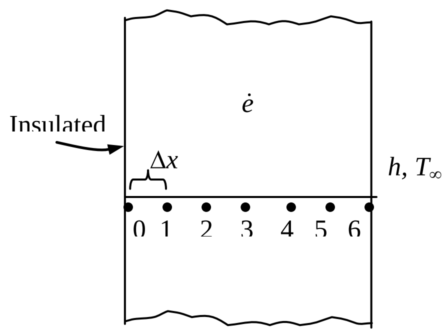
Properties The conductivity and diffusivity are given to be k = 28 W/m·°C and $\alpha = 12.5\times10^{-6}\ \text{m}^2/\text{s}$.

Analysis The nodal spacing is given to be Δx = 0.015 m. Then the number of nodes becomes $M = L/\Delta x + 1$ = 0.09/0.015+1 = 7. This problem involves 7 unknown nodal temperatures, and thus we need to have 7 equations. Node 0 is on insulated boundary, and thus we can treat it as an interior note by using the mirror image concept. Nodes 1, 2, and 3 are interior nodes, and thus for them we can use the general explicit finite difference relation expressed as

$$T_{m-1}^i - 2T_m^i + T_{m+1}^i + \frac{\dot{e}_m^i \Delta x^2}{k} = \frac{T_m^{i+1} - T_m^i}{\tau} \quad \rightarrow \quad T_m^{i+1} = \tau(T_{m-1}^i + T_{m+1}^i) + (1-2\tau)T_m^i + \tau\frac{\dot{e}_m^i \Delta x^2}{k}$$

The finite difference equation for node 4 on the right surface subjected to convection is obtained by applying an energy balance on the half volume element about node 4 and taking the direction of all heat transfers to be towards the node under consideration:

Node 0 (insulated) : $T_0^{i+1} = \tau(T_1^i + T_1^i) + (1-2\tau)T_0^i + \tau\frac{\dot{e}_0 \Delta x^2}{k}$



Node 1 (interior) : $T_1^{i+1} = \tau(T_0^i + T_2^i) + (1-2\tau)T_1^i + \tau\frac{\dot{e}_0 \Delta x^2}{k}$

Node 2 (interior) : $T_2^{i+1} = \tau(T_1^i + T_3^i) + (1-2\tau)T_2^i + \tau\frac{\dot{e}_0 \Delta x^2}{k}$

Node 3 (interior) : $T_3^{i+1} = \tau(T_2^i + T_4^i) + (1-2\tau)T_3^i + \tau\frac{\dot{e}_0 \Delta x^2}{k}$

Node 4 (interior) : $T_4^{i+1} = \tau(T_3^i + T_5^i) + (1-2\tau)T_4^i + \tau\frac{\dot{e}_0 \Delta x^2}{k}$

Node 5 (interior) : $T_5^{i+1} = \tau(T_4^i + T_6^i) + (1-2\tau)T_5^i + \tau\frac{\dot{e}_0 \Delta x^2}{k}$

Node 6 (convection) : $h(T_\infty - T_6^i) + k\frac{T_5^i - T_6^i}{\Delta x} + \dot{e}_0\frac{\Delta x}{2} = \rho\frac{\Delta x}{2}c_p\frac{T_6^{i+1} - T_6^i}{\Delta t}$

or $$T_6^{i+1} = \left(1 - 2\tau - 2\tau\frac{h\Delta x}{k}\right)T_6^i + 2\tau T_5^i + 2\tau\frac{h\Delta x}{k}T_\infty + \tau\frac{\dot{e}_0(\Delta x)^2}{k}$$

where

$$\Delta x = 0.015\ \text{m}, \dot{e}_0 = 10^6\ \text{W/m}^3,\ k = 28\ \text{W/m·°C},\ h = 35\ \text{W/m}^2\cdot\text{°C},\ T_\infty = 20\text{°C}\text{ , and }\alpha = 12.5\times10^{-6}\ \text{m}^2\text{/s.}$$

The upper limit of the time step Δt is determined from the stability criteria that requires all primary coefficients to be greater than or equal to zero. The coefficient of T_4^i is smaller in this case, and thus the stability criteria for this problem can be expressed as

$$1 - 2\tau - 2\tau\frac{h\Delta x}{k} \geq 0 \quad \rightarrow \quad \tau \leq \frac{1}{2(1 + h\Delta x/k)} \quad \rightarrow \quad \Delta t \leq \frac{\Delta x^2}{2\alpha(1 + h\Delta x/k)}$$

since $\tau = \alpha\Delta t/\Delta x^2$. Substituting the given quantities, the maximum allowable the time step becomes

$$\Delta t \leq \frac{(0.015\ \text{m})^2}{2(12.5\times10^{-6}\ \text{m}^2\text{/s})[1 + (35\ \text{W/m}^2\text{.°C})(0.015\ \text{m})/(28\ \text{W/m.°C})]} = 8.8\ \text{s}$$

Therefore, any time step less than 8.8 s can be used to solve this problem. For convenience, let us choose the time step to be Δt = 7.5 s. Then the mesh Fourier number becomes

$$\tau = \frac{\alpha \Delta t}{\Delta x^2} = \frac{(12.5 \times 10^{-6}\ \text{m}^2/\text{s})(7.5\,\text{s})}{(0.015\,\text{m})^2} = 0.4167$$

Substituting this value of τ and other given quantities, the nodal temperatures after 5×60/7.5 = 40 time steps (5 min) are determined to be

After 5 min:

T_0 = **229.9°C**, T_1 = **229.7°C**, T_2 = **229.0°C**, T_3 = **227.7°C**, T_4 = **225.8°C**, T_5 = **223.3°C**, and T_6 = **220.0 °C**

(*b*) The time needed for transient operation to be established is determined by increasing the number of time steps until the nodal temperatures no longer change. Using EES, we increased time steps for 16.8 hours (60500 seconds). The temperatures seem to remain constant at about this time. Then, the nodal temperatures under steady conditions are

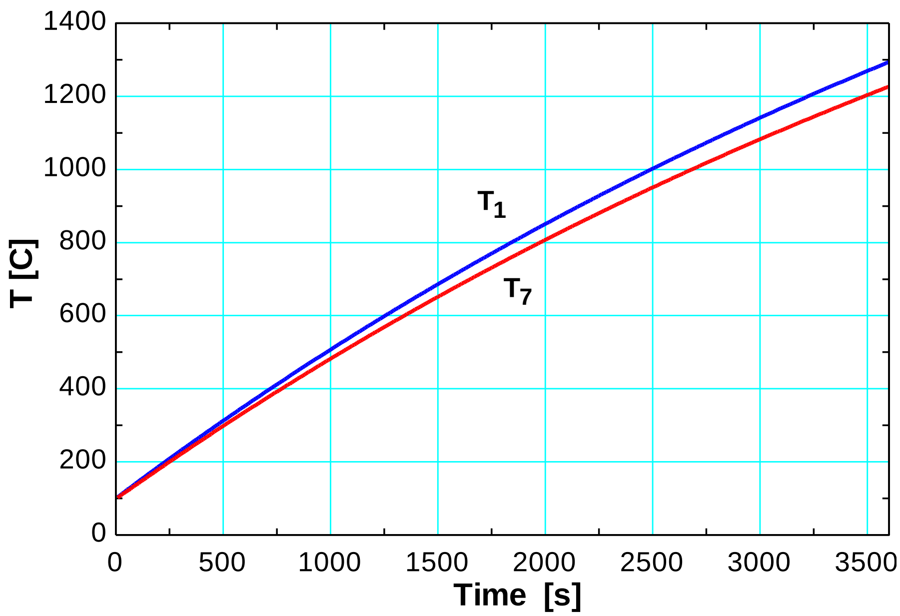
T_0 = **2736°C**, T_1 = **2732°C**, T_2 = **2720°C**, T_4 = **2700°C**, T_5 = **2672°C**, T_6 = **2636°C**, and T_7 = **2591 °C**

5-108 Prob. 5-107 is reconsidered. The effect of the cooling time on the temperatures of the left and right sides of the plate is to be investigated.

Analysis The problem is solved using EES, and the solution is given below.

```
"GIVEN"
L=0.09 [m]
k=28 [W/m-C]
alpha=12.5E-6 [m^2/s]
T_i=100 [C]
g_dot=1E6 [W/m^3]
T_infinity=20 [C]
h=35 [W/m^2-C]
DELTAx=0.015 [m]
"time=300 [s]"

"ANALYSIS"
M=L/DELTAx+1 "Number of nodes"
"DELTAt=7.5 [s]"
tau=(alpha*DELTAt)/DELTAx^2

"The technique is to store the temperatures in the parametric table and recover them (as old temperatures) using
the variable ROW.  The first row contains the initial values so Solve Table must begin at row 2. Use the
DUPLICATE statement to reduce the number of equations that need to be typed.  Column 1 contains the time,
column 2 the value of T[1], column 3, the value of T[2], etc., and column 9 the Row."
Time=TableValue(Row-1,#Time)+DELTAt
Duplicate i=1,7
    T_old[i]=TableValue(Row-1,#T[i])
end

"Using the explicit finite difference approach, the six equations for the six unknown temperatures are determined
to be"
T[1]=tau*(T_old[2]+T_old[2])+(1-2*tau)*T_old[1]+tau*(g_dot*DELTAx^2)/k "Node 0, insulated"
T[2]=tau*(T_old[1]+T_old[3])+(1-2*tau)*T_old[2]+tau*(g_dot*DELTAx^2)/k "Node 1"
T[3]=tau*(T_old[2]+T_old[4])+(1-2*tau)*T_old[3]+tau*(g_dot*DELTAx^2)/k "Node 2"
T[4]=tau*(T_old[3]+T_old[5])+(1-2*tau)*T_old[4]+tau*(g_dot*DELTAx^2)/k "Node 3"
T[5]=tau*(T_old[4]+T_old[6])+(1-2*tau)*T_old[5]+tau*(g_dot*DELTAx^2)/k "Node 4"
T[6]=tau*(T_old[5]+T_old[7])+(1-2*tau)*T_old[6]+tau*(g_dot*DELTAx^2)/k "Node 5"
T[7]=(1-2*tau-
2*tau*(h*DELTAx)/k)*T_old[7]+2*tau*T_old[6]+2*tau*(h*DELTAx)/k*T_infinity+tau*(g_dot*DELTAx^2)/k "Node 6,
convection"
```

Time [s]	T_1 [C]	T_2 [C]	T_3 [C]	T_4 [C]	T_5 [C]	T_6 [C]	T_7 [C]	Row
0	100	100	100	100	100	100	100	1
7.5	103.3	103.3	103.3	103.3	103.3	103.3	103.3	2
15	106.7	106.7	106.7	106.7	106.7	106.2	106.7	3
22.5	110	110	110	110	109.8	109.3	109.8	4
30	113.4	113.4	113.4	113.3	113.1	112.3	113.1	5
37.5	116.7	116.7	116.7	116.6	116.2	115.5	116.2	6
45	120.1	120.1	120	119.8	119.4	118.5	119.4	7
52.5	123.4	123.4	123.3	123.1	122.6	121.7	122.6	8
60	126.8	126.7	126.6	126.3	125.7	124.8	125.7	9
67.5	130.1	130	129.9	129.5	128.9	127.9	128.9	10
…	…	…	…	…	…	…	…	…
…	…	…	…	…	…	…	…	…
3525	1276	1274	1268	1259	1246	1230	1209	471
3533	1277	1276	1270	1261	1248	1232	1211	472
3540	1279	1277	1272	1263	1250	1233	1213	473
3548	1281	1279	1274	1265	1252	1235	1215	474
3555	1283	1281	1276	1266	1254	1237	1216	475
3563	1285	1283	1277	1268	1255	1239	1218	476
3570	1287	1285	1279	1270	1257	1240	1220	477
3578	1288	1287	1281	1272	1259	1242	1222	478
3585	1290	1288	1283	1274	1261	1244	1223	479
3593	1292	1290	1285	1275	1262	1246	1225	480
3600	1294	1292	1286	1277	1264	1247	1227	481



5-109E A plain window glass initially at a uniform temperature is subjected to convection on both sides. The transient finite difference formulation of this problem is to be obtained, and it is to be determined how long it will take for the fog on the windows to clear up (i.e., for the inner surface temperature of the window glass to reach 54°F).

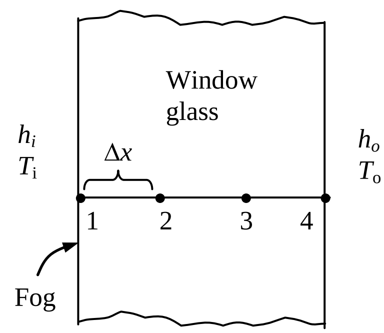
Assumptions **1** Heat transfer is one-dimensional since the window is large relative to its thickness. **2** Thermal conductivity is constant. **3** Radiation heat transfer is negligible.

Properties The conductivity and diffusivity are given to be k = 0.48 Btu/h.ft·°F and $\alpha = 4.2\times10^{-6}\ \text{ft}^2/\text{s}$.

Analysis The nodal spacing is given to be Δx = 0.125 in. Then the number of nodes becomes $M = L/\Delta x + 1$ = 0.375/0.125+1 = 4. This problem involves 4 unknown nodal temperatures, and thus we need to have 4 equations. Nodes 2 and 3 are interior nodes, and thus for them we can use the general explicit finite difference relation expressed as

$$T_{m-1}^i - 2T_m^i + T_{m+1}^i + \frac{\dot{e}_m^i \Delta x^2}{k} = \frac{T_m^{i+1} - T_m^i}{\tau} \quad \rightarrow \quad T_m^{i+1} = \tau(T_{m-1}^i + T_{m+1}^i) + (1-2\tau)T_m^i$$

since there is no heat generation. The finite difference equation for nodes 1 and 4 on the surfaces subjected to convection is obtained by applying an energy balance on the half volume element about the node, and taking the direction of all heat transfers to be towards the node under consideration:



$$\text{Node 1 (convection)}: \ h_i(T_i - T_1^i) + k\frac{T_2^i - T_1^i}{\Delta x} = \rho\frac{\Delta x}{2}c_p\frac{T_1^{i+1} - T_1^i}{\Delta t}$$

or

$$T_1^{i+1} = \left(1 - 2\tau - 2\tau\frac{h_i\Delta x}{k}\right)T_1^i + 2\tau T_2^i + 2\tau\frac{h_i\Delta x}{k}T_i$$

$$\text{Node 2 (interior)}: \ T_2^{i+1} = \tau(T_1^i + T_3^i) + (1-2\tau)T_2^i$$

$$\text{Node 3 (interior)}: \ T_3^{i+1} = \tau(T_2^i + T_4^i) + (1-2\tau)T_3^i$$

$$\text{Node 4 (convection)}: \ h_o(T_o - T_4^i) + k\frac{T_3^i - T_4^i}{\Delta x} = \rho\frac{\Delta x}{2}c\frac{T_4^{i+1} - T_4^i}{\Delta t}$$

or

$$T_4^{i+1} = \left(1 - 2\tau - 2\tau\frac{h_o\Delta x}{k}\right)T_4^i + 2\tau T_3^i + 2\tau\frac{h_o\Delta x}{k}T_o$$

where Δx = 0.125/12 ft , k = 0.48 Btu/h.ft·°F, h_i = 1.2 Btu/h.ft^2·°F, T_i =35+2*(t/60)°F (t in seconds), h_o = 2.6 Btu/h.ft^2·°F, and T_o =35°F. The upper limit of the time step Δt is determined from the stability criteria that requires all primary coefficients to be greater than or equal to zero. The coefficient of T_4^i is smaller in this case, and thus the stability criteria for this problem can be expressed as

$$1 - 2\tau - 2\tau\frac{h\Delta x}{k} \geq 0 \quad \rightarrow \quad \tau \leq \frac{1}{2(1 + h\Delta x/k)} \quad \rightarrow \quad \Delta t \leq \frac{\Delta x^2}{2\alpha(1 + h\Delta x/k)}$$

since $\tau = \alpha\Delta t/\Delta x^2$. Substituting the given quantities, the maximum allowable time step becomes

$$\Delta t \leq \frac{(0.125/12\ \text{ft})^2}{2(4.2\times10^{-6}\ \text{ft}^2/\text{s})[1 + (2.6\ \text{Btu/h.ft}^2.°\text{F})(0.125/12\ \text{m})/(0.48\ \text{Btu/h.ft.°F})]} = 12.2\ \text{s}$$

Therefore, any time step less than 12.2 s can be used to solve this problem. For convenience, let us choose the time step to be Δt = 10 s. Then the mesh Fourier number becomes

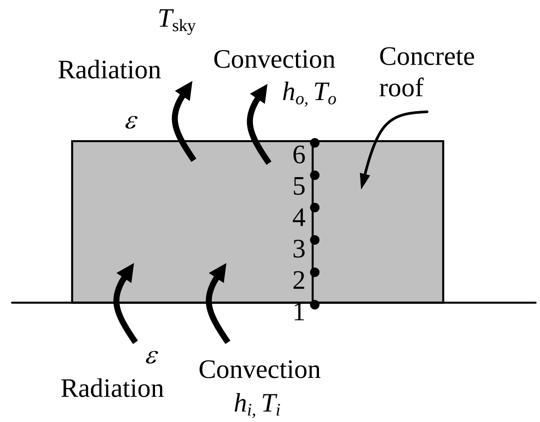
$$\tau = \frac{\alpha\Delta t}{\Delta x^2} = \frac{(4.2\times10^{-6}\ \text{ft}^2/\text{s})(10\ \text{s})}{(0.125/12\ \text{ft})^2} = 0.3871$$

Substituting this value of τ and other given quantities, the time needed for the inner surface temperature of the window glass to reach 54°F to avoid fogging is determined to be *never*. This is because steady conditions are reached in about 156 min, and the inner surface temperature at that time is determined to be 48.0°F. Therefore, the window will be fogged at all times.

5-110 The roof of a house initially at a uniform temperature is subjected to convection and radiation on both sides. The temperatures of the inner and outer surfaces of the roof at 6 am in the morning as well as the average rate of heat transfer through the roof during that night are to be determined.

Assumptions **1** Heat transfer is one-dimensional. **2** Thermal properties, heat transfer coefficients, and the indoor and outdoor temperatures are constant. **3** Radiation heat transfer is significant.

Properties The conductivity and diffusivity are given to be k = 1.4 W/m.°C and $\alpha = 0.69 \times 10^{-6}\ \text{m}^2/\text{s}$. The emissivity of both surfaces of the concrete roof is 0.9.



Analysis The nodal spacing is given to be Δx = 0.03 m. Then the number of nodes becomes $M = L/\Delta x + 1$ = 0.15/0.03+1 = 6. This problem involves 6 unknown nodal temperatures, and thus we need to have 6 equations. Nodes 2, 3, 4, and 5 are interior nodes, and thus for them we can use the general explicit finite difference relation expressed as

$$T_{m-1}^i - 2T_m^i + T_{m+1}^i + \frac{\dot{e}_m^i \Delta x^2}{k} = \frac{T_m^{i+1} - T_m^i}{\tau}$$

$$\rightarrow T_m^{i+1} = \tau(T_{m-1}^i + T_{m+1}^i) + (1-2\tau)T_m^i + \tau \frac{\dot{e}_m^i \Delta x^2}{k}$$

The finite difference equations for nodes 1 and 6 subjected to convection and radiation are obtained from an energy balance by taking the direction of all heat transfers to be towards the node under consideration:

$$\text{Node 1 (convection)}: \quad h_i(T_i - T_1^i) + k\frac{T_2^i - T_1^i}{\Delta x} + \varepsilon\sigma\left[T_{wall}^4 - (T_1^i + 273)^4\right] = \rho\frac{\Delta x}{2}c_p\frac{T_1^{i+1} - T_1^i}{\Delta t}$$

$$\text{Node 2 (interior)}: \quad T_2^{i+1} = \tau(T_1^i + T_3^i) + (1-2\tau)T_2^i$$

$$\text{Node 3 (interior)}: \quad T_3^{i+1} = \tau(T_2^i + T_4^i) + (1-2\tau)T_3^i$$

$$\text{Node 4 (interior)}: \quad T_4^{i+1} = \tau(T_3^i + T_5^i) + (1-2\tau)T_4^i$$

$$\text{Node 5 (interior)}: \quad T_5^{i+1} = \tau(T_4^i + T_6^i) + (1-2\tau)T_5^i$$

$$\text{Node 6 (convection)}: \quad h_o(T_0 - T_6^i) + k\frac{T_5^i - T_6^i}{\Delta x} + \varepsilon\sigma\left[T_{sky}^4 - (T_6^i + 273)^4\right] = \rho\frac{\Delta x}{2}c_p\frac{T_6^{i+1} - T_6^i}{\Delta t}$$

where k = 1.4 W/m.°C, $\alpha = k/\rho c_p = 0.69 \times 10^{-6}\ \text{m}^2/\text{s}$, T_i = 20°C, T_{wall} = 293 K, T_o = 6°C, T_{sky} =260 K, h_i = 5 W/m^2.°C, h_o = 12 W/m^2.°C, Δx = 0.03 m, and Δt = 5 min. Also, the mesh Fourier number is

$$\tau = \frac{\alpha \Delta t}{\Delta x^2} = \frac{(0.69 \times 10^{-6}\ \text{m}^2/\text{s})(300\ \text{s})}{(0.03\ \text{m})^2} = 0.230$$

Substituting this value of τ and other given quantities, the inner and outer surface temperatures of the roof after 12×(60/5) = 144 time steps (12 h) are determined to be T_1 = **10.3°C** and T_6 = **-0.97°C.**

(b) The average temperature of the inner surface of the roof can be taken to be

$$T_{1,avg} = \frac{T_{1@6\,\text{PM}} + T_{1@6\,\text{AM}}}{2} = \frac{18 + 10.3}{2} = 14.15°\text{C}$$

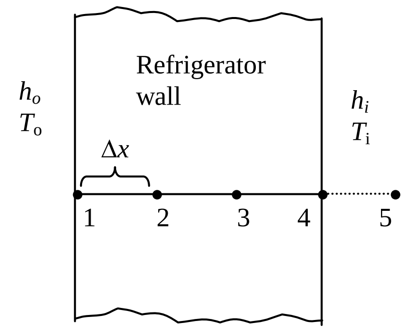
Then the average rate of heat loss through the roof that night becomes

$$\begin{aligned}
\dot{Q}_{avg} &= h_i A_s (T_i - T_{1,ave}) + \varepsilon\sigma A_s\left[T_{wall}^4 - (T_1^i + 273)^4\right] \\
&= (5\ \text{W/m}^2\cdot°\text{C})(18 \times 32\ \text{m}^2)(20 - 14.15)°\text{C} \\
&\quad + 0.9(18 \times 32\ \text{m}^2)(5.67 \times 10^{-8}\ \text{W/m}^2\cdot\text{K}^4)[(293\ \text{K})^4 - (14.15 + 273\ \text{K})^4] \\
&= \mathbf{33{,}640\ W}
\end{aligned}$$

5-111 A refrigerator whose walls are constructed of 3-cm thick urethane insulation malfunctions, and stops running for 6 h. The temperature inside the refrigerator at the end of this 6 h period is to be determined.

Assumptions **1** Heat transfer is one-dimensional since the walls are large relative to their thickness. **2** Thermal properties, heat transfer coefficients, and the outdoor temperature are constant. **3** Radiation heat transfer is negligible. **4** The temperature of the contents of the refrigerator, including the air inside, rises uniformly during this period. **5** The local atmospheric pressure is 1 atm. **6** The space occupied by food and the corner effects are negligible. **7** Heat transfer through the bottom surface of the refrigerator is negligible.

Properties The conductivity and diffusivity are given to be k = 0.026 W/m·K and $\alpha = 0.36\times10^{-6}$ m^2/s. The average specific heat of food items is given to be 3.6 kJ/kg·K. The specific heat and density of air at 1 atm and 3°C are c_p = 1.006 kJ/kg·K and ρ = 1.28 kg/m^3 (Table A-15).



Analysis The nodal spacing is given to be Δx = 0.01 m. Then the number of nodes becomes $M = L/\Delta x + 1$ = 0.03/0.01+1 = 4. This problem involves 4 unknown nodal temperatures, and thus we need to have 4 equations. Nodes 2 and 3 are interior nodes, and thus for them we can use the general explicit finite difference relation expressed as

$$T_{m-1}^i - 2T_m^i + T_{m+1}^i + \frac{\dot{e}_m^i \Delta x^2}{k} = \frac{T_m^{i+1} - T_m^i}{\tau} \;\rightarrow\; T_m^{i+1} = \tau(T_{m-1}^i + T_{m+1}^i) + (1-2\tau)T_m^i + \tau\frac{\dot{e}_m^i \Delta x^2}{k}$$

The finite difference equations for nodes 1 and 4 subjected to convection and radiation are obtained from an energy balance by taking the direction of all heat transfers to be towards the node under consideration:

$$\text{Node 1 (convection)}: \quad h_o(T_0 - T_1^i) + k\frac{T_2^i - T_1^i}{\Delta x} = \rho\frac{\Delta x}{2}c_p\frac{T_1^{i+1} - T_1^i}{\Delta t}$$

$$\text{Node 2 (interior)}: \quad T_2^{i+1} = \tau(T_1^i + T_3^i) + (1-2\tau)T_2^i$$

$$\text{Node 3 (interior)}: \quad T_3^{i+1} = \tau(T_2^i + T_4^i) + (1-2\tau)T_3^i$$

$$\text{Node 4 (convection)}: \quad h_i(T_5^i - T_4^i) + k\frac{T_3^i - T_4^i}{\Delta x} = \rho\frac{\Delta x}{2}c_p\frac{T_4^{i+1} - T_4^i}{\Delta t}$$

where $T_5 = T_i$ = 3°C (initially), T_o = 25°C, h_i = 6 W/m^2·K, h_o = 9 W/m^2·K, Δx = 0.01 m, and Δt = 1 min. Also, the mesh Fourier number is

$$\tau = \frac{\alpha\Delta t}{\Delta x^2} = \frac{(0.36\times10^{-6}\text{ m}^2/\text{s})(60\text{ s})}{(0.01\text{ m})^2} = 0.216$$

The volume of the refrigerator cavity and the mass of air inside are

$$\mathbf{V} = (1.80-0.03)(0.8-0.03)(0.7-0.03) = 0.913\text{ m}^3$$

$$m_{air} = \rho\mathbf{V} = (1.28\text{ kg/m}^3)(0.913\text{ m}^3) = 1.17\text{ kg}$$

Energy balance for the air space of the refrigerator can be expressed as

$$\text{Node 5 (refrig. air)}: \quad h_i A_i(T_4^i - T_5^i) = (mc_p\Delta T)_{air} + (mc_p\Delta T)_{food}$$

or

$$h_i A_i(T_4^i - T_5^i) = \left[(mc_p)_{air} + (mc_p)_{food}\right]\frac{T_5^{i+1} - T_5^i}{\Delta t}$$

where $A_i = 2(1.77\times0.77) + 2(1.77\times0.67) + (0.77\times0.67) = 5.6135\text{ m}^2$

Substituting, temperatures of the refrigerated space after 6×60 = 360 time steps (6 h) is determined to be

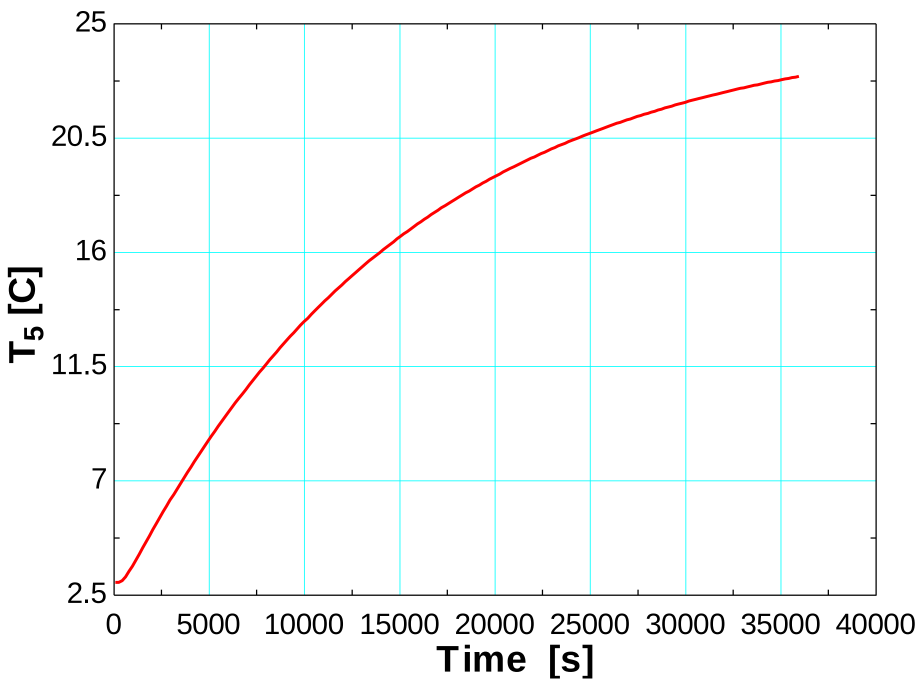
$T_{in} = T_5 =$ **19.6°C**

5–112 Prob. 5-111 is reconsidered. The temperature inside the refrigerator as a function of heating time is to be plotted.

Analysis The problem is solved using EES, and the solution is given below.

```
"GIVEN"
t_ins=0.03 [m]
k=0.026 [W/m-C]
alpha=0.36E-6 [m^2/s]
T_i=3 [C]
h_i=6 [W/m^2-C]
h_o=9 [W/m^2-C]
T_infinity=25 [C]
m_food=15 [kg]
C_food=3600 [J/kg-C]
DELTAx=0.01 [m]
DELTAt=60 [s]
time=6*3600 [s]

"PROPERTIES"
rho_air=density(air, T=T_i, P=101.3)
C_air=CP(air, T=T_i)*Convert(kJ/kg-C, J/kg-C)

"ANALYSIS"
M=t_ins/DELTAx+1 "Number of nodes"
tau=(alpha*DELTAt)/DELTAx^2
RhoC=k/alpha "RhoC=rho*C"

"The technique is to store the temperatures in the parametric table and recover them (as old temperatures)
using the variable ROW.  The first row contains the initial values so Solve Table must begin at row 2.
Use the DUPLICATE statement to reduce the number of equations that need to be typed.  Column 1
contains the time, column 2 the value of T[1], column 3, the value of T[2], etc., and column 7 the Row."
Time=TableValue('Table 1',Row-1,#Time)+DELTAt
Duplicate i=1,5
    T_old[i]=TableValue('Table 1',Row-1,#T[i])
end

"Using the explicit finite difference approach, the six equations for the six unknown temperatures are
determined to be"
h_o*(T_infinity-T_old[1])+k*(T_old[2]-T_old[1])/DELTAx=RhoC*DELTAx/2*(T[1]-T_old[1])/DELTAt "Node 1,
convection"
T[2]=tau*(T_old[1]+T_old[3])+(1-2*tau)*T_old[2] "Node 2"
T[3]=tau*(T_old[2]+T_old[4])+(1-2*tau)*T_old[3] "Node 3"
h_i*(T_old[5]-T_old[4])+k*(T_old[3]-T_old[4])/DELTAx=RhoC*DELTAx/2*(T[4]-T_old[4])/DELTAt "Node 4,
convection"
h_i*A_i*(T_old[4]-T_old[5])=m_air*C_air*(T[5]-T_old[5])/DELTAt+m_food*C_food*(T[5]-T_old[5])/DELTAt
"Node 5, refrig. air"

A_i=2*(1.8-0.03)*(0.8-0.03)+2*(1.8-0.03)*(0.7-0.03)+(0.8-0.03)*(0.7-0.03)
m_air=rho_air*V_air
V_air=(1.8-0.03)*(0.8-0.03)*(0.7-0.03)
```

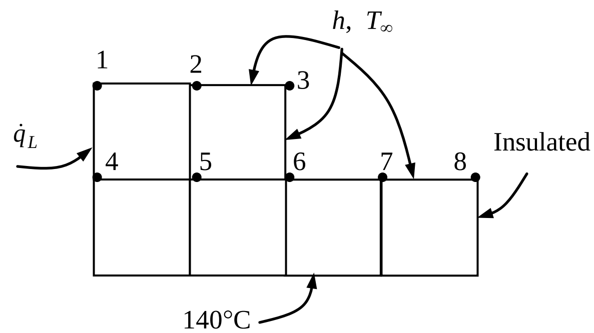
Time [s]	T_1 [C]	T_2 [C]	T_3 [C]	T_4 [C]	T_5 [C]	Row
0	3	3	3	3	3	1
60	35.9	3	3	3	3	2
120	5.389	10.11	3	3	3	3
180	36.75	7.552	4.535	3	3	4
240	6.563	13.21	4.855	3.663	3	5
300	37	9.968	6.402	3.517	3.024	6
360	7.374	15.04	6.549	4.272	3.042	7
420	37.04	11.55	7.891	4.03	3.087	8
480	8.021	16.27	7.847	4.758	3.122	9
540	36.97	12.67	8.998	4.461	3.182	10
…	…	…	…	…	…	…
…	…	…	…	…	…	…
35460	24.85	24.23	23.65	23.09	22.86	592
35520	24.81	24.24	23.65	23.1	22.87	593
35580	24.85	24.23	23.66	23.11	22.88	594
35640	24.81	24.24	23.67	23.12	22.88	595
35700	24.85	24.24	23.67	23.12	22.89	596
35760	24.81	24.25	23.68	23.13	22.9	597
35820	24.85	24.25	23.68	23.14	22.91	598
35880	24.81	24.26	23.69	23.15	22.92	599
35940	24.85	24.25	23.69	23.15	22.93	600
36000	24.82	24.26	23.7	23.16	22.94	601



5-113 Heat conduction through a long L-shaped solid bar with specified boundary conditions is considered. The temperature at the top corner (node #3) of the body after 2, 5, and 30 min is to be determined with the transient explicit finite difference method.

Assumptions **1** Heat transfer through the body is given to be transient and two-dimensional. **2** Thermal conductivity is constant. **3** Heat generation is uniform.

Properties The conductivity and diffusivity are given to be k = 15 W/m·°C and $\alpha = 3.2\times10^{-6}\ \text{m}^2/\text{s}$.



Analysis The nodal spacing is given to be Δ*x*=Δ*x*=*l*=0.015 m. The explicit finite difference equations are determined on the basis of the energy balance for the transient case expressed as

$$\sum_{\text{All sides}} \dot{Q}^i + \dot{E}^{\,i}_{\text{element}} = \rho V_{\text{element}} c_p \frac{T_m^{i+1} - T_m^i}{\Delta t}$$

The quantities $h, T_\infty, \dot{e},$ and $\dot{q}_R$ do not change with time, and thus we do not need to use the superscript i for them. Also, the energy balance expressions can be simplified using the definitions of thermal diffusivity $\alpha = k/\rho c_p$ and the dimensionless mesh Fourier number $\tau = \alpha\Delta t / l^2$ where $\Delta x = \Delta y = l$. We note that all nodes are boundary nodes except node 5 that is an interior node. Therefore, we will have to rely on energy balances to obtain the finite difference equations. Using energy balances, the finite difference equations for each of the 8 nodes are obtained as follows:

Node 1: $$\dot{q}_L \frac{l}{2} + h\frac{l}{2}(T_\infty - T_1^i) + k\frac{l}{2}\frac{T_2^i - T_1^i}{l} + k\frac{l}{2}\frac{T_4^i - T_1^i}{l} + \dot{e}_0\frac{l^2}{4} = \rho\frac{l^2}{4}c\frac{T_1^{i+1} - T_1^i}{\Delta t}$$

Node 2: $$hl(T_\infty - T_2^i) + k\frac{l}{2}\frac{T_1^i - T_2^i}{l} + k\frac{l}{2}\frac{T_3^i - T_2^i}{l} + kl\frac{T_5^i - T_2^i}{l} + \dot{e}_0\frac{l^2}{2} = \rho\frac{l^2}{2}c_p\frac{T_2^{i+1} - T_2^i}{\Delta t}$$

Node 3: $$hl(T_\infty - T_3^i) + k\frac{l}{2}\frac{T_2^i - T_3^i}{l} + k\frac{l}{2}\frac{T_6^i - T_3^i}{l} + \dot{e}_0\frac{l^2}{4} = \rho\frac{l^2}{4}c\frac{T_3^{i+1} - T_3^i}{\Delta t}$$

(It can be rearranged as $$T_3^{i+1} = \left(1 - 4\tau - 4\tau\frac{hl}{k}\right)T_3^i + 2\tau\left(T_4^i + T_6^i + 2\frac{hl}{k}T_\infty + \frac{\dot{e}_0 l^2}{2k}\right)$$)

Node 4: $$\dot{q}_L l + k\frac{l}{2}\frac{T_1^i - T_4^i}{l} + k\frac{l}{2}\frac{140 - T_4^i}{l} + kl\frac{T_5^i - T_4^i}{l} + \dot{e}_0\frac{l^2}{2} = \rho\frac{l^2}{2}c\frac{T_4^{i+1} - T_4^i}{\Delta t}$$

Node 5 (interior): $$T_5^{i+1} = (1 - 4\tau)T_5^i + \tau\left(T_2^i + T_4^i + T_6^i + 140 + \frac{\dot{e}_0 l^2}{k}\right)$$

Node 6: $$hl(T_\infty - T_6^i) + k\frac{l}{2}\frac{T_3^i - T_6^i}{l} + kl\frac{T_5^i - T_6^i}{l} + kl\frac{140 - T_6^i}{l} + k\frac{l}{2}\frac{T_7^i - T_6^i}{l} + \dot{e}_0\frac{3l^2}{4} = \rho\frac{3l^2}{4}c\frac{T_6^{i+1} - T_6^i}{\Delta t}$$

Node 7: $$hl(T_\infty - T_7^i) + k\frac{l}{2}\frac{T_6^i - T_7^i}{l} + k\frac{l}{2}\frac{T_8^i - T_7^i}{l} + kl\frac{140 - T_7^i}{l} + \dot{e}_0\frac{l^2}{2} = \rho\frac{l^2}{2}c\frac{T_7^{i+1} - T_7^i}{\Delta t}$$

Node 8: $$h\frac{l}{2}(T_\infty - T_8^i) + k\frac{l}{2}\frac{T_7^i - T_8^i}{l} + k\frac{l}{2}\frac{140 - T_8^i}{l} + \dot{e}_0\frac{l^2}{4} = \rho\frac{l^2}{4}c\frac{T_8^{i+1} - T_8^i}{\Delta t}$$

where

$\dot{e}_0 = 2\times10^7\ \text{W/m}^3$, $\dot{q}_L = 8000\ \text{W/m}^2$, l = 0.015 m, k =15 W/m·°C, h = 80 W/m²·°C, and T_∞ =25°C.

The upper limit of the time step Δt is determined from the stability criteria that requires the coefficient of T_m^i in the T_m^{i+1} expression (the primary coefficient) be greater than or equal to zero for all nodes. The smallest primary coefficient in the 8 equations above is the coefficient of T_3^i in the T_3^{i+1} expression since it is exposed to most convection per unit volume (this can be verified), and thus the stability criteria for this problem can be expressed as

$$1-4\tau-4\tau\frac{hl}{k}\geq 0 \quad \rightarrow \quad \tau\leq\frac{1}{4(1+hl/k)} \quad \rightarrow \quad \Delta t\leq\frac{l^2}{4\alpha(1+hl/k)}$$

since $\tau=\alpha\Delta t/l^2$. Substituting the given quantities, the maximum allowable value of the time step is determined to be

$$\Delta t\leq\frac{(0.015\,\text{m})^2}{4(3.2\times10^{-6}\ \text{m}^2/\text{s})[1+(80\ \text{W/m}^2.°\text{C})(0.015\,\text{m})/(15\ \text{W/m}.°\text{C})]}=16.3\,\text{s}$$

Therefore, any time step less than 16.3 s can be used to solve this problem. For convenience, we choose the time step to be Δt = 15 s. Then the mesh Fourier number becomes

$$\tau=\frac{\alpha\Delta t}{l^2}=\frac{(3.2\times10^{-6}\ \text{m}^2/\text{s})(15\,\text{s})}{(0.015\,\text{m})^2}=0.2133 \qquad (\text{for}\quad \Delta t = 15\text{ s})$$

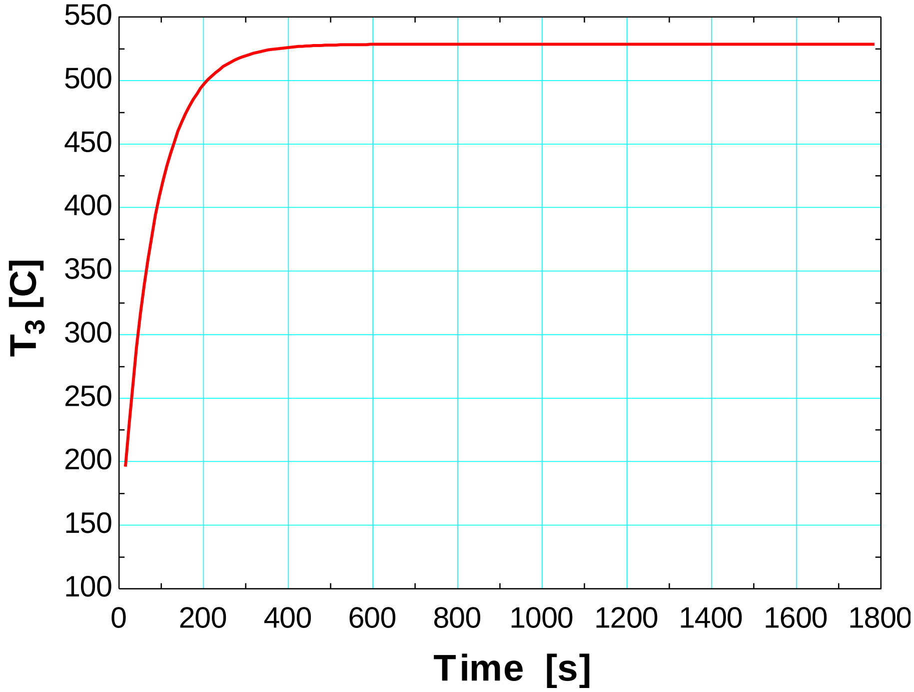
Using the specified initial condition as the solution at time t = 0 (for i = 0), sweeping through the 9 equations above will give the solution at intervals of 15 s. Using a computer, the solution at the upper corner node (node 3) is determined to be **441, 520**, and **529°C** at 2, 5, and 30 min, respectively. It can be shown that the steady state solution at node 3 is 531°C.

5-114 Prob. 5-113 is reconsidered. The temperature at the top corner as a function of heating time is to be plotted.

Analysis The problem is solved using EES, and the solution is given below.

```
"GIVEN"
T_i=140 [C]
k=15 [W/m-C]
alpha=3.2E-6 [m^2/s]
e_dot=2E7 [W/m^3]
T_bottom=140 [C]
T_infinity=25 [C]
h=80 [W/m^2-C]
q_dot_L=8000 [W/m^2]
DELTAx=0.015 [m]
DELTAy=0.015 [m]
time=120 [s]

"ANALYSIS"
l=DELTAx
DELTAt=15 [s]
tau=(alpha*DELTAt)/l^2
RhoC=k/alpha "RhoC=rho*C"

"The technique is to store the temperatures in the parametric table and recover them (as old temperatures)
using the variable ROW.  The first row contains the initial values so Solve Table must begin at row 2.
Use the DUPLICATE statement to reduce the number of equations that need to be typed.  Column 1
contains the time, column 2 the value of T[1], column 3, the value of T[2], etc., and column 10 the Row."
Time=TableValue('Table 1',Row-1,#Time)+DELTAt
Duplicate i=1,8
    T_old[i]=TableValue('Table 1',Row-1,#T[i])
end

"Using the explicit finite difference approach, the eight equations for the eight unknown temperatures are
determined to be"
q_dot_L*l/2+h*l/2*(T_infinity-T_old[1])+k*l/2*(T_old[2]-T_old[1])/l+k*l/2*(T_old[4]-
T_old[1])/l+e_dot*l^2/4=RhoC*l^2/4*(T[1]-T_old[1])/DELTAt "Node 1"
h*l*(T_infinity-T_old[2])+k*l/2*(T_old[1]-T_old[2])/l+k*l/2*(T_old[3]-T_old[2])/l+k*l*(T_old[5]-
T_old[2])/l+e_dot*l^2/2=RhoC*l^2/2*(T[2]-T_old[2])/DELTAt "Node 2"
h*l*(T_infinity-T_old[3])+k*l/2*(T_old[2]-T_old[3])/l+k*l/2*(T_old[6]-T_old[3])/l+e_dot*l^2/4=RhoC*l^2/4*(T[3]-
T_old[3])/DELTAt "Node 3"
q_dot_L*l+k*l/2*(T_old[1]-T_old[4])/l+k*l/2*(T_bottom-T_old[4])/l+k*l*(T_old[5]-
T_old[4])/l+e_dot*l^2/2=RhoC*l^2/2*(T[4]-T_old[4])/DELTAt "Node 4"
T[5]=(1-4*tau)*T_old[5]+tau*(T_old[2]+T_old[4]+T_old[6]+T_bottom+e_dot*l^2/k) "Node 5"
h*l*(T_infinity-T_old[6])+k*l/2*(T_old[3]-T_old[6])/l+k*l*(T_old[5]-T_old[6])/l+k*l*(T_bottom-
T_old[6])/l+k*l/2*(T_old[7]-T_old[6])/l+e_dot*3/4*l^2=RhoC*3/4*l^2*(T[6]-T_old[6])/DELTAt "Node 6"
h*l*(T_infinity-T_old[7])+k*l/2*(T_old[6]-T_old[7])/l+k*l/2*(T_old[8]-T_old[7])/l+k*l*(T_bottom-
T_old[7])/l+e_dot*l^2/2=RhoC*l^2/2*(T[7]-T_old[7])/DELTAt "Node 7"
h*l/2*(T_infinity-T_old[8])+k*l/2*(T_old[7]-T_old[8])/l+k*l/2*(T_bottom-
T_old[8])/l+e_dot*l^2/4=RhoC*l^2/4*(T[8]-T_old[8])/DELTAt "Node 8"
```

Time [s]	T_1 [C]	T_2 [C]	T_3 [C]	T_4 [C]	T_5 [C]	T_6 [C]	T_7 [C]	T_8 [C]	Row
0	140	140	140	140	140	140	140	140	1
15	203.5	200.1	196.1	207.4	204	201.4	200.1	200.1	2
30	265	259.7	252.4	258.2	253.7	243.7	232.7	232.5	3
45	319	312.7	300.3	299.9	293.5	275.7	252.4	250.1	4
60	365.5	357.4	340.3	334.6	326.4	300.7	265.2	260.4	5
75	404.6	394.9	373.2	363.6	353.5	320.6	274.1	267	6
90	437.4	426.1	400.3	387.8	375.9	336.7	280.8	271.6	7
105	464.7	451.9	422.5	407.9	394.5	349.9	286	275	8
120	487.4	473.3	440.9	424.5	409.8	360.7	290.1	277.5	9
135	506.2	491	456.1	438.4	422.5	369.6	293.4	279.6	10
…	…	…	…	…	…	…	…	…	…
…	…	…	…	…	…	…	…	…	…
1650	596.3	575.7	528.5	504.6	483.1	411.9	308.8	288.9	111
1665	596.3	575.7	528.5	504.6	483.1	411.9	308.8	288.9	112
1680	596.3	575.7	528.5	504.6	483.1	411.9	308.8	288.9	113
1695	596.3	575.7	528.5	504.6	483.1	411.9	308.8	288.9	114
1710	596.3	575.7	528.5	504.6	483.1	411.9	308.8	288.9	115
1725	596.3	575.7	528.5	504.6	483.1	411.9	308.8	288.9	116
1740	596.3	575.7	528.5	504.6	483.1	411.9	308.8	288.9	117
1755	596.3	575.7	528.5	504.6	483.1	411.9	308.8	288.9	118
1770	596.3	575.7	528.5	504.6	483.1	411.9	308.8	288.9	119
1785	596.3	575.7	528.5	504.6	483.1	411.9	308.8	288.9	120

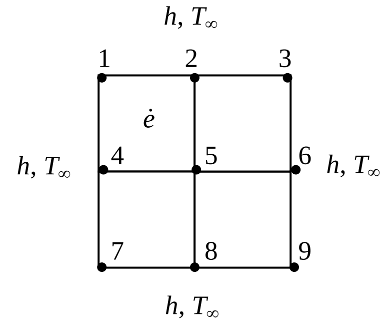


5-115 A long solid bar is subjected to transient two-dimensional heat transfer. The centerline temperature of the bar after 20 min and after steady conditions are established are to be determined.

Assumptions **1** Heat transfer through the body is given to be transient and two-dimensional. **2** Heat is generated uniformly in the body. **3** The heat transfer coefficient also includes the radiation effects.

Properties The conductivity and diffusivity are given to be k = 28 W/m·°C and $\alpha = 12\times10^{-6}\ \text{m}^2/\text{s}$.

Analysis The nodal spacing is given to be Δx=Δx=l=0.1 m. The explicit finite difference equations are determined on the basis of the energy balance for the transient case expressed as



$$\sum_{\text{All sides}} \dot{Q}^i + \dot{E}^{\text{i}}_{\text{element}} = \rho V_{\text{element}} c_p \frac{T_m^{i+1} - T_m^i}{\Delta t}$$

The quantities h, T_∞, and $\dot{e}_0$ do not change with time, and thus we do not need to use the superscript i for them. The general explicit finite difference form of an interior node for transient two-dimensional heat conduction is expressed as

$$T_{\text{node}}^{i+1} = \tau(T_{\text{left}}^{\text{i}} + T_{\text{top}}^{\text{i}} + T_{\text{right}}^{\text{i}} + T_{\text{bottom}}^{\text{i}}) + (1-4\tau)T_{\text{node}}^{\text{i}} + \tau\frac{\dot{e}_{\text{node}}^{\text{i}} l^2}{k}$$

There is symmetry about the vertical, horizontal, and diagonal lines passing through the center. Therefore, $T_1 = T_3 = T_7 = T_9$ and $T_2 = T_4 = T_6 = T_8$, and T_1, T_2, and T_5 are the only 3 unknown nodal temperatures, and thus we need only 3 equations to determine them uniquely. Also, we can replace the symmetry lines by insulation and utilize the mirror-image concept when writing the finite difference equations for the interior nodes. The finite difference equations for boundary nodes are obtained by applying an energy balance on the volume elements and taking the direction of all heat transfers to be towards the node under consideration:

Node 1: $$hl(T_\infty - T_1^i) + k\frac{l}{2}\frac{T_2^i - T_1^i}{l} + k\frac{l}{2}\frac{T_4^i - T_1^i}{l} + \dot{e}_0\frac{l^2}{4} = \rho\frac{l^2}{4}c\frac{T_1^{i+1} - T_1^i}{\Delta t}$$

Node 2: $$h\frac{l}{2}(T_\infty - T_2^i) + k\frac{l}{2}\frac{T_1^i - T_2^i}{l} + k\frac{l}{2}\frac{T_5^i - T_2^i}{l} + \dot{e}_0\frac{l^2}{4} = \rho\frac{l^2}{4}c\frac{T_2^{i+1} - T_2^i}{\Delta t}$$

Node 5 (interior): $$T_5^{i+1} = (1-4\tau)T_5^i + \tau\left(4T_2^i + \frac{\dot{e}_0 l^2}{k}\right)$$

where $\dot{e}_0 = 8\times10^5\ \text{W/m}^3$, l = 0.1 m, and k = 28 W/m·°C, h = 45 W/m^2·°C, and T_∞ = 30°C.

The upper limit of the time step Δt is determined from the stability criteria that requires the coefficient of T_m^i in the T_m^{i+1} expression (the primary coefficient) be greater than or equal to zero for all nodes. The smallest primary coefficient in the 3 equations above is the coefficient of T_1^i in the T_1^{i+1} expression since it is exposed to most convection per unit volume (this can be verified), and thus the stability criteria for this problem can be expressed as

$$1 - 4\tau - 4\tau\frac{hl}{k} \geq 0 \quad\rightarrow\quad \tau \leq \frac{1}{4(1+hl/k)} \quad\rightarrow\quad \Delta t \leq \frac{l^2}{4\alpha(1+hl/k)}$$

since $\tau = \alpha\Delta t / l^2$. Substituting the given quantities, the maximum allowable value of the time step is determined to be

$$\Delta t \leq \frac{(0.1\,\text{m})^2}{4(12\times10^{-6}\ \text{m}^2/\text{s})[1 + (45\ \text{W/m}^2.°\text{C})(0.1\,\text{m})/(28\ \text{W/m}.°\text{C})]} = 179\ \text{s}$$

Therefore, any time step less than 179 s can be used to solve this problem. For convenience, we choose the time step to be Δt = 60 s. Then the mesh Fourier number becomes

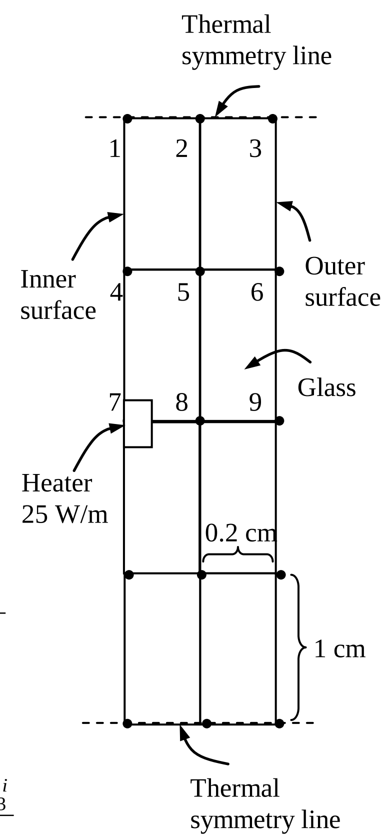
$$\tau = \frac{\alpha\Delta t}{l^2} = \frac{(12\times10^{-6}\ \text{m}^2/\text{s})(60\ \text{s})}{(0.1\,\text{m})^2} = 0.072 \qquad \text{(for } \Delta t = 60\ \text{s)}$$

Using the specified initial condition as the solution at time t = 0 (for i = 0), sweeping through the 3 equations above will give the solution at intervals of 1 min. Using a computer, the solution at the center node (node 5) is determined to be 227.5°C, 312.0°C, **387.6°C**, 455.1°C, 515.5°C, 617.7°C, 699.3°C, and 764.5°C at 10, 15, 20, 25, 30, 40, 50, and 60 min, respectively. Continuing in this manner, it is observed that steady conditions are reached in the medium after about 6 hours for which the temperature at the center node is **1023°C**.

5-116 The formation of fog on the glass surfaces of a car is to be prevented by attaching electric resistance heaters to the inner surfaces. The temperature distribution throughout the glass 15 min after the strip heaters are turned on and also when steady conditions are reached are to be determined using the explicit method.

Assumptions **1** Heat transfer through the glass is given to be transient and two-dimensional. **2** Thermal conductivity is constant. **3** There is heat generation only at the inner surface, which will be treated as prescribed heat flux.

Properties The conductivity and diffusivity are given to be k = 0.84 W/m·°C and $\alpha = 0.39 \times 10^{-6}\ \text{m}^2/\text{s}$.



Analysis The nodal spacing is given to be Δx = 0.2 cm and Δy = 1 cm. The explicit finite difference equations are determined on the basis of the energy balance for the transient case expressed as

$$\sum_{\text{All sides}} \dot{Q}^i + \dot{E}^i_{\text{gen,element}} = \rho \boldsymbol{V}_{\text{element}} c_p \frac{T_m^{i+1} - T_m^i}{\Delta t}$$

We consider only 9 nodes because of symmetry. Note that we do not have a square mesh in this case, and thus we will have to rely on energy balances to obtain the finite difference equations. Using energy balances, the finite difference equations for each of the 9 nodes are obtained as follows:

Node 1: $$h_i \frac{\Delta y}{2}(T_i - T_1^i) + k\frac{\Delta x}{2}\frac{T_4^i - T_1^i}{\Delta y} + k\frac{\Delta y}{2}\frac{T_2^i - T_1^i}{\Delta x} = \rho c_p \frac{\Delta x}{2}\frac{\Delta y}{2}\frac{T_1^{i+1} - T_1^i}{\Delta t}$$

Node 2: $$k\frac{\Delta y}{2}\frac{T_1^i - T_2^i}{\Delta x} + k\frac{\Delta y}{2}\frac{T_3^i - T_2^i}{\Delta x} + k\Delta x\frac{T_5^i - T_2^i}{\Delta y} = \rho c_p \Delta x\frac{\Delta y}{2}\frac{T_2^{i+1} - T_2^i}{\Delta t}$$

Node 3: $$h_o \frac{\Delta y}{2}(T_o - T_3^i) + k\frac{\Delta x}{2}\frac{T_6^i - T_3^i}{\Delta y} + k\frac{\Delta y}{2}\frac{T_2^i - T_3^i}{\Delta x} = \rho c_p \frac{\Delta x}{2}\frac{\Delta y}{2}\frac{T_3^{i+1} - T_3^i}{\Delta t}$$

Node 4: $$h_i \Delta y(T_i - T_4^i) + k\frac{\Delta x}{2}\frac{T_1^i - T_4^i}{\Delta y} + k\frac{\Delta x}{2}\frac{T_7^i - T_4^i}{\Delta y} + k\Delta y\frac{T_5^i - T_4^i}{\Delta x} = \rho c_p \Delta y\frac{\Delta x}{2}\frac{T_4^{i+1} - T_4^i}{\Delta t}$$

Node 5: $$k\Delta y\frac{T_4^i - T_5^i}{\Delta x} + k\Delta y\frac{T_6^i - T_5^i}{\Delta x} + k\Delta x\frac{T_8^i - T_5^i}{\Delta y} + k\Delta x\frac{T_2^i - T_5^i}{\Delta y} = \rho c_p \Delta x \Delta y\frac{T_5^{i+1} - T_5^i}{\Delta t}$$

Node 6: $$h_o \Delta y(T_i - T_6^i) + k\frac{\Delta x}{2}\frac{T_3^i - T_6^i}{\Delta y} + k\frac{\Delta x}{2}\frac{T_9^i - T_6^i}{\Delta y} + k\Delta y\frac{T_5^i - T_6^i}{\Delta x} = \rho c_p \Delta y\frac{\Delta x}{2}\frac{T_6^{i+1} - T_6^i}{\Delta t}$$

Node 7: $$12.5\text{ W} + h_i \frac{\Delta y}{2}(T_i - T_7^i) + k\frac{\Delta x}{2}\frac{T_4^i - T_7^i}{\Delta y} + k\frac{\Delta y}{2}\frac{T_8^i - T_7^i}{\Delta x} = \rho c_p \frac{\Delta x}{2}\frac{\Delta y}{2}\frac{T_7^{i+1} - T_7^i}{\Delta t}$$

Node 8: $$k\frac{\Delta y}{2}\frac{T_7^i - T_8^i}{\Delta x} + k\frac{\Delta y}{2}\frac{T_9^i - T_8^i}{\Delta x} + k\Delta x\frac{T_5^i - T_8^i}{\Delta y} = \rho c_p \Delta x\frac{\Delta y}{2}\frac{T_8^{i+1} - T_8^i}{\Delta t}$$

Node 9: $$h_o \frac{\Delta y}{2}(T_o - T_9^i) + k\frac{\Delta x}{2}\frac{T_6^i - T_9^i}{\Delta y} + k\frac{\Delta y}{2}\frac{T_8^i - T_9^i}{\Delta x} = \rho c_p \frac{\Delta x}{2}\frac{\Delta y}{2}\frac{T_9^{i+1} - T_9^i}{\Delta t}$$

where

k = 0.84 W/m.°C, $\alpha = k/\rho c = 0.39 \times 10^{-6}\ \text{m}^2/\text{s}$, $T_i = T_o$ = -3°C h_i = 6 W/m^2.°C,

h_o = 20 W/m^2.°C, Δx = 0.002 m, and Δy = 0.01 m.

The upper limit of the time step Δt is determined from the stability criteria that requires the coefficient of T_m^i in the T_m^{i+1} expression (the primary coefficient) be greater than or equal to zero for all nodes. The smallest primary coefficient in the 9 equations above is the coefficient of T_6^i in the T_6^{i+1} expression since it is exposed to most convection per unit volume (this can be verified). The equation for node 6 can be rearranged as

$$T_6^{i+1} = \left[1 - 2\alpha\Delta t\left(\frac{h_o}{k\Delta x} + \frac{1}{\Delta y^2} + \frac{1}{\Delta x^2}\right)\right]T_6^i + 2\alpha\Delta t\left(\frac{h_o}{k\Delta x}T_0 + \frac{T_3^i + T_9^i}{\Delta y^2} + \frac{T_5^i}{\Delta x^2}\right)$$

Therefore, the stability criteria for this problem can be expressed as

$$1 - 2\alpha\Delta t\left(\frac{h_o}{k\Delta x} + \frac{1}{\Delta y^2} + \frac{1}{\Delta x^2}\right) \geq 0 \;\rightarrow\; \Delta \mathrm{t} \leq \frac{1}{2\alpha\left(\frac{h_o}{k\Delta x} + \frac{1}{\Delta y^2} + \frac{1}{\Delta x^2}\right)}$$

Substituting the given quantities, the maximum allowable value of the time step is determined to be

or,

$$\Delta \mathrm{t} \leq \frac{1}{2\times(0.39\times10^{6}\ \mathrm{m}^2/s)\left(\frac{20\ \mathrm{W/m^2\cdot{}^\circ C}}{(0.84\ \mathrm{W/m\cdot{}^\circ C})(0.002\ \mathrm{m})} + \frac{1}{(0.002\ \mathrm{m})^2} + \frac{1}{(0.01\ \mathrm{m})^2}\right)} = 4.7\ \mathrm{s}$$

Therefore, any time step less than 4.8 s can be used to solve this problem. For convenience, we choose the time step to be Δt = 4 s. Then the temperature distribution throughout the glass 15 min after the strip heaters are turned on and when steady conditions are reached are determined to be (from the EES solutions in CD)

15 min: T_1 = 10.2°C, T_2 = 10.2°C, T_3 = 9.8°C, T_4 = 16.0°C, T_5 = 15.9°C,

T_6 = 15.3°C, T_7 = 41.1°C, T_8 = 36.8°C, T_9 = 34.3°C

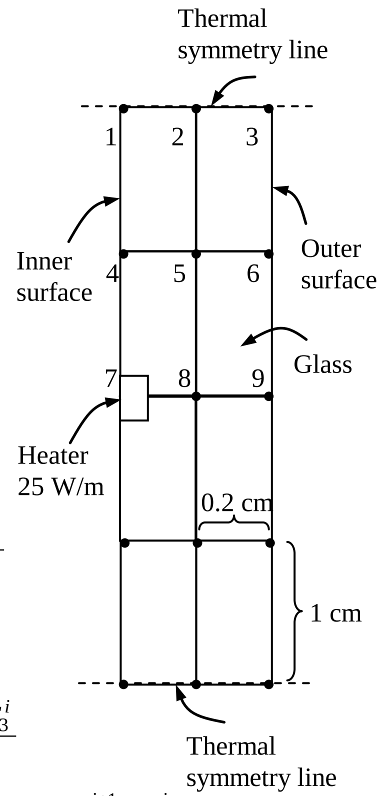
Steady-state: T_1 = 11.8°C, T_2 = 11.7°C, T_3 = 11.3°C, T_4 = 17.6°C, T_5 = 17.5°C,

T_6 = 16.8°C, T_7 = 42.7°C, T_8 = 38.4°C, T_9 = 35.8°C

Discussion Steady operating conditions are reached in about 20 min.

5-117 The formation of fog on the glass surfaces of a car is to be prevented by attaching electric resistance heaters to the inner surfaces. The temperature distribution throughout the glass 15 min after the strip heaters are turned on and also when steady conditions are reached are to be determined using the implicit method with a time step of Δt = 1 min.

Assumptions **1** Heat transfer through the glass is given to be transient and two-dimensional. **2** Thermal conductivity is constant. **3** There is heat generation only at the inner surface, which will be treated as prescribed heat flux.



Properties The conductivity and diffusivity are given to be k = 0.84 W/m·°C and $\alpha = 0.39 \times 10^{-6}\ \text{m}^2/\text{s}$.

Analysis The nodal spacing is given to be Δx = 0.2 cm and Δy = 1 cm. The implicit finite difference equations are determined on the basis of the energy balance for the transient case expressed as

$$\sum_{\text{All sides}} \dot{Q}^{i+1} + \dot{E}^{i+1}_{\text{gen,element}} = \rho \mathcal{V}_{\text{element}} c_p \frac{T_m^{i+1} - T_m^i}{\Delta t}$$

We consider only 9 nodes because of symmetry. Note that we do not have a square mesh in this case, and thus we will have to rely on energy balances to obtain the finite difference equations. Using energy balances, the finite difference equations for each of the 9 nodes are obtained as follows:

Node 1: $$h_i \frac{\Delta y}{2}(T_i - T_1^{i+1}) + k\frac{\Delta x}{2}\frac{T_4^{i+1} - T_1^{i+1}}{\Delta y} + k\frac{\Delta y}{2}\frac{T_2^{i+1} - T_1^{i+1}}{\Delta x} = \rho c_p \frac{\Delta x}{2}\frac{\Delta y}{2}\frac{T_1^{i+1} - T_1^i}{\Delta t}$$

Node 2: $$k\frac{\Delta y}{2}\frac{T_1^{i+1} - T_2^{i+1}}{\Delta x} + k\frac{\Delta y}{2}\frac{T_3^{i+1} - T_2^{i+1}}{\Delta x} + k\Delta x\frac{T_5^{i+1} - T_2^{i+1}}{\Delta y} = \rho c_p \Delta x \frac{\Delta y}{2}\frac{T_2^{i+1} - T_2^i}{\Delta t}$$

Node 3: $$h_o \frac{\Delta y}{2}(T_o - T_3^{i+1}) + k\frac{\Delta x}{2}\frac{T_6^{i+1} - T_3^{i+1}}{\Delta y} + k\frac{\Delta y}{2}\frac{T_2^{i+1} - T_3^{i+1}}{\Delta x} = \rho c_p \frac{\Delta x}{2}\frac{\Delta y}{2}\frac{T_3^{i+1} - T_3^i}{\Delta t}$$

N4: $$h_i \Delta y(T_i - T_4^{i+1}) + k\frac{\Delta x}{2}\frac{T_1^{i+1} - T_4^{i+1}}{\Delta y} + k\frac{\Delta x}{2}\frac{T_7^{i+1} - T_4^{i+1}}{\Delta y} + k\Delta y\frac{T_5^{i+1} - T_4^{i+1}}{\Delta x} = \rho c_p \Delta y \frac{\Delta x}{2}\frac{T_4^{i+1} - T_4^i}{\Delta t}$$

Node 5: $$k\Delta y\frac{T_4^{i+1} - T_5^{i+1}}{\Delta x} + k\Delta y\frac{T_6^{i+1} - T_5^{i+1}}{\Delta x} + k\Delta x\frac{T_8^{i+1} - T_5^{i+1}}{\Delta y} + k\Delta x\frac{T_2^{i+1} - T_5^{i+1}}{\Delta y} = \rho c_p \Delta x \Delta y \frac{T_5^{i+1} - T_5^i}{\Delta t}$$

N6: $$h_o \Delta y(T_i - T_6^{i+1}) + k\frac{\Delta x}{2}\frac{T_3^{i+1} - T_6^{i+1}}{\Delta y} + k\frac{\Delta x}{2}\frac{T_9^{i+1} - T_6^{i+1}}{\Delta y} + k\Delta y\frac{T_5^{i+1} - T_6^{i+1}}{\Delta x} = \rho c_p \Delta y \frac{\Delta x}{2}\frac{T_6^{i+1} - T_6^i}{\Delta t}$$

Node 7: $$12.5\ \text{W} + h_i \frac{\Delta y}{2}(T_i - T_7^{i+1}) + k\frac{\Delta x}{2}\frac{T_4^{i+1} - T_7^{i+1}}{\Delta y} + k\frac{\Delta y}{2}\frac{T_8^{i+1} - T_7^{i+1}}{\Delta x} = \rho c_p \frac{\Delta x}{2}\frac{\Delta y}{2}\frac{T_7^{i+1} - T_7^i}{\Delta t}$$

Node 8: $$k\frac{\Delta y}{2}\frac{T_7^{i+1} - T_8^{i+1}}{\Delta x} + k\frac{\Delta y}{2}\frac{T_9^{i+1} - T_8^{i+1}}{\Delta x} + k\Delta x\frac{T_5^{i+1} - T_8^{i+1}}{\Delta y} = \rho c_p \Delta x \frac{\Delta y}{2}\frac{T_8^{i+1} - T_8^i}{\Delta t}$$

Node 9: $$h_o \frac{\Delta y}{2}(T_o - T_9^{i+1}) + k\frac{\Delta x}{2}\frac{T_6^{i+1} - T_9^{i+1}}{\Delta y} + k\frac{\Delta y}{2}\frac{T_8^{i+1} - T_9^{i+1}}{\Delta x} = \rho c_p \frac{\Delta x}{2}\frac{\Delta y}{2}\frac{T_9^{i+1} - T_9^i}{\Delta t}$$

where k = 0.84 W/m.°C, $\alpha = k/\rho c_p = 0.39 \times 10^{-6}\ \text{m}^2/\text{s}$, $T_i = T_o$ = -3°C h_i = 6 W/m^2.°C, h_o = 20 W/m^2.°C, Δx = 0.002 m, and Δy = 0.01 m. Taking time step to be Δt = 1 min, the temperature distribution throughout the glass 15 min after the strip heaters are turned on and when steady conditions are reached are determined to be (from the EES solutions in the CD)

15 min: T_1 = 10.2°C, T_2 = 10.2°C, T_3 = 9.8°C, T_4 = 16.0°C, T_5 = 15.9°C,
T_6 = 15.3°C, T_7 = 41.1°C, T_8 = 36.8°C, T_9 = 34.3°C

Steady-state: T_1 = 11.8°C, T_2 = 11.7°C, T_3 = 11.3°C, T_4 = 17.6°C, T_5 = 17.5°C,
T_6 = 16.8°C, T_7 = 42.7°C, T_8 = 38.4°C, T_9 = 35.8°C

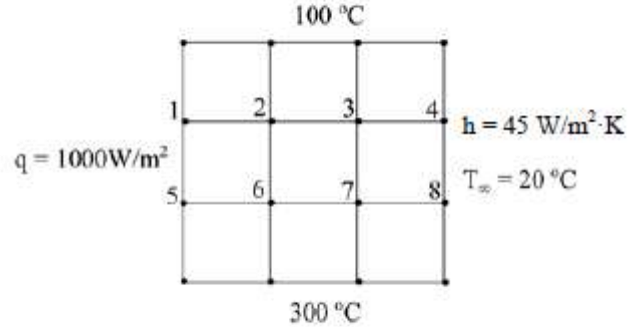
Discussion Steady operating conditions are reached in about 20 min.

5-118 Square cross section geometry is subjected to four different boundary conditions. The transient temperature distribution within the geometry is to be determined after 15 seconds using the explicit finite difference formulation.

Assumptions **1** Two-dimensional transient heat conduction without internal heat generation. **2** Thermal properties are constant.

Properties Thermal conductivity is given as k=20 W/m·K and the thermal diffusivity value is $\alpha = 6.694 \times 10^{-6}$ m^2/s.

Analysis: (a) There are 4 internal nodes (node 2, 3, 6 and 7) and 4 boundary nodes (node 1, 4, 5 and 8). Thus we need to have 8 equations for 8 unknown temperatures. Finite difference equations for internal nodes can be expressed directly using Eq. (5-60) while for the boundary nodes energy balance is carried out around the node volume.



The explicit finite difference formulations for different nodes are as follows.

Node 1: (Left boundary node)

$$\dot{q}\Delta y + k\frac{\Delta x}{2}\frac{\left(T_5^i - T_1^i\right)}{\Delta y} + k\frac{\Delta x}{2}\frac{\left(100 - T_1^i\right)}{\Delta y} + k\Delta y\frac{\left(T_2^i - T_1^i\right)}{\Delta x} = \rho c_p \Delta y\frac{\Delta x}{2}\frac{\left(T_1^{i+1} - T_1^i\right)}{\Delta t}$$

$$\Rightarrow T_1^{i+1} = (1-4\tau)T_1^i + \tau\left(100 + T_5^i + 2T_2^i + \frac{2\dot{q}\Delta y}{k}\right)$$

Node 2: (Interior node) $T_2^{i+1} = (1-4\tau)T_2^i + \tau\left(100 + T_6^i + T_1^i + T_3^i\right)$

Node 3: (Interior node) $T_3^{i+1} = (1-4\tau)T_3^i + \tau\left(100 + T_2^i + T_4^i + T_7^i\right)$

Node 4: (Right boundary node)

$$k\frac{\Delta x}{2}\frac{\left(100 - T_4^i\right)}{\Delta y} + k\frac{\Delta x}{2}\frac{\left(T_8^i - T_4^i\right)}{\Delta y} + k\Delta y\frac{\left(T_3^i - T_4^i\right)}{\Delta x} + h\Delta y\left(T_\infty - T_4^i\right) = \rho c_p \Delta y\frac{\Delta x}{2}\frac{\left(T_4^{i+1} - T_4^i\right)}{\Delta t}$$

$$\Rightarrow T_4^{i+1} = \left(1 - 4\tau - \frac{2\tau\Delta y h}{k}\right)T_4^i + \tau\left(100 + T_8^i + 2T_3^i + \frac{2h\Delta y}{k}T_\infty\right)$$

Node 5: (Left boundary node)

$$\dot{q}\Delta y + k\frac{\Delta x}{2}\frac{\left(T_1^i - T_5^i\right)}{\Delta y} + k\frac{\Delta x}{2}\frac{\left(300 - T_5^i\right)}{\Delta y} + k\Delta y\frac{\left(T_{62}^i - T_1^i\right)}{\Delta x} = \rho c_p \Delta y\frac{\Delta x}{2}\frac{\left(T_5^{i+1} - T_5^i\right)}{\Delta t}$$

$$\Rightarrow T_5^{i+1} = (1-4\tau)T_5^i + \tau\left(300 + T_1^i + 2T_6^i + \frac{2\dot{q}\Delta y}{k}\right)$$

Node 6: (Interior node) $T_6^{i+1} = (1-4\tau)T_6^i + \tau\left(300 + T_5^i + T_2^i + T_7^i\right)$

Node 7: (Interior node) $T_7^{i+1} = (1-4\tau)T_7^i + \tau\left(300 + T_6^i + T_3^i + T_8^i\right)$

Node 8: (Right boundary node)

$$k\frac{\Delta x}{2}\frac{\left(300 - T_8^i\right)}{\Delta y} + k\frac{\Delta x}{2}\frac{\left(T_4^i - T_8^i\right)}{\Delta y} + k\Delta y\frac{\left(T_7^i - T_8^i\right)}{\Delta x} + h\Delta y\left(T_\infty - T_8^i\right) = \rho c_p \Delta y\frac{\Delta x}{2}\frac{\left(T_8^{i+1} - T_8^i\right)}{\Delta t}$$

$$\Rightarrow T_8^{i+1} = \left(1 - 4\tau - \frac{2\tau\Delta y h}{k}\right)T_8^i + \tau\left(300 + T_4^i + 2T_7^i + \frac{2h\Delta y}{k}T_\infty\right)$$

where $\Delta x = \Delta y = 0.01m$, $\mathrm{h = 45(W/m^2 \cdot K)}$, $\mathrm{k = 20(W/m \cdot K)}$ and $T_\infty = 20^o C$

Next we need to determine the upper limit of the time step from the stability criterion which requires the primary coefficient of T_m^i in the expression of T_m^{i+1} to be greater than or equal to zero at all nodes. The smallest primary coefficient is for node at right boundary exposed to convective environment. Thus using the stability criterion we get

$$\tau \le \frac{1}{2(2+h\Delta y/k)} \Rightarrow \Delta t \le \frac{\Delta y^2}{2\alpha(2+h\Delta y/k)}$$

Thus the maximum allowable time step is

$$\Delta t \le \frac{0.01^2(\mathrm{m}^2)}{2\times 6.694\times 10^{-6}(\mathrm{m}^2/\mathrm{s})\left(2+45(\mathrm{W/m^2\cdot K})\times 0.01(\mathrm{m})/20(\mathrm{W/m\cdot K})\right)} \le 3.69(\mathrm{s})$$

For convenience let's select a time step of $\Delta t = 3s$ .This gives a mesh Fourier number of

$$\tau = \frac{\alpha\Delta t}{\Delta y^2} = 0.20082$$

(b) The temperature distribution is obtained by running the following EES code.

```
"Given data"
k = 20 [W/mC] " Thermal conductivity"
h = 45 [W/m^2C] " Convective heat transfer coefficient"
T_infi = 20 [C] " Convective environment temperature"
q_dot = 1000 [W/m^2] " Heat flux at left boundary"
DELTAt = 3 [s] "Time step"
DELTAy = 0.01 [m] " Mesh size"
tau = 0.20082 "Mesh Fourier number"
"The technique is to store the temperatures in the parametric table and recover them (as old temperatures) using the variable ROW.  The first row contains the initial values so Solve Table must begin at row 2. Use the DUPLICATE statement to reduce the number of equations that need to be typed.  Column 1 contains the time, column 2 the value of T[1], column 3, the value of T[2], etc., and column 9 the Row.To start the Solve Table at 2 go to 'Calculate' and select 'Solve table' (or hit F3) and make the 'First Run Number' as 2. The initial temperatures and the initial time '0' can be set manually in the parametric table"

Row = TableRun#
Time = TableValue('Table 1',Row-1,#Time)+DELTAt
        Duplicate i=1,8
                T_old[i] = TableValue('Table 1',Row-1,#T[i])
         End
"Finite difference Equation"
"Node 1" T[1]=(1-4*tau)*T_old[1]+tau*(100+T_old[5]+2*T_old[2]+2*q_dot*DELTAy/k)
"Node 2" T[2] = (1-4*tau)*T_old[2]+tau*(100+T_old[6]+T_old[1]+T_old[3])
"Node 3" T[3] = (1-4*tau)*T_old[3]+tau*(100+T_old[2]+T_old[4]+T_old[7])
"Node4"T[4] = (1-4*tau-2*tau*h*DELTAy/k)*T_old[4]+tau*(100+T_old[8]+2*T_old[3]+2*h*DELTAy*T_infi/k)
"Node 5" T[5] = (1-4*tau)*T_old[5]+tau*(300+T_old[1]+2*T_old[6]+2*q_dot*DELTAy/k)
"Node 6" T[6] = (1-4*tau)*T_old[6]+tau*(300+T_old[5]+T_old[2]+T_old[7])
"Node 7" T[7] = (1-4*tau)*T_old[7]+tau*(300+T_old[6]+T_old[3]+T_old[8])
"Node8"T[8] = (1-4*tau-tau*2*h*DELTAy/k)*T_old[8]+tau*(300+T_old[4]+2*T_old[7]+2*h*DELTAy*T_infi/k)
```

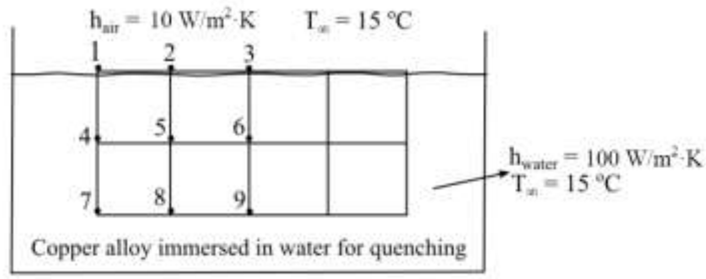
Temperature distribution at different time steps.

	Nodal temperature, ^{o}C							
Time (s)	T_1	T_2	T_3	T_4	T_5	T_6	T_7	T_8
0	200.0	200.0	200.0	200.0	200.0	200.0	200.0	200.0
3	180.0	179.8	179.8	178.2	220.1	219.9	219.9	218.3
6	172.0	171.7	171.4	169.6	228.1	227.8	227.5	225.4
9	168.8	168.4	167.9	166.1	231.2	230.9	230.3	228.0
12	167.4	167.1	166.4	164.5	232.4	232.0	231.2	228.9
15	**166.9**	**166.5**	**165.7**	**163.8**	**232.8**	**232.4**	**231.5**	**229.1**

5-119 A copper alloy with known thermal properties and initially heated to 800°C is subjected to quenching in water with top surface exposed to air environment. The temperature distribution in the copper alloy after 10 min is to be determined using explicit finite difference formulation.

Assumptions **1** Two-dimensional transient heat transfer without internal heat generation. 2 Constant thermal properties 3 The top surface of the copper alloy is always in contact with the air environment.

Properties Thermal conductivity is given as k = 120 W/m·K and the thermal diffusivity value is $\alpha = 3.91 \times 10^{-6}$ m^2/s.



Analysis The nodes 1, 2 and 3 are exposed to air environment while nodes 4, 7, 8 and 9 are in contact with the water environment. Since there is geometrical symmetry we need to find the temperatures for only the nodes shown in figure. Node 5 can be treated as an internal node and for all other nodes, the temperature is found using energy balance at the node volume. The explicit finite difference formulation at different nodes is as follows

Node 1: $$\left(h_1\frac{\Delta x}{2}+h_2\frac{\Delta y}{2}\right)\left(T_\infty - T_1^i\right)+k\frac{\Delta x}{2}\frac{\left(T_4^i-T_1^i\right)}{\Delta y}+k\frac{\Delta y}{2}\frac{\left(T_2^i-T_1^i\right)}{\Delta x}=\rho\frac{\Delta x}{2}\frac{\Delta y}{2}c_p\frac{\left(T_1^{i+1}-T_1^i\right)}{\Delta t}$$

$$\Rightarrow T_1^{i+1}=\left(1-4\tau-2\tau\frac{(h_1+h_2)l}{k}\right)T_1^i+2\tau\left(T_4^i+T_2^i+\frac{l}{k}(h_1+h_2)T_\infty\right)$$

Node 2: $$h_1\Delta x\left(T_\infty-T_2^i\right)+k\frac{\Delta y}{2}\frac{\left(T_1^i-T_2^i\right)}{\Delta x}+k\frac{\Delta y}{2}\frac{\left(T_3^i-T_2^i\right)}{\Delta x}+k\Delta x\frac{\left(T_5^i-T_2^i\right)}{\Delta y}=\rho\Delta x\frac{\Delta y}{2}c_p\frac{\left(T_2^{i+1}-T_2^i\right)}{\Delta t}$$

$$\Rightarrow T_2^{i+1}=\left(1-4\tau-2\tau\frac{h_1 l}{k}\right)T_2^i+\tau\left(T_1^i+T_3^i+2T_5^i+2\frac{h_1 l}{k}T_\infty\right)$$

Node 3: $$h_1\Delta x\left(T_\infty-T_3^i\right)+2k\frac{\Delta y}{2}\frac{\left(T_2^i-T_3^i\right)}{\Delta x}+k\Delta x\frac{\left(T_6^i-T_3^i\right)}{\Delta y}=\rho\Delta x\frac{\Delta y}{2}c_p\frac{\left(T_3^{i+1}-T_3^i\right)}{\Delta t}$$

$$\Rightarrow T_3^{i+1}=\left(1-4\tau-2\tau\frac{h_1 l}{k}\right)T_3^i+\tau\left(2T_2^i+2T_6^i+2\frac{h_1 l}{k}T_\infty\right)$$

Node 4: $$h_2\Delta y\left(T_\infty-T_4^i\right)+k\frac{\Delta x}{2}\frac{\left(T_1^i-T_4^i\right)}{\Delta y}+k\frac{\Delta x}{2}\frac{\left(T_7^i-T_4^i\right)}{\Delta y}+k\Delta y\frac{\left(T_5^i-T_4^i\right)}{\Delta x}=\rho\Delta y\frac{\Delta x}{2}c_p\frac{\left(T_4^{i+1}-T_4^i\right)}{\Delta t}$$

$$\Rightarrow T_4^{i+1}=\left(1-4\tau-2\tau\frac{h_2 l}{k}\right)T_4^i+\tau\left(T_1^i+T_7^i+2T_5^i+2\frac{h_2 l}{k}T_\infty\right)$$

Node 5: $$T_4^i+T_2^i+T_6^i+T_8^i-4T_5^i=\rho\Delta x\Delta y c_p\frac{\left(T_5^{i+1}-T_5^i\right)}{\Delta t}$$

$$\Rightarrow T_5^{i+1}=(1-4\tau)T_5^i+\tau\left(T_2^i+T_8^i+T_6^i+T_4^i\right)$$

Node 6: $$k\Delta x\frac{\left(T_9^i-T_6^i\right)}{\Delta y}+2k\Delta y\frac{\left(T_5^i-T_6^i\right)}{\Delta x}+k\Delta x\frac{\left(T_3^i-T_6^i\right)}{\Delta y}=\rho\Delta x\Delta y c_p\frac{\left(T_6^{i+1}-T_6^i\right)}{\Delta t}$$

$$\Rightarrow T_6^{i+1}=(1-4\tau)T_6^i+\tau\left(2T_5^i+T_3^i+T_9^i\right)$$

Node 7: $$h_2\left(\frac{\Delta y}{2}+\frac{\Delta x}{2}\right)\left(T_\infty-T_7^i\right)+k\frac{\Delta x}{2}\frac{\left(T_4^i-T_7^i\right)}{\Delta y}+k\frac{\Delta y}{2}\frac{\left(T_8^i-T_7^i\right)}{\Delta y}=\rho\frac{\Delta y}{2}\frac{\Delta x}{2}c_p\frac{\left(T_7^{i+1}-T_7^i\right)}{\Delta t}$$

$$\Rightarrow T_7^{i+1}=\left(1-4\tau-4\tau\frac{h_2 l}{k}\right)T_7^i+2\tau\left(T_4^i+T_8^i+2\frac{h_2 l}{k}T_\infty\right)$$

Node 8: $h_2\Delta x\left(T_\infty - T_8^i\right) + k\dfrac{\Delta y}{2}\dfrac{\left(T_7^i - T_8^i\right)}{\Delta x} + k\dfrac{\Delta y}{2}\dfrac{\left(T_9^i - T_8^i\right)}{\Delta x} + k\Delta x\dfrac{\left(T_5^i - T_8^i\right)}{\Delta y} = \rho\dfrac{\Delta y}{2}\Delta x c_p\dfrac{\left(T_8^{i+1} - T_8^i\right)}{\Delta t}$

$$\Rightarrow T_8^{i+1} = \left(1 - 4\tau - 2\tau\frac{h_2 l}{k}\right)T_8^i + \tau\left(2T_5^i + T_7^i + T_9^i + 2\frac{h_2 l}{k}T_\infty\right)$$

Node 9: $h_2\Delta x\left(T_\infty - T_9^i\right) + 2k\dfrac{\Delta y}{2}\dfrac{\left(T_8^i - T_9^i\right)}{\Delta x} + k\Delta x\dfrac{\left(T_6^i - T_9^i\right)}{\Delta y} = \rho\dfrac{\Delta y}{2}\Delta x c_p\dfrac{\left(T_9^{i+1} - T_9^i\right)}{\Delta t}$

$$\Rightarrow T_9^{i+1} = \left(1 - 4\tau - 2\tau\frac{h_2 l}{k}\right)T_9^i + \tau\left(2T_6^i + 2T_8^i + 2\frac{h_2 l}{k}T_\infty\right)$$

The unknown nodal temperatures are found by running the following EES program.

```
"Given"
l = 0.1 [m] "mesh size"
alpha = 3.91e-6 "Thermal diffusivity [m^2/s]"
DELTAt = 10 [s] "Time step"
k = 120 [W/moC] "Thermal conductivity"
h_1 = 10 [W/m^2oC] "Convective heat transfer coefficient of air"
h_2 = 100 [W/m^2oC] "Convective heat transfer coefficient of water"
T_infi = 15 [C] " Ambient temperature"

tau = (alpha*DELTAt)/l^2 "Mesh Fourier number"

"The technique is to store the temperatures in the parametric table and recover them (as old temperatures) using the variable ROW.  The first row contains the initial values so Solve Table must begin at row 2. Use the DUPLICATE statement to reduce the number of equations that need to be typed.  Column 1 contains the time, column 2 the value of T[1], column 3, the value of T[2], etc., and column 9 the Row.

To start the Solve Table at 2 go to 'Calculate' and select 'Solve table' (or hit F3) and make the 'First Run Number' as 2. The initial temperatures and the initial time '0' can be set manually in the parametric table"

Row = TableRun#
Time = TableValue('Table 1',Row-1,#Time)+DELTAt
        Duplicate i=1,9
                T_old[i] = TableValue('Table 1',Row-1,#T[i])
         end

"Finite difference formulation"
"Node 1" T[1] = (1-4*tau-2*tau*(h_1+h_2)*l/k)*T_old[1]+2*tau*(T_old[4]+T_old[2]+(h_1+h_2)*l*T_infi/k)
"Node 2" T[2] = (1-4*tau-2*tau*h_1*l/k)*T_old[2]+tau*(T_old[1]+T_old[3]+2*T_old[5]+2*h_1*l*T_infi/k)
"Node 3" T[3] = (1-4*tau-2*tau*h_1*l/k)*T_old[3]+tau*(2*T_old[2]+2*T_old[6]+2*h_1*l*T_infi/k)
"Node 4" T[4] = (1-4*tau-2*tau*h_2*l/k)*T_old[4] + tau*(T_old[1]+T_old[7]+2*T_old[5]+2*h_2*l*T_infi/k)
"Node 5" T[5] = (1-4*tau)*T_old[5]+tau*(T_old[2]+T_old[4]+T_old[6]+T_old[8])
"Node 6" T[6] = (1-4*tau)*T_old[6]+tau*(2*T_old[5]+T_old[3]+T_old[9])
"Node 7" T[7] = (1-4*tau-4*tau*h_2*l/k)*T_old[7]+2*tau*(T_old[4]+T_old[8]+2*h_2*l*T_infi/k)
"Node 8" T[8] = (1-4*tau-2*tau*h_2*l/k)*T_old[8]+tau*(T_old[7]+T_old[9]+2*T_old[5]+2*h_2*l*T_infi/k)
"Node 9" T[9] = (1-4*tau-2*tau*h_2*l/k)*T_old[9]+tau*(2*T_old[8]+2*T_old[6]+2*h_2*l*T_infi/k)
```

Temperature distribution in the copper alloy block after 10 min is as follows,

T_1 = **772.7°C**, T_2 = **794.5°C**, T_3 = **796.7°C**, T_4 = **772.7°C**, T_5 = **794.5°C**,

T_6 = **796.7°C**, T_7 = **751.7°C**, T_8 = **772.9°C**, T_9 = **775.1°C**.

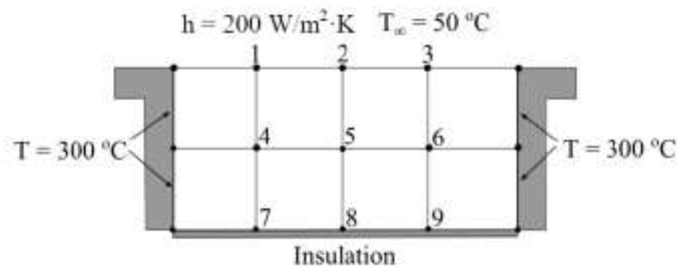
Discussion If the mass and/or heat capacity of the fluid used for quenching is not large enough then the heat released by the metal parts is absorbed by the fluid thus raising its temperature considerably. In such a case constant temperature of the convective environment may not be a good approximation as it will alter the rate of heat transfer by convection. Note that thermal diffusivity of the material dictates the temperature distribution and the quenching time to attain specified temperature. Use of flowing water (forced convective cooling) instead of stagnant water bath may further drop the temperature of copper alloy significantly.

5-120 A ceramic strip at a specified initial temperature is exposed to convective environment at top and constant temperature boundary conditions at its two sides. Using implicit finite difference formulation, the temperature distribution in the ceramic strip is to be determined after 12 seconds.

Assumptions **1** Two-dimensional transient heat conduction without heat generation. **2** Thermal properties of the ceramic strip stays constant.

Properties Thermal properties of ceramic strip are given as: k = 3 W/m·K, ρ = 1600 kg/m^3 and c_p = 800 J/kg·K.

Analysis The finite difference equation at each node is developed by doing an energy balance assuming all heat transfer is to the control volume of the node under consideration. Nodes 1, 2 and 3 are the boundary nodes exposed to the convection environment. Nodes 7, 8 and 9 are the boundary nodes at the insulated surface. The implicit finite difference formulation for all the nodes is as follows



Node 1: $$h\Delta x\left(T_\infty - T_1^{i+1}\right) + k\frac{\Delta y}{2}\frac{\left(300 - T_1^{i+1}\right)}{\Delta x} + k\frac{\Delta y}{2}\frac{\left(T_2^{i+1} - T_1^{i+1}\right)}{\Delta x} + k\Delta x\frac{\left(T_4^{i+1} - T_1^{i+1}\right)}{\Delta y} = \rho c_p \Delta x \frac{\Delta y}{2}\frac{\left(T_1^{i+1} - T_1^{i}\right)}{\Delta t}$$

Node 2: $$h\Delta x\left(T_\infty - T_2^{i+1}\right) + 2k\frac{\Delta y}{2}\frac{\left(T_1^{i+1} - T_2^{i+1}\right)}{\Delta x} + k\Delta x\frac{\left(T_5^{i+1} - T_1^{i+1}\right)}{\Delta y} = \rho c_p \Delta x \frac{\Delta y}{2}\frac{\left(T_2^{i+1} - T_2^{i}\right)}{\Delta t}$$

Node 4: $$k\Delta y\frac{\left(300 - T_4^{i+1}\right)}{\Delta x} + k\Delta y\frac{\left(T_5^{i+1} - T_4^{i+1}\right)}{\Delta x} + k\Delta x\frac{\left(T_1^{i+1} - T_4^{i+1}\right)}{\Delta y} + k\Delta x\frac{\left(T_7^{i+1} - T_4^{i+1}\right)}{\Delta y} = \rho c_p \Delta x \Delta y\frac{\left(T_4^{i+1} - T_4^{i}\right)}{\Delta t}$$

Node 5: $$k\Delta x\frac{\left(T_2^{i+1} - T_5^{i+1}\right)}{\Delta y} + k\Delta x\frac{\left(T_8^{i+1} - T_5^{i+1}\right)}{\Delta y} + 2k\Delta y\frac{\left(T_4^{i+1} - T_5^{i+1}\right)}{\Delta x} = \rho c_p \Delta x \Delta y\frac{\left(T_5^{i+1} - T_5^{i}\right)}{\Delta t}$$

Node 7: $$k\frac{\Delta y}{2}\frac{\left(300 - T_7^{i+1}\right)}{\Delta x} + k\frac{\Delta y}{2}\frac{\left(T_8^{i+1} - T_7^{i+1}\right)}{\Delta x} + 2k\Delta x\frac{\left(T_4^{i+1} - T_7^{i+1}\right)}{\Delta y} = \rho c_p \Delta x \Delta y\frac{\left(T_7^{i+1} - T_7^{i}\right)}{\Delta t}$$

Node 8: $$2k\frac{\Delta y}{2}\frac{\left(T_7^{i+1} - T_8^{i+1}\right)}{\Delta x} + 2k\Delta x\frac{\left(T_5^{i+1} - T_8^{i+1}\right)}{\Delta y} = \rho c_p \Delta x \Delta y\frac{\left(T_8^{i+1} - T_8^{i}\right)}{\Delta t}$$

The unknown temperatures at theses nodes are determined by running the following EES code.

```
"Given data"
k = 3 [W/mC] "Thermal conductivity of ceramic strip"
h = 200 [W/m^2C] "Convective heat transfer coefficient"
T_infi = 50 [C] " Ambient temperature"
DELTAx = 0.01 [m] "Mesh size in x direction"
DELTAy = 0.01 [m] "Mesh size in y direction"
DELTAt = 2 [s] "Time step"
rho = 1600 [kg/m^3] "Density of ceramic strip"
c = 800 [J/kgK] " Specific heat capacity of ceramic strip"
alpha = k/(rho*c)  "Thermal diffusivity of ceramic strip [m^2/s]"
tau = DELTAx^2/(alpha*DELTAt)"Mesh Fourier number"

"The technique is to store the temperatures in the parametric table and recover them (as old temperatures) using the variable ROW.  The first row contains the initial values so Solve Table must begin at row 2. Use the DUPLICATE statement to reduce the number of equations that need to be typed.  Column 1 contains the time, column 2 the value of T[1], column 3, the value of T[2], etc., and column 9 the Row.
```

To start the Solve Table at 2 go to 'Calculate' and select 'Solve table' (or hit F3) and make the 'First Run Number' as 2. The initial temperatures and the initial time ‘0’ can be set manually in the parametric table"

```
Row = TableRun#
Time = TableValue('Table 1',Row-1,#Time)+DELTAt
Duplicate i = 1,9
T_old[i] = TableValue('Table 1', Row-1,#T[i])
End
 "Implicit finite difference formulation"
"Node 1"
h*DELTAx*(T_infi-T[1])+k*DELTAy/2*(300-T[1])/DELTAx+k*DELTAy/2*(T[2]-T[1])/DELTAx+k*DELTAx*(T[4]-
T[1])/DELTAy = rho*c*DELTAx*DELTAy/(2*DELTAt)*(T[1]-T_old[1])
"Node 2"
h*DELTAx*(T_infi-T[2])+k*DELTAy/2*(T[1]-T[2])/DELTAx+k*DELTAy/2*(T[3]-T[2])/DELTAx+k*DELTAx*(T[5]-
T[2])/DELTAy = rho*c*DELTAx*DELTAy/(2*DELTAt)*(T[2]-T_old[2])
"Node 4"
k*DELTAy*(300-T[4])/DELTAx+k*DELTAx*(T[1]-T[4])/DELTAy+k*DELTAy*(T[5]-T[4])/DELTAx+k*DELTAx*(T[7]-
T[4])/DELTAy = rho*c*DELTAx*DELTAy/DELTAt*(T[4]-T_old[4])
"Node 5"
k*DELTAy*(T[4]-T[5])/DELTAx+k*DELTAy*(T[6]-T[5])/DELTAx+k*DELTAx*(T[2]-T[5])/DELTAy+k*DELTAx*(T[8]-
T[5])/DELTAy = rho*c*DELTAx*DELTAy/DELTAt*(T[5]-T_old[5])
"Node 7"
k*DELTAy/2*(300-T[7])/DELTAx+k*DELTAy/2*(T[8]-T[7])/DELTAx+2*k*DELTAx*(T[4]-T[7])/DELTAy =
rho*c*DELTAx*DELTAy/(1*DELTAt)*(T[7]-T_old[7])
"Node 8"
k*DELTAy/2*(T[7]-T[8])/DELTAx+k*DELTAy/2*(T[9]-T[8])/DELTAx+2*k*DELTAx*(T[5]-T[8])/DELTAy =
rho*c*DELTAx*DELTAy/(1*DELTAt)*(T[8]-T_old[8])

"Due to symmetry"
T[3] = T[1]
T[6] = T[4]
T[9] = T[7]
```

Temperature distribution in the ceramic strip after 12 seconds is as follows

T_1 = **246.6°C,** T_2 = **241.3°C,** T_3 = **246.6°C,** T_4 = **292.8°C,** T_5 = **291.6°C,** T_6 = **292.8°C,** T_7 = **298.4°C,**

T_8 = **298.1°C,** T_9 = **298.4°C.**

Due to symmetry we have,

$T_1 = T_3$, $T_4 = T_6$ and $T_7 = T_9$

Discussion Usually the ceramic strips in electronic components are embedded into high thermal conductivity material. In such cases the temperature drop at their interface must be accounted.

Special Topic: Controlling the Numerical Error

5-121C The results obtained using a numerical method differ from the exact results obtained analytically because the results obtained by a numerical method are approximate. The difference between a numerical solution and the exact solution (the error) is primarily due to two sources: The *discretization error* (also called the *truncation* or *formulation* error) which is caused by the approximations used in the formulation of the numerical method, and the *round-off error* which is caused by the computers' representing a number by using a limited number of significant digits and continuously rounding (or chopping) off the digits it cannot retain.

5-122C The *discretization error* (also called the *truncation* or *formulation* error) is due to replacing the derivatives by differences in each step, or replacing the actual temperature distribution between two adjacent nodes by a straight line segment. The difference between the two solutions at each time step is called the *local discretization error*. The total discretization error at any step is called the *global* or *accumulated discretization error*. The local and global discretization errors are identical for the first time step.

5-123C Yes, the global (accumulated) discretization error be less than the local error during a step. The global discretization error usually increases with increasing number of steps, but the opposite may occur when the solution function changes direction frequently, giving rise to local discretization errors of opposite signs which tend to cancel each other.

5-124C The Taylor series expansion of the temperature at a specified nodal point m about time t_i is

$$T(x_m, t_i + \Delta t) = T(x_m, t_i) + \Delta t \frac{\partial T(x_m, t_i)}{\partial t} + \frac{1}{2}\Delta t^2 \frac{\partial^2 T(x_m, t_i)}{\partial t^2} + \cdots$$

The finite difference formulation of the time derivative at the same nodal point is expressed as

$$\frac{\partial T(x_m, t_i)}{\partial t} \cong \frac{T(x_m, t_i + \Delta t) - T(x_m, t_i)}{\Delta t} = \frac{T_m^{i+1} - T_m^i}{\Delta t} \quad \text{or} \quad T(x_m, t_i + \Delta t) \cong T(x_m, t_i) + \Delta t \frac{\partial T(x_m, t_i)}{\partial t}$$

which resembles the Taylor series expansion terminated after the first two terms.

5-125C The Taylor series expansion of the temperature at a specified nodal point *m* about time t_i is

$$T(x_m, t_i + \Delta t) = T(x_m, t_i) + \Delta t \frac{\partial T(x_m, t_i)}{\partial t} + \frac{1}{2}\Delta t^2 \frac{\partial^2 T(x_m, t_i)}{\partial t^2} + \cdots$$

The finite difference formulation of the time derivative at the same nodal point is expressed as

$$\frac{\partial T(x_m, t_i)}{\partial t} \cong \frac{T(x_m, t_i + \Delta t) - T(x_m, t_i)}{\Delta t} = \frac{T_m^{i+1} - T_m^i}{\Delta t} \quad \text{or} \quad T(x_m, t_i + \Delta t) \cong T(x_m, t_i) + \Delta t \frac{\partial T(x_m, t_i)}{\partial t}$$

which resembles the Taylor series expansion terminated after the first two terms. Therefore, the 3rd and following terms in the Taylor series expansion represent the error involved in the finite difference approximation. For a sufficiently small time step, these terms decay rapidly as the order of derivative increases, and their contributions become smaller and smaller. The first term neglected in the Taylor series expansion is proportional to $(\Delta t)^2$, and thus the local discretization error is also proportional to $(\Delta t)^2$.

The global discretization error is proportional to the step size to Δt itself since, at the worst case, the accumulated discretization error after I time steps during a time period t_0 is $I\Delta t^2 = (t_0 / \Delta t)\Delta t^2 = t_0 \Delta t$ which is proportional to Δt.

5-126C The *round-off error* is caused by retaining a limited number of digits during calculations. It depends on the number of calculations, the method of rounding off, the type of the computer, and even the sequence of calculations. Calculations that involve the alternate addition of small and large numbers are most susceptible to round-off error.

5-127C As the step size is decreased, the discretization error decreases but the round-off error increases.

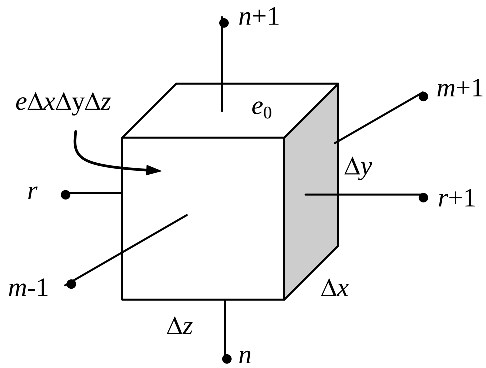
5-128C The round-off error can be reduced by avoiding extremely small mess sizes (smaller than necessary to keep the discretization error in check) and sequencing the terms in the program such that the addition of small and large numbers is avoided.

5-129C A practical way of checking if the round-off error has been significant in calculations is to repeat the calculations using double precision holding the mesh size and the size of the time step constant. If the changes are not significant, we conclude that the round-off error is not a problem.

5-130C A practical way of checking if the discretization error has been significant in calculations is to start the calculations with a reasonable mesh size Δx (and time step size Δt for transient problems), based on experience, and then to repeat the calculations using a mesh size of $\Delta x/2$. If the results obtained by halving the mesh size do not differ significantly from the results obtained with the full mesh size, we conclude that the discretization error is at an acceptable level.

Review Problems

5-131 Starting with an energy balance on a volume element, the steady three-dimensional finite difference equation for a general interior node in rectangular coordinates for $T(x, y, z)$ for the case of constant thermal conductivity and uniform heat generation is to be obtained.



Analysis We consider a *volume element* of size $\Delta x \times \Delta y \times \Delta z$ centered about a general interior node (m, n, r) in a region in which heat is generated at a constant rate of $\dot{e}_0$ and the thermal conductivity k is variable. Assuming the direction of heat conduction to be *towards* the node under consideration at all surfaces, the energy balance on the volume element can be expressed as

$$\dot{Q}_{\text{cond,left}} + \dot{Q}_{\text{cond,top}} + \dot{Q}_{\text{cond,right}} + \dot{Q}_{\text{cond,bottom}} + \dot{Q}_{\text{cond,front}} + \dot{Q}_{\text{cond,back}} + \dot{E}_{\text{gen,element}} = \frac{\Delta E_{\text{element}}}{\Delta t} = 0$$

for the *steady* case. Again assuming the temperatures between the adjacent nodes to vary linearly, the energy balance relation above becomes

$$k(\Delta y \times \Delta z)\frac{T_{m-1,n,r} - T_{m,n,r}}{\Delta x} + k(\Delta x \times \Delta z)\frac{T_{m,n+1,r} - T_{m,n,r}}{\Delta y}$$
$$+ k(\Delta y \times \Delta z)\frac{T_{m+1,n,r} - T_{m,n,r}}{\Delta x} + k(\Delta x \times \Delta z)\frac{T_{m,n-1,r} - T_{m,n,r}}{\Delta y}$$
$$+ k(\Delta x \times \Delta y)\frac{T_{m,n,r-1} - T_{m,n,r}}{\Delta z} + k(\Delta x \times \Delta y)\frac{T_{m,n,r+1} - T_{m,n,r}}{\Delta z} + \dot{e}_0(\Delta x \times \Delta y \times \Delta z) = 0$$

Dividing each term by $k\,\Delta x \times \Delta y \times \Delta z$ and simplifying gives

$$\frac{T_{m-1,n,r} - 2T_{m,n,r} + T_{m+1,n,r}}{\Delta x^2} + \frac{T_{m,n-1,r} - 2T_{m,n,r} + T_{m,n+1,r}}{\Delta y^2} + \frac{T_{m,n,r-1} - 2T_{m,n,r} + T_{m,n,r+1}}{\Delta z^2} + \frac{\dot{e}_0}{k} = 0$$

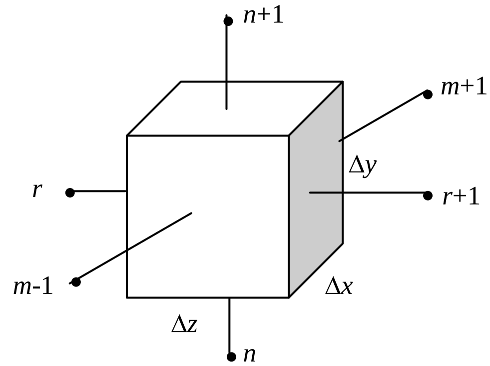
For a cubic mesh with $\Delta x = \Delta y = \Delta z = l$, and the relation above simplifies to

$$T_{m-1,n,r} + T_{m+1,n,r} + T_{m,n-1,r} + T_{m,n-1,r} + T_{m,n,r-1} + T_{m,n,r+1} - 6T_{m,n,r} + \frac{\dot{e}_0 l^2}{k} = 0$$

It can also be expressed in the following easy-to-remember form:

$$T_{\text{left}} + T_{\text{top}} + T_{\text{right}} + T_{\text{bottom}} + T_{\text{front}} + T_{\text{back}} - 6T_{\text{node}} + \frac{\dot{e}_0 l^2}{k} = 0$$

5-132 Starting with an energy balance on a volume element, the three-dimensional transient *explicit* finite difference equation for a general interior node in rectangular coordinates for $T(x, y, z, t)$ for the case of constant thermal conductivity k and no heat generation is to be obtained.



Analysis We consider a rectangular region in which heat conduction is significant in the x and y directions. There is no heat generation in the medium, and the thermal conductivity k of the medium is constant. Now we divide the x-y-z region into a *mesh* of nodal points which are spaced Δx, Δy, and Δz apart in the x, y, and z directions, respectively, and consider a general interior node (m, n, r) whose coordinates are $x = m\Delta x$, $y = n\Delta y$, are $z = r\Delta z$. Noting that the volume element centered about the general interior node (m, n, r) involves heat conduction from six sides (right, left, front, rear, top, and bottom) and expressing them at previous time step i, the transient explicit finite difference formulation for a general interior node can be expressed as

$$k(\Delta y\times\Delta z)\frac{T^i_{m-1,n,r}-T^i_{m,n,r}}{\Delta x}+k(\Delta x\times\Delta z)\frac{T^i_{m,n+1,r}-T^i_{m,n,r}}{\Delta y}+k(\Delta y\times\Delta z)\frac{T^i_{m+1,n,r}-T^i_{m,n,r}}{\Delta x}$$

$$+k(\Delta x\times\Delta z)\frac{T^i_{m,n-1,r}-T^i_{m,n,r}}{\Delta y}+k(\Delta x\times\Delta y)\frac{T^i_{m,n,r-1}-T^i_{m,n,r}}{\Delta z}+k(\Delta x\times\Delta y)\frac{T^i_{m,n,r+1}-T^i_{m,n,r}}{\Delta z}$$

$$=\rho(\Delta x\times\Delta y\times 1)c\frac{T^{i+1}_{m,n}-T^i_{m,n}}{\Delta t}$$

Taking a cubic mesh ($\Delta x = \Delta y = \Delta z = l$) and dividing each term by k gives, after simplifying,

$$T^i_{m-1,n,r}+T^i_{m+1,n,r}+T^i_{m,n+1,r}+T^i_{m,n-1,r}+T^i_{m,n,r-1}+T^i_{m,n,r+1}-6T^i_{m,n,r}=\frac{T^{i+1}_{m,n,r}-T^i_{m,n,r}}{\tau}$$

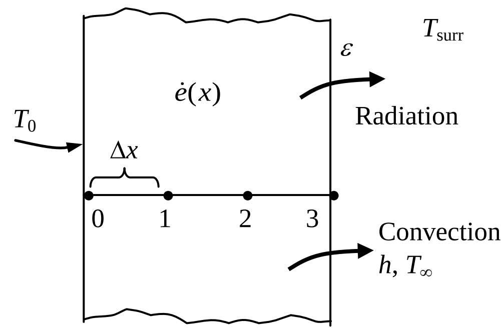
where $\alpha = k/\rho c$ is the thermal diffusivity of the material and $\tau = \alpha\Delta t/l^2$ is the dimensionless mesh Fourier number. It can also be expressed in terms of the temperatures at the neighboring nodes in the following easy-to-remember form:

$$T^i_{\text{left}}+T^i_{\text{top}}+T^i_{\text{right}}+T^i_{\text{bottom}}+T^i_{\text{front}}+T^i_{\text{back}}-6T^i_{\text{node}}=\frac{T^{i+1}_{\text{node}}-T^i_{\text{node}}}{\tau}$$

Discussion We note that setting $T^{i+1}_{\text{node}} = T^i_{\text{node}}$ gives the steady finite difference formulation.

5-133 A plane wall with variable heat generation and constant thermal conductivity is subjected to combined convection and radiation at the right (node 3) and specified temperature at the left boundary (node 0). The finite difference formulation of the right boundary node (node 3) and the finite difference formulation for the rate of heat transfer at the left boundary (node 0) are to be determined.

Assumptions **1** Heat transfer through the wall is given to be steady and one-dimensional. **2** The thermal conductivity is given to be constant.



Analysis Using the energy balance approach and taking the direction of all heat transfers to be towards the node under consideration, the finite difference formulations become

Right boundary node (all temperatures are in K):

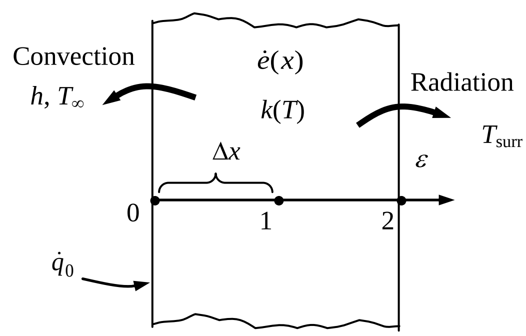
$$\varepsilon\sigma A(T_{\text{surr}}^4 - T_3^4) + hA(T_\infty - T_3) + kA\frac{T_2 - T_3}{\Delta x} + \dot{e}_3(A\Delta x/2) = 0$$

Heat transfer at left surface:

$$\dot{Q}_{\text{left surface}} + kA\frac{T_1 - T_0}{\Delta x} + \dot{e}_0(A\Delta x/2) = 0$$

5-134 A plane wall with variable heat generation and variable thermal conductivity is subjected to uniform heat flux $\dot{q}_0$ and convection at the left (node 0) and radiation at the right boundary (node 2). The explicit transient finite difference formulation of the problem using the energy balance approach method is to be determined.

Assumptions **1** Heat transfer through the wall is given to be transient, and the thermal conductivity and heat generation to be variables. **2** Heat transfer is one-dimensional since the plate is large relative to its thickness. **3** Radiation from the left surface, and convection from the right surface are negligible.



Analysis Using the energy balance approach and taking the direction of all heat transfers to be towards the node under consideration, the *explicit* finite difference formulations become

Left boundary node (node 0):

$$k_0^i A\frac{T_1^i - T_0^i}{\Delta x} + \dot{q}_0 A + hA(T_\infty - T_0^i) + \dot{e}_0^i(A\Delta x/2) = \rho A\frac{\Delta x}{2}c_p\frac{T_0^{i+1} - T_0^i}{\Delta t}$$

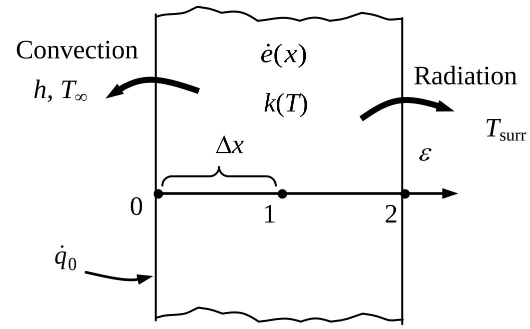
Interior node (node 1):

$$k_1^i A\frac{T_0^i - T_1^i}{\Delta x} + k_1^i A\frac{T_2^i - T_1^i}{\Delta x} + \dot{e}_1^i(A\Delta x) = \rho A\Delta x c_p\frac{T_1^{i+1} - T_1^i}{\Delta t}$$

Right boundary node (node 2):

$$k_2^i A\frac{T_1^i - T_2^i}{\Delta x} + \varepsilon\sigma A[(T_{surr}^i + 273)^4 - (T_2^i + 273)^4] + \dot{e}_2^i(A\Delta x/2) = \rho A\frac{\Delta x}{2}c_p\frac{T_2^{i+1} - T_2^i}{\Delta t}$$

5-135 A plane wall with variable heat generation and variable thermal conductivity is subjected to uniform heat flux $\dot{q}_0$ and convection at the left (node 0) and radiation at the right boundary (node 2). The implicit transient finite difference formulation of the problem using the energy balance approach method is to be determined.



Assumptions **1** Heat transfer through the wall is given to be transient, and the thermal conductivity and heat generation to be variables. **2** Heat transfer is one-dimensional since the plate is large relative to its thickness. **3** Radiation from the left surface, and convection from the right surface are negligible.

Analysis Using the energy balance approach and taking the direction of all heat transfers to be towards the node under consideration, the *implicit* finite difference formulations become

Left boundary node (node 0):

$$k_0^{i+1}A\frac{T_1^{i+1}-T_0^{i+1}}{\Delta x}+\dot{q}_0A+hA(T_\infty-T_0^{i+1})+\dot{e}_0^{i+1}(A\Delta x/2)=\rho A\frac{\Delta x}{2}c_p\frac{T_0^{i+1}-T_0^i}{\Delta t}$$

Interior node (node 1):

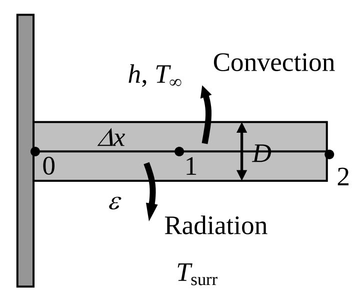
$$k_1^{i+1}A\frac{T_0^{i+1}-T_1^{i+1}}{\Delta x}+k_1^{i+1}A\frac{T_2^{i+1}-T_1^{i+1}}{\Delta x}+\dot{e}_1^{i+1}(A\Delta x)=\rho A\Delta x c_p\frac{T_1^{i+1}-T_1^i}{\Delta t}$$

Right boundary node (node 2):

$$k_2^{i+1}A\frac{T_1^{i+1}-T_2^{i+1}}{\Delta x}+\varepsilon\sigma A[(T_{surr}^{i+1}+273)^4-(T_2^{i+1}+273)^4]+\dot{e}_2^{i+1}(A\Delta x/2)=\rho A\frac{\Delta x}{2}c_p\frac{T_2^{i+1}-T_2^i}{\Delta t}$$

5-136 A pin fin with convection and radiation heat transfer from its tip is considered. The complete finite difference formulation for the determination of nodal temperatures is to be obtained.

Assumptions **1** Heat transfer through the pin fin is given to be steady and one-dimensional, and the thermal conductivity to be constant. **2** Convection heat transfer coefficient and emissivity are constant and uniform.



Assumptions **1** Heat transfer through the wall is given to be steady and one-dimensional, and the thermal conductivity and heat generation to be variable. **2** Convection heat transfer at the right surface is negligible.

Analysis The nodal network consists of 3 nodes, and the base temperature T_0 at node 0 is specified. Therefore, there are two unknowns T_1 and T_2, and we need two equations to determine them. Using the energy balance approach and taking the direction of all heat transfers to be towards the node under consideration, the finite difference formulations become

Node 1 (at midpoint):

$$kA\frac{T_0-T_1}{\Delta x}+kA\frac{T_2-T_1}{\Delta x}+h(p\Delta x)(T_\infty-T_1)+\varepsilon\sigma(p\Delta x)[T_{\text{surr}}^4-(T_1+273)^4]=0$$

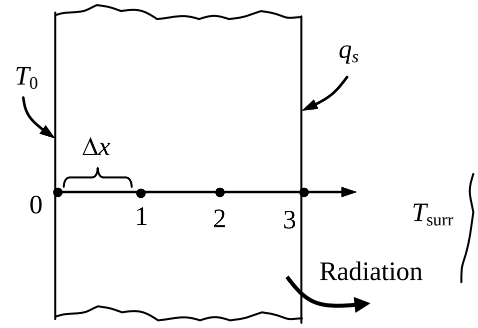
Node 2 (at fin tip):

$$kA\frac{T_1-T_2}{\Delta x}+h(p\Delta x/2+A)(T_\infty-T_2)+\varepsilon\sigma(p\Delta x/2+A)[T_{\text{surr}}^4-(T_2+273)^4]=0$$

where $A=\pi D^2/4$ is the cross-sectional area and $p=\pi D$ is the perimeter of the fin.

5-137E A plane wall in space is subjected to specified temperature on one side and radiation and heat flux on the other. The finite difference formulation of this problem is to be obtained, and the nodal temperatures under steady conditions are to be determined.

Assumptions **1** Heat transfer through the wall is given to be steady and one-dimensional. **2** Thermal conductivity is constant. **3** There is no heat generation. **4** There is no convection in space.



Properties The properties of the wall are given to be k=1.2 Btu/h·ft·°F, ε = 0.80, and α_s = 0.6.

Analysis The nodal spacing is given to be Δx = 0.1 ft. Then the number of nodes becomes $M = L/\Delta x + 1$ = 0.3/0.1+1 = 4. The left surface temperature is given to be T_0 = 520 R = 60°F. This problem involves 3 unknown nodal temperatures, and thus we need to have 3 equations to determine them uniquely. Nodes 1 and 2 are interior nodes, and thus for them we can use the general finite difference relation expressed as

$$\frac{T_{m-1} - 2T_m + T_{m+1}}{\Delta x^2} + \frac{\dot{e}_m}{k} = 0 \rightarrow T_{m-1} - 2T_m + T_{m+1} = 0 \quad (\text{since } \dot{e} = 0), \quad \text{for } m = 1 \text{ and } 2$$

The finite difference equation for node 3 on the right surface subjected to convection and solar heat flux is obtained by applying an energy balance on the half volume element about node 3 and taking the direction of all heat transfers to be towards the node under consideration:

$$\text{Node 1 (interior)}: \qquad T_0 - 2T_1 + T_2 = 0$$

$$\text{Node 2 (interior)}: \qquad T_1 - 2T_2 + T_3 = 0$$

$$\text{Node 3 (right surface)}: \qquad \alpha_s \dot{q}_s + \varepsilon\sigma[T_{\text{space}}^4 - (T_3 + 460)^4] + k\frac{T_2 - T_3}{\Delta x} = 0$$

where k = 1.2 Btu/h.ft.°F, ε = 0.80, α_s = 0.60, $\dot{q}_s = 350$ Btu/h.ft^2, T_{space} = 0 R, and σ = 0.1714×10^{-8} Btu/h.ft^2.R^4 The system of 3 equations with 3 unknown temperatures constitute the finite difference formulation of the problem.

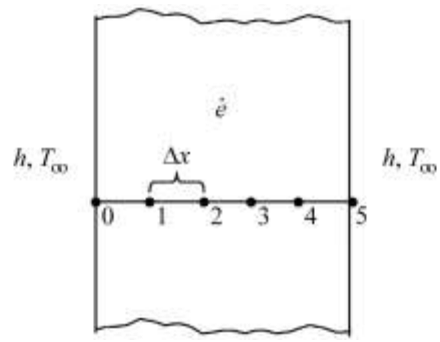
(*b*) The nodal temperatures under steady conditions are determined by solving the 3 equations above simultaneously with an equation solver to be

T_1 = **67.6°F = 527.6 R**, T_2 = **75.2°F = 535.2 R**, and T_3 = **82.8°F = 542.8 R**

5-138 A nuclear fuel element, modeled as a plane wall, generates 3×10^7 W/m^3 of heat uniformly with both side surfaces cooled by liquid. The finite difference equations and the nodal temperatures are to be determined, and the surface temperatures of both sides of the fuel element are to be compared with analytical solution.

Assumptions **1** Heat transfer through the nuclear fuel element is steady and one-dimensional. **2** Thermal properties are constant. **3** Heat transfer by radiation is negligible.

Properties The thermal conductivity is given as 57 W/m·K.



Analysis (*a*) The nodal spacing is given as Δx = 8 mm. Then the number of nodes *M* becomes

$$M = \frac{L}{\Delta x} + 1 = \frac{40\,\text{mm}}{8\,\text{mm}} + 1 = 6$$

There are 6 unknown nodal temperatures, thus we need to have 6 equations to determine them uniquely. The finite difference equation for node 0 on the left surface subjected to convection is obtained by applying an energy balance on the half volume element about that node:

$$h(T_\infty - T_0) + k\frac{T_1 - T_0}{\Delta x} + \dot{e}_0\frac{\Delta x}{2} = 0 \quad \rightarrow \quad T_1 - \left(1 + \frac{h}{k}\Delta x\right)T_0 + \dot{e}_0\frac{\Delta x^2}{2k} + \frac{h}{k}\Delta x T_\infty = 0$$

Nodes 1, 2, 3, and 4 are interior nodes, and we can use the general finite difference relation expressed as

$$\frac{T_{m-1} - 2T_m + T_{m+1}}{\Delta x^2} + \frac{\dot{e}_m}{k} = 0 \quad \rightarrow \quad T_{m-1} - 2T_m + T_{m+1} + \frac{\dot{e}_m}{k}\Delta x^2 = 0$$

The finite difference equation for node 5 on the right surface subjected to convection is obtained by applying an energy balance on the half volume element about that node:

$$k\frac{T_4 - T_5}{\Delta x} + \dot{e}_5\frac{\Delta x}{2} + h(T_\infty - T_5) = 0 \quad \rightarrow \quad T_4 - \left(1 + \frac{h}{k}\Delta x\right)T_5 + \frac{\Delta x^2}{2k}\dot{e}_5 + \frac{h}{k}\Delta x T_\infty = 0$$

Then

m = 0: $\quad T_1 - (1 + h\Delta x / k)T_0 + (\Delta x^2 \dot{e}_0)/(2k) + (h\Delta x / k)T_\infty = 0$

m = 1: $\quad T_0 - 2T_1 + T_2 + (\dot{e}_1 / k)\Delta x^2 = 0$

m = 2: $\quad T_1 - 2T_2 + T_3 + (\dot{e}_2 / k)\Delta x^2 = 0$

m = 3: $\quad T_2 - 2T_3 + T_4 + (\dot{e}_3 / k)\Delta x^2 = 0$

m = 4: $\quad T_3 - 2T_4 + T_5 + (\dot{e}_4 / k)\Delta x^2 = 0$

m = 5: $\quad T_4 - (1 + h\Delta x / k)T_5 + (\Delta x^2 \dot{e}_5)/(2k) + (h\Delta x / k)T_\infty = 0$

(*b*) The nodal temperatures under steady conditions are determined by solving the 6 equations above simultaneously with an equation solver. Copy the following lines and paste on a blank EES screen to solve the above equations:

```
e_gen=3E7
h=8000
k=57
Dx=8E-3
T_inf=80
T_1-(1+h*Dx/k)*T_0+(Dx^2*e_gen)/(2*k)+(h*Dx/k)*T_inf=0
T_0-2*T_1+T_2+(e_gen/k)*Dx^2=0
T_1-2*T_2+T_3+(e_gen/k)*Dx^2=0
T_2-2*T_3+T_4+(e_gen/k)*Dx^2=0
T_3-2*T_4+T_5+(e_gen/k)*Dx^2=0
T_4-(1+h*Dx/k)*T_5+(Dx^2*e_gen)/(2*k)+(h*Dx/k)*T_inf=0
```

Solving by EES software, we get

$$T_0 = \mathbf{155\,°C}, \quad T_1 = \mathbf{222\,°C}, \quad T_2 = \mathbf{256\,°C}$$

$$T_3 = \mathbf{256\,°C}, \quad T_4 = \mathbf{222\,°C}, \quad T_5 = \mathbf{155\,°C}$$

(*c*) Using the analytical solution from Chapter 2, for a plane wall of thickness 2*L* with heat generation, the surface temperature exposed to convection can be determined using

$$T_{s,\,\text{plane wall}} = T_\infty + \frac{\dot{e}_{\text{gen}} L}{h} = 80\,°\text{C} + \frac{(3\times10^7\ \text{W/m}^3)(0.02\ \text{m})}{8000\ \text{W/m}^2\cdot\text{K}} = \mathbf{155\,°C} \quad \text{(for both sides)}$$

The analytical solution matches exactly with the results obtained using numerical method for both sides of the surface temperatures, $T_0 = T_5 = \mathbf{155\,°C}$.

Discussion Since both side of the fuel element are exposed to the same liquid temperature and convection heat transfer coefficient, it is possible to solve half of the plane wall by treating the centerline of the plane wall as symmetry line and get the same results.

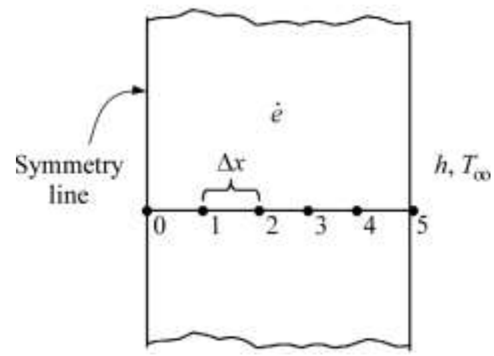
5-139 A fuel element, modeled as a plane wall, generates 5×10^7 W/m^3 of heat uniformly with both side surfaces cooled by liquid. The finite difference equations and the nodal temperatures are to be determined by making use of the symmetry line of the plane wall.

Assumptions **1** Heat transfer through the fuel element is steady and one-dimensional. **2** Thermal properties are constant. **3** Heat transfer by radiation is negligible.

Properties The thermal conductivity is given as 67 W/m·K.

Analysis (*a*) The nodal spacing is given to be Δx = 4 mm. Then the number of nodes *M* becomes

$$M = \frac{L}{\Delta x} + 1 = \frac{20\text{ mm}}{4\text{ mm}} + 1 = 6$$



There are 6 unknown nodal temperatures, thus we need to have 6 equations to determine them uniquely. The finite difference equation for node 0 on the symmetry line is obtained by applying an energy balance on the half volume element about that node (the symmetry boundary is similar to the insulated boundary):

$$k\frac{T_1 - T_0}{\Delta x} + \dot{e}_0 \frac{\Delta x}{2} = 0 \quad \rightarrow \quad T_1 - T_0 + \dot{e}_0 \frac{\Delta x^2}{2k} = 0$$

Nodes 1, 2, 3, and 4 are interior nodes, and we can use the general finite difference relation expressed as

$$\frac{T_{m-1} - 2T_m + T_{m+1}}{\Delta x^2} + \frac{\dot{e}_m}{k} = 0 \quad \rightarrow \quad T_{m-1} - 2T_m + T_{m+1} + \frac{\dot{e}_m}{k}\Delta x^2 = 0$$

The finite difference equation for node 5 on the right surface subjected to convection is obtained by applying an energy balance on the half volume element about that node:

$$k\frac{T_4 - T_5}{\Delta x} + \dot{e}_5 \frac{\Delta x}{2} + h(T_\infty - T_5) = 0 \quad \rightarrow \quad T_4 - \left(1 + \frac{h}{k}\Delta x\right)T_5 + \frac{\Delta x^2}{2k}\dot{e}_5 + \frac{h}{k}\Delta x T_\infty = 0$$

Then

$m = 0$: $\quad T_1 - T_0 + (\dot{e}_0 \Delta x^2)/(2k) = 0$

$m = 1$: $\quad T_0 - 2T_1 + T_2 + (\dot{e}_1 / k)\Delta x^2 = 0$

$m = 2$: $\quad T_1 - 2T_2 + T_3 + (\dot{e}_2 / k)\Delta x^2 = 0$

$m = 3$: $\quad T_2 - 2T_3 + T_4 + (\dot{e}_3 / k)\Delta x^2 = 0$

$m = 4$: $\quad T_3 - 2T_4 + T_5 + (\dot{e}_4 / k)\Delta x^2 = 0$

$m = 5$: $\quad T_4 - (1 + h\Delta x / k)T_5 + (\Delta x^2 \dot{e}_5)/(2k) + (h\Delta x / k)T_\infty = 0$

(*b*) The nodal temperatures under steady conditions are determined by solving the 6 equations above simultaneously with an equation solver. Copy the following lines and paste on a blank EES screen to solve the above equations:

```
e_gen=5E7
h=5000
k=67
Dx=4E-3
T_inf=90
T_1-T_0+(Dx^2*e_gen)/(2*k)=0
T_0-2*T_1+T_2+(e_gen/k)*Dx^2=0
T_1-2*T_2+T_3+(e_gen/k)*Dx^2=0
T_2-2*T_3+T_4+(e_gen/k)*Dx^2=0
T_3-2*T_4+T_5+(e_gen/k)*Dx^2=0
T_4-(1+h*Dx/k)*T_5+(Dx^2*e_gen)/(2*k)+(h*Dx/k)*T_inf=0
```

Solving by EES software, we get

$$T_0 = \mathbf{439\,°C}, \quad T_1 = \mathbf{433\,°C}, \quad T_2 = \mathbf{415\,°C}$$

$$T_3 = \mathbf{386\,°C}, \quad T_4 = \mathbf{344\,°C}, \quad T_5 = \mathbf{290\,°C}$$

Discussion Using the analytical solution from Chapter 2, for a plane wall of thickness 2*L* with heat generation, the surface temperature exposed to convection can be determined using

$$T_{s,\,\text{plane wall}} = T_\infty + \frac{\dot{e}_{\text{gen}} L}{h} = 90\,°\text{C} + \frac{(5\times10^7\ \text{W/m}^3)(0.02\ \text{m})}{5000\ \text{W/m}^2\cdot\text{K}} = 290\ °\text{C}$$

The analytical solution matches exactly with the results obtained using numerical method for both sides of the surface temperatures, $T_5 = 290\ °\text{C}$.

5-140 A square cross section with uniform heat generation is undergoing a steady two-dimensional heat transfer. The top and right surfaces are subjected to convection while the left and bottom surfaces maintain a constant temperature. The finite difference equations and the nodal temperatures are to be determined.

Assumptions **1** Steady heat conduction is two-dimensional. **2** Thermal properties are constant. **3** The heat generation in the body is uniform.

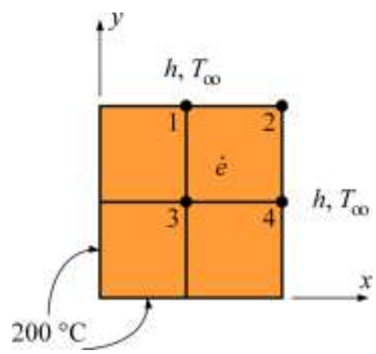
Properties The conductivity is given to be $k = 25$ W/m·K.

Analysis (*a*) There is symmetry about the diagonal line passing through the center. Therefore, $T_1 = T_4$, and the unknown temperatures are T_1, T_2, and T_3. Thus, we need to have 3 equations to determine them uniquely.

Node 1: $$h\Delta x(T_\infty - T_1) + k\frac{\Delta y}{2}\frac{T_2 - T_1}{\Delta x} + k\Delta x\frac{T_3 - T_1}{\Delta y} + k\frac{\Delta y}{2}\frac{200 - T_1}{\Delta x} + \dot{e}_1\Delta x\frac{\Delta y}{2} = 0$$

Node 2: $$h\left(\frac{\Delta x}{2} + \frac{\Delta y}{2}\right)(T_\infty - T_2) + k\frac{\Delta x}{2}\frac{T_4 - T_2}{\Delta y} + k\frac{\Delta y}{2}\frac{T_1 - T_2}{\Delta x} + \dot{e}_2\frac{\Delta x}{2}\frac{\Delta y}{2} = 0$$

Node 3: $$\frac{200 - 2T_3 + T_4}{\Delta x^2} + \frac{200 - 2T_3 + T_1}{\Delta y^2} + \frac{\dot{e}_{m,n}}{k} = 0$$



Or

Node 1: $$T_1 = \frac{1}{4 + 2h\Delta x/k}\left(200 + T_2 + 2T_3 + 2\frac{h\Delta x}{k}T_\infty + \frac{\dot{e}_1\Delta x^2}{k}\right)$$

Node 2: $$T_2 = \frac{1}{2 + 2h\Delta x/k}\left(2T_1 + 2\frac{h\Delta x}{k}T_\infty + \frac{\dot{e}_2\Delta x^2}{2k}\right)$$

Node 3: $$T_3 = 0.25(400 + 2T_1 + \dot{e}_3\Delta x^2/k)$$

Then

Node 1: $$T_1 = (200 + T_2 + 2T_3 + 0.2T_\infty + 12)/4.2$$

Node 2: $$T_2 = (2T_1 + 0.2T_\infty + 6)/2.2$$

Node 3: $$T_3 = 0.25(400 + 2T_1 + 12)$$

where $\dot{e}_{\text{node}}\Delta x^2/k = 12\,°\text{C}$ and $2h\Delta x/k = 0.2$

(*b*) By letting the initial guesses as $T_1 = T_2 = T_3 = 200$°C, the results obtained from successive iterations are listed in the following table:

Iteration	Nodal temperature,°C		
	T_1	T_2	T_3
1	198.1	191.9	202.0
2	197.1	191.0	201.6
3	196.7	190.6	201.4
4	196.5	190.5	201.3
5	196.4	190.4	201.2
6	196.4	190.3	201.2
7	196.4	190.3	201.2

Hence, the converged nodal temperatures are

$T_1 = T_4 =$ **196.4°C**, $T_2 =$ **190.3°C**, $T_3 =$ **201.2°C**

Discussion The finite difference equations can also be calculated using the EES. Copy the following lines and paste on a blank EES screen to solve the above equations:

```
T_1=(200+T_2+2*T_3+0.2*100+12)/4.2
T_2=(2*T_1+0.2*100+6)/2.2
T_3=0.25*(400+2*T_1+12)
```

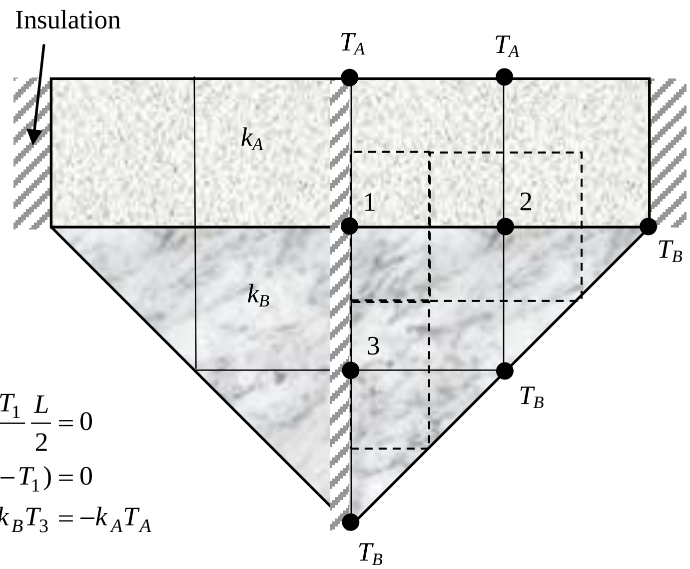
Solving by EES software, we get the same results:

$T_1 = T_4 =$ **196.4°C**, $T_2 =$ **190.3°C**, $T_3 =$ **201.2°C**

5-141 A two-dimensional bar shown in the figure is considered. The simplest form of the matrix equation is to be written and the grid notes with energy balance equations are to be identified on the figure.

Assumptions **1** Heat transfer through the body is given to be steady and two-dimensional. **2** Thermal conductivity is constant. **3** There is no heat generation.

Analysis From symmetry, we have only three unknown temperatures at nodes 1, 2, and 3. The finite difference formulations are



Node 1:

$$k_A \frac{T_A - T_1}{L}\frac{L}{2} + k_A \frac{T_2 - T_1}{L}\frac{L}{2} + k_B \frac{T_2 - T_1}{L}\frac{L}{2} + k_B \frac{T_3 - T_1}{L}\frac{L}{2} = 0$$

$$k_A(T_A - T_1) + k_A(T_2 - T_1) + k_B(T_2 - T_1) + k_B(T_3 - T_1) = 0$$

$$-2(k_A + k_B)T_1 + (k_A + k_B)T_2 + k_B T_3 = -k_A T_A$$

Node 2:

$$k_A \frac{T_A - T_2}{L}L + k_A \frac{T_1 - T_2}{L}\frac{L}{2} + k_B \frac{T_1 - T_2}{L}\frac{L}{2} + k_B \frac{T_B - T_2}{L}L + k_B \frac{T_B - T_2}{L}\frac{L}{2} + k_A \frac{T_B - T_2}{L}\frac{L}{2} = 0$$

$$2k_A(T_A - T_2) + k_A(T_1 - T_2) + k_B(T_1 - T_2) + 2k_B(T_B - T_2) + k_B(T_B - T_2) + k_A(T_B - T_2) = 0$$

$$(k_A + k_B)T_1 - 4(k_A + k_B)T_2 = -2k_A T_A - (k_A + 3k_B)T_B$$

Node 3:

$$k_B \frac{T_1 - T_3}{L}\frac{L}{2} + k_B \frac{T_B - T_3}{L}L + k_B \frac{T_B - T_3}{L}\frac{L}{2} = 0$$

$$T_1 - T_3 + 2(T_B - T_3) + T_B - T_3 = 0$$

$$T_1 + -4T_3 = -3T_B$$

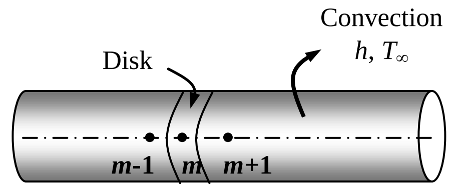
The matrix equation is

$$\begin{bmatrix} -2(k_A + k_B) & k_A + k_B & k_B \\ k_A + k_B & -4(k_A + k_B) & 0 \\ 1 & 0 & -4 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} = \begin{bmatrix} k_A T_A \\ -2k_A T_A - (k_A + 3k_B)T_B \\ -3T_B \end{bmatrix}$$

Discussion Note that the results do not depend on L (size of the system). If you don't use the symmetry and get a 4×4 linear system, two of the equations must be equivalent.

5-142 Starting with an energy balance on a disk volume element, the one-dimensional transient implicit finite difference equation for a general interior node for $T(z,t)$ in a cylinder whose side surface is subjected to convection with a convection coefficient of h and an ambient temperature of T_∞ for the case of constant thermal conductivity with uniform heat generation is to be obtained.

Analysis We consider transient one-dimensional heat conduction in the axial z direction in a cylindrical rod of constant cross-sectional area A with constant heat generation $\dot{e}_0$ and constant conductivity k with a mesh size of Δz in the z direction. Noting that the volume element of a general interior node m involves heat conduction from two sides, convection from its lateral surface, and the volume of the element is $V_{\text{element}} = A\Delta z$, the transient implicit finite difference formulation for an interior node can be expressed as



$$hp\Delta z(T_\infty - T_m^{i+1}) + kA\frac{T_{m-1}^{i+1} - T_m^{i+1}}{\Delta z} + kA\frac{T_{m+1}^{i+1} - T_m^{i+1}}{\Delta z} + \dot{e}_0 A\Delta z = \rho A\Delta z c_p \frac{T_m^{i+1} - T_m^i}{\Delta t}$$

where $A = \pi D^2/4$ is the cross-sectional area. Multiplying both sides of equation by $\Delta z/(kA)$,

$$\frac{hp\Delta z^2}{kA}(T_\infty - T_m^{i+1}) + (T_{m-1}^{i+1} - T_m^{i+1}) + (T_{m+1}^{i+1} - T_m^{i+1}) + \frac{\dot{e}_0\Delta z^2}{k} = \frac{\rho\Delta z^2 c_p}{k\Delta t}(T_m^{i+1} - T_m^i)$$

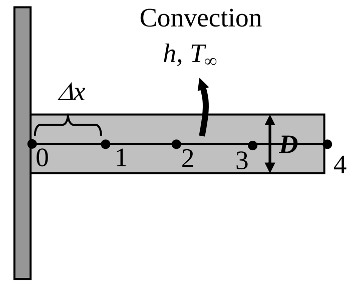
Using the definitions of *thermal diffusivity* $\alpha = k/\rho c_p$ and the dimensionless *mesh Fourier number* $\tau = \dfrac{\alpha\Delta t}{\Delta z^2}$ the equation reduces to

$$\frac{hp\Delta z^2}{kA}(T_\infty - T_m^{i+1}) + (T_{m-1}^{i+1} + T_{m+1}^{i+1} - 2T_m^{i+1}) + \frac{\dot{e}_0\Delta z^2}{k} = \frac{(T_m^{i+1} - T_m^i)}{\tau}$$

Discussion We note that setting $T_{\text{m}}^{i+1} = T_{\text{m}}^{i}$ gives the steady finite difference formulation.

5-143 A hot surface is to be cooled by aluminum pin fins. The nodal temperatures after 10 min are to be determined using the explicit finite difference method. Also to be determined is the time it takes for steady conditions to be reached.

Assumptions **1** Heat transfer through the pin fin is given to be one-dimensional. **2** The thermal properties of the fin are constant. **3** Convection heat transfer coefficient is constant and uniform. **4** Radiation heat transfer is negligible. **5** Heat loss from the fin tip is considered.



Analysis The nodal network of this problem consists of 5 nodes, and the base temperature T_0 at node 0 is specified. Therefore, there are 4 unknown nodal temperatures, and we need 4 equations to determine them. Using the energy balance approach and taking the direction of all heat transfers to be towards the node under consideration, the explicit transient finite difference formulations become

Node 1 (interior): $$hp\Delta x(T_\infty - T_1^i) + kA\frac{T_2^i - T_1^i}{\Delta x} + kA\frac{T_0 - T_1^i}{\Delta x} = \rho A\Delta x c_p \frac{T_1^{i+1} - T_1^i}{\Delta t}$$

Node 2 (interior): $$hp\Delta x(T_\infty - T_2^i) + kA\frac{T_3^i - T_2^i}{\Delta x} + kA\frac{T_1^i - T_2^i}{\Delta x} = \rho A\Delta x c_p \frac{T_2^{i+1} - T_2^i}{\Delta t}$$

Node 3 (interior): $$hp\Delta x(T_\infty - T_3^i) + kA\frac{T_4^i - T_3^i}{\Delta x} + kA\frac{T_2^i - T_3^i}{\Delta x} = \rho A\Delta x c_p \frac{T_3^{i+1} - T_3^i}{\Delta t}$$

Node 4 (fin tip): $$h(p\Delta x/2 + A)(T_\infty - T_4^i) + kA\frac{T_3^i - T_4^i}{\Delta x} = \rho A(\Delta x/2) c_p \frac{T_4^{i+1} - T_4^i}{\Delta t}$$

where $A = \pi D^2/4$ is the cross-sectional area and $p = \pi D$ is the perimeter of the fin. Also, D = 0.008 m, k = 237 W/m.°C, $\alpha = k/\rho c_p = 97.1\times10^{-6}\ \text{m}^2/\text{s}$, Δx = 0.02 m, T_∞ = 15°C, $T_0 = T_i$ = 120°C, h_o = 35 W/m^2.°C, and Δt = 1 s. Also, the mesh Fourier number is

$$\tau = \frac{\alpha\Delta t}{\Delta x^2} = \frac{(97.1\times10^{-6}\ \text{m}^2/\text{s})(1\,\text{s})}{(0.02\,\text{m})^2} = 0.24275$$

Substituting these values, the nodal temperatures along the fin after 10×60 = 600 time steps (4 h) are determined to be

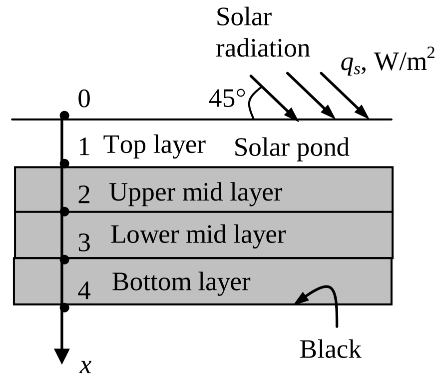
T_0 = **120°C**, T_1 = **110.6°C**, T_2 = **103.9°C**, T_3 = **100.0°C**, and T_4 = **98.5°C.**

Printing the temperatures after each time step and examining them, we observe that the nodal temperatures stop changing after about 3.8 min. Thus we conclude that steady conditions are reached after **3.8 min.**

5-144 A large pond is initially at a uniform temperature. Solar energy is incident on the pond surface at for 4 h The temperature distribution in the pond under the most favorable conditions is to be determined.

Assumptions **1** Heat transfer is one-dimensional since the pond is large relative to its depth. **2** Thermal properties, heat transfer coefficients, and the indoor temperatures are constant. **3** Radiation heat transfer is significant. **4** There are no convection currents in the water. **5** The given time step Δt = 15 min is less than the critical time step so that the stability criteria is satisfied. **6** All heat losses from the pond are negligible. **7** Heat generation due to absorption of radiation is uniform in each layer.

Properties The conductivity and diffusivity are given to be k = 0.61 W/m.°C and $\alpha = 0.15\times10^{-6}\ \text{m}^2/\text{s}$. The volumetric absorption coefficients of water are as given in the problem.



Analysis The nodal spacing is given to be Δx = 0.25 m. Then the number of nodes becomes $M = L/\Delta x + 1$ = 1/0.25+1 = 4. This problem involves 5 unknown nodal temperatures, and thus we need to have 5 equations. Nodes 2, 3, and 4 are interior nodes, and thus for them we can use the general explicit finite difference relation expressed as

$$T_{m-1}^i - 2T_m^i + T_{m+1}^i + \frac{\dot{e}_m^i \Delta x^2}{k} = \frac{T_m^{i+1} - T_m^i}{\tau} \quad \rightarrow \quad T_m^{i+1} = \tau(T_{m-1}^i + T_{m+1}^i) + (1-2\tau)T_m^i + \tau\frac{\dot{e}_m^i \Delta x^2}{k}$$

Node 0 can also be treated as an interior node by using the mirror image concept. The finite difference equation for node 4 subjected to heat flux is obtained from an energy balance by taking the direction of all heat transfers to be towards the node:

$$\text{Node 0 (insulation)}: \quad T_0^{i+1} = \tau(T_1^i + T_1^i) + (1-2\tau)T_0^i + \tau\dot{e}_0(\Delta x)^2/k$$

$$\text{Node 0 (insulation)}: \quad T_1^{i+1} = \tau(T_0^i + T_2^i) + (1-2\tau)T_1^i + \tau\dot{e}_1(\Delta x)^2/k$$

$$\text{Node 2 (interior)}: \quad T_2^{i+1} = \tau(T_1^i + T_3^i) + (1-2\tau)T_2^i + \tau\dot{e}_2(\Delta x)^2/k$$

$$\text{Node 3 (interior)}: \quad T_3^{i+1} = \tau(T_2^i + T_4^i) + (1-2\tau)T_3^i + \tau\dot{e}_3(\Delta x)^2/k$$

$$\text{Node 6 (convection)}: \quad \dot{q}_b + k\frac{T_3^i - T_4^i}{\Delta x} + \tau\dot{e}_4(\Delta x)^2/k = \rho\frac{\Delta x}{2}c_p\frac{T_4^{i+1} - T_4^i}{\Delta t}$$

where k = 0.61 W/m.°C, $\alpha = k/\rho c_p = 0.15\times10^{-6}\ \text{m}^2/\text{s}$, Δx = 0.25 m, and Δt = 15 min = 900 s. Also, the mesh Fourier number is

$$\tau = \frac{\alpha\Delta t}{\Delta x^2} = \frac{(0.15\times10^{-6}\ \text{m}^2/\text{s})(900\ \text{s})}{(0.25\ \text{m})^2} = 0.002160$$

The values of heat generation rates at the nodal points are determined as follows:

$$\dot{e}_0 = \frac{\dot{E}_0}{\text{Volume}} = \frac{0.473\times500\ \text{W}}{(1\ \text{m}^2)(0.25\ \text{m})} = 946\ \text{W/m}^3$$

$$\dot{e}_1 = \frac{\dot{E}_1}{\text{Volume}} = \frac{[(0.473+0.061)/2]\times500\ \text{W}}{(1\ \text{m}^2)(0.25\ \text{m})} = 534\ \text{W/m}^3$$

$$\dot{e}_4 = \frac{\dot{E}_4}{\text{Volume}} = \frac{0.024\times500\ \text{W}}{(1\ \text{m}^2)(0.25\ \text{m})} = 48\ \text{W/m}^3$$

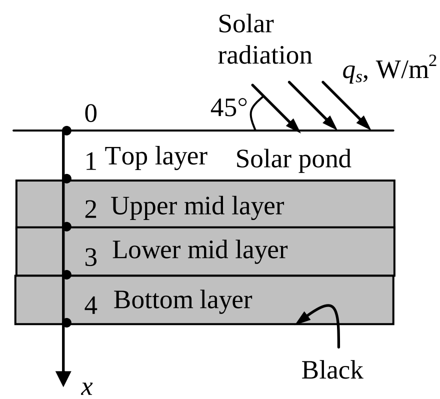
Also, the heat flux at the bottom surface is $\dot{q}_b = 0.379\times500\ \text{W/m}^2 = 189.5\ \text{W/m}^2$. Substituting these values, the nodal temperatures in the pond after 4×(60/15) = 16 time steps (4 h) are determined to be

T_0 = **18.3°C**, T_1 = **16.9°C**, T_2 = **15.4°C**, T_3 = **15.3°C**, and T_4 = **20.2°C**

5-145 A large 1-m deep pond is initially at a uniform temperature of 15°C throughout. Solar energy is incident on the pond surface at 45° at an average rate of 500 W/m^2 for a period of 4 h The temperature distribution in the pond under the most favorable conditions is to be determined.

Assumptions **1** Heat transfer is one-dimensional since the pond is large relative to its depth. **2** Thermal properties, heat transfer coefficients, and the indoor temperatures are constant. **3** Radiation heat transfer is significant. **4** There are no convection currents in the water. **5** The given time step Δt = 15 min is less than the critical time step so that the stability criteria is satisfied. **6** All heat losses from the pond are negligible. **7** Heat generation due to absorption of radiation is uniform in each layer.

Properties The conductivity and diffusivity are given to be k = 0.61 W/m.°C and $\alpha = 0.15\times10^{-6}\ \text{m}^2/\text{s}$. The volumetric absorption coefficients of water are as given in the problem.



Analysis The nodal spacing is given to be Δx = 0.25 m. Then the number of nodes becomes $M = L/\Delta x + 1$ = 1/0.25+1 = 4. This problem involves 5 unknown nodal temperatures, and thus we need to have 5 equations. Nodes 2, 3, and 4 are interior nodes, and thus for them we can use the general explicit finite difference relation expressed as

$$T_{m-1}^i - 2T_m^i + T_{m+1}^i + \frac{\dot{e}_m^i \Delta x^2}{k} = \frac{T_m^{i+1} - T_m^i}{\tau}$$

$$\rightarrow \quad T_m^{i+1} = \tau(T_{m-1}^i + T_{m+1}^i) + (1-2\tau)T_m^i + \tau\frac{\dot{e}_m^i \Delta x^2}{k}$$

Node 0 can also be treated as an interior node by using the mirror image concept. The finite difference equation for node 4 subjected to heat flux is obtained from an energy balance by taking the direction of all heat transfers to be towards the node:

$$\text{Node 0 (insulation)}: \quad T_0^{i+1} = \tau(T_1^i + T_1^i) + (1-2\tau)T_0^i + \tau\dot{e}_0(\Delta x)^2/k$$

$$\text{Node 0 (insulation)}: \quad T_1^{i+1} = \tau(T_0^i + T_2^i) + (1-2\tau)T_1^i + \tau\dot{e}_1(\Delta x)^2/k$$

$$\text{Node 2 (interior)}: \quad T_2^{i+1} = \tau(T_1^i + T_3^i) + (1-2\tau)T_2^i + \tau\dot{e}_2(\Delta x)^2/k$$

$$\text{Node 3 (interior)}: \quad T_3^{i+1} = \tau(T_2^i + T_4^i) + (1-2\tau)T_3^i + \tau\dot{e}_3(\Delta x)^2/k$$

$$\text{Node 6 (convection)}: \quad \dot{q}_b + k\frac{T_3^i - T_4^i}{\Delta x} + \tau\dot{e}_4(\Delta x)^2/k = \rho\frac{\Delta x}{2}c_p\frac{T_4^{i+1} - T_4^i}{\Delta t}$$

where k = 0.61 W/m.°C, $\alpha = k/\rho c_p = 0.15\times10^{-6}\ \text{m}^2/\text{s}$, Δx = 0.25 m, and Δt = 15 min = 900 s. Also, the mesh Fourier number is

$$\tau = \frac{\alpha\Delta t}{\Delta x^2} = \frac{(0.15\times10^{-6}\ \text{m}^2/\text{s})(900\ \text{s})}{(0.25\ \text{m})^2} = 0.002160$$

The absorption of solar radiation is given to be $\dot{e}(x) = \dot{q}_s(0.859 - 3.415x + 6.704x^2 - 6.339x^3 + 2.278x^4)$

where $\dot{q}_s$ is the solar flux incident on the surface of the pond in W/m^2, and x is the distance form the free surface of the pond, in m. Then the values of heat generation rates at the nodal points are determined to be

Node 0 (x = 0): $\dot{e}_0 = 500(0.859 - 3.415\times0 + 6.704\times0^2 - 6.339\times0^3 + 2.278\times0^4) = 429.5\ \text{W/m}^3$

Node 1(x=0.25): $\dot{e}_1 = 500(0.859 - 3.415\times0.25 + 6.704\times0.25^2 - 6.339\times0.25^3 + 2.278\times0.25^4) = 167.1\ \text{W/m}^3$

Node 2 (x=0.50): $\dot{e}_2 = 500(0.859 - 3.415\times0.5 + 6.704\times0.5^2 - 6.339\times0.5^3 + 2.278\times0.5^4) = 88.8\ \text{W/m}^3$

Node3 (x=0.75): $\dot{e}_3 = 500(0.859 - 3.415\times0.75 + 6.704\times0.75^2 - 6.339\times0.75^3 + 2.278\times0.75^4) = 57.6\ \text{W/m}^3$

Node 4 (x = 1.00): $\dot{e}_4 = 500(0.859 - 3.415\times1 + 6.704\times1^2 - 6.339\times1^3 + 2.278\times1^4) = 43.5\ \text{W/m}^3$

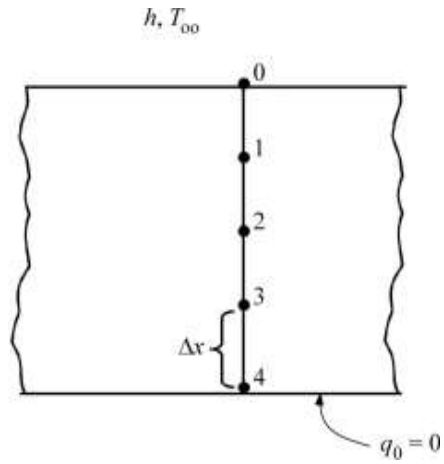
Also, the heat flux at the bottom surface is $\dot{q}_b = 0.379\times500\ \text{W/m}^2 = 189.5\ \text{W/m}^2$. Substituting these values, the nodal temperatures in the pond after 4×(60/15) = 16 time steps (4 h) are determined to be

T_0 = **16.5°C**, T_1 = **15.6°C**, T_2 = **15.3°C**, T_3 = **15.3°C**, and T_4 = **20.2°C**

5-146 A hot brass plate is having its upper surface cooled by impinging jet while its lower surface is insulated. The explicit finite difference equations, the maximum allowable value of the time step, and the temperature at the center plane of the brass plate after 1 minute of cooling are to be determined.

Assumptions **1** Transient heat conduction is one-dimensional. **2** Thermal properties are constant. **3** Convection heat transfer coefficient is uniform. **4** Heat transfer by radiation is negligible. **5** There is no heat generation.

Properties The properties of the brass plate are given as ρ = 8530 kg/m^3, c_p = 380 J/kg·K, k = 110 W/m·K, and α = 33.9 × 10^{-6} m^2/s.



Analysis The nodal spacing is given to be Δx = 2.5 cm. Then the number of nodes becomes $M = L/\Delta x + 1 = 10/2.5 + 1 = 5$. This problem involves 5 unknown nodal temperatures, and thus we need to have 5 equations. The finite difference equation for node 0 on the top surface subjected to convection is obtained by applying an energy balance on the half volume element about node 0 and taking the direction of all heat transfers to be towards the node under consideration:

$$h(T_\infty - T_0^i) + k\frac{T_1^i - T_0^i}{\Delta x} = \rho\frac{\Delta x}{2}c_p\frac{T_0^{i+1} - T_0^i}{\Delta t}$$

or

$$T_0^{i+1} = \left(1 - 2\tau - 2\tau\frac{h\Delta x}{k}\right)T_0^i + \tau\left(2T_1^i + 2\frac{h\Delta x}{k}T_\infty\right)$$

Node 4 is on insulated boundary, and thus we can treat it as an interior node by using the mirror image concept. Nodes 1, 2, and 3 are interior nodes, and thus for them we can use the general explicit finite difference relation expressed as

$$T_{m-1}^i - 2T_m^i + T_{m+1}^i = \frac{T_m^{i+1} - T_m^i}{\tau}$$

or $\quad T_m^{i+1} = \tau(T_{m-1}^i + T_{m+1}^i) + (1 - 2\tau)T_m^i$

Thus, the explicit finite difference equations are

Node 0: $\quad T_0^{i+1} = \left(1 - 2\tau - 2\tau\frac{h\Delta x}{k}\right)T_0^i + \tau\left(2T_1^i + 2\frac{h\Delta x}{k}T_\infty\right)$

Node 1: $\quad T_1^{i+1} = \tau(T_0^i + T_2^i) + (1 - 2\tau)T_1^i$

Node 2: $\quad T_2^{i+1} = \tau(T_1^i + T_3^i) + (1 - 2\tau)T_2^i$

Node 3: $\quad T_3^{i+1} = \tau(T_2^i + T_4^i) + (1 - 2\tau)T_3^i$

Node 4: $\quad T_4^{i+1} = \tau(T_3^i + T_3^i) + (1 - 2\tau)T_4^i$

where

Δx = 2.5 cm, k = 110 W/m·K, h = 220 W/m^2·K, T_∞ = 15°C, α = 33.9 × 10^{-6} m^2/s, and $h\Delta x/k$ = 0.05.

(*b*) The upper limit of the time step Δt is determined from the stability criterion that requires all primary coefficients to be greater than or equal to zero. The coefficient of T_0^i is smaller in this case, and thus the stability criterion for this problem can be expressed as

$$1 - 2\tau - 2\tau\frac{h\Delta x}{k} \geq 0 \quad\rightarrow\quad \tau \leq \frac{1}{2(1 + h\Delta x/k)} \quad\rightarrow\quad \Delta t \leq \frac{\Delta x^2}{2\alpha(1 + h\Delta x/k)}$$

Since $\tau = \alpha\Delta t/\Delta x^2$. Substituting the given quantities, the maximum allowable value of the time step is determined to be

$$\Delta t \leq \frac{(0.025\,\text{m})^2}{2(33.9\times 10^{-6}\ \text{m}^2/\text{s})[1 + (220\ \text{W/m}^2\cdot\text{K})(0.025\,\text{m})/(110\ \text{W/m}\cdot\text{K})]} = 8.779\,\text{s}$$

Therefore, any time step less than 8.779 s can be used to solve this problem. For convenience, let us choose the time step to be Δt = 6 s. Then the mesh Fourier number becomes

$$\tau = \frac{\alpha \Delta t}{\Delta x^2} = \frac{(33.9\times10^{-6}\ \text{m}^2/\text{s})(6\,\text{s})}{(0.025\,\text{m})^2} = 0.32544 \qquad (\text{for } \Delta t = 6\text{ s})$$

(c) With the initial nodal temperatures of 650°C, the results obtained from successive iterations are listed in the following table:

Time step, i	Time, s	Nodal temperature,°C				
		T_0^i	T_1^i	T_2^i	T_3^i	T_4^i
0	0	650	650	650	650	650
1	6	629.3	650	650	650	650
2	12	622.8	643.3	650	650	650
3	18	616.3	638.8	647.8	650	650
4	24	611.4	634.4	645.6	649.3	650
5	30	607.0	630.6	643.2	648.3	649.5
6	36	603.1	627.0	640.7	647.0	648.7
7	42	599.5	623.7	638.3	645.5	647.6
8	48	596.2	620.6	635.9	643.9	646.3
9	54	593.2	617.6	633.5	642.1	644.7
10	60	590.3	614.8	**631.1**	640.1	643.0

The temperature at the center plane of the brass plate after 1 minute of cooling is

$$T_2^{10} = T(0.05\,\text{m}, 60\,\text{s}) = \mathbf{631.1°C}$$

(d) From Chapter 4, the approximate analytical solution is given as

$$\theta_{\text{wall}} = \frac{T(x,t)-T_\infty}{T_i - T_\infty} = A_1 e^{-\lambda_1^2\tau}\cos(\lambda_1 x/L)$$

where

$$Bi = \frac{hL}{k} = \frac{(220\ \text{W/m}^2\cdot\text{K})(0.10\,\text{m})}{110\ \text{W/m}\cdot\text{K}} = 0.2$$

$$\tau = \frac{\alpha t}{L^2} = \frac{(33.9\times10^{-6}\ \text{m}^2/\text{s})(60\,\text{s})}{(0.10\,\text{m})^2} = 0.2034 > 0.2$$

$$\lambda_1 = 0.4328 \quad \text{and} \quad A_1 = 1.0311 \ \ (\text{from Table 4-2})$$

Hence,

$$T(x,t) = (T_i - T_\infty)A_1 e^{-\lambda_1^2\tau}\cos(\lambda_1 x/L) + T_\infty$$

$$\begin{aligned} T(0.05\,\text{m}, 180\,\text{s}) &= (650\,°\text{C} - 15\,°\text{C})(1.0311)e^{-(0.4328)^2(0.2034)}\cos[(0.4328)(0.5)] + 15\,°\text{C} \\ &= \mathbf{630.6\,°C} \end{aligned}$$

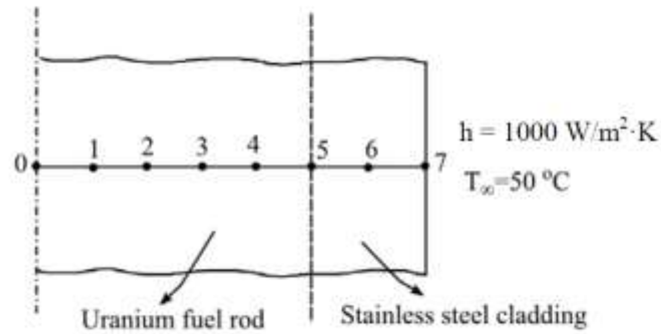
Discussion The comparison between the approximate analytical and numerical solutions is within ±0.08% agreement.

5-147 A uranium nuclear fuel rod experiencing a uniform heat generation and enclosed in a stainless steel cladding is cooled by pressurized water at specified conditions. The temperature distribution in the nuclear rod and the cladding material is to be determined at different time spans.

Assumptions **1** One-dimensional transient heat transfer with constant thermal conductivity. **2** Perfect contact between the fuel rod and cladding material. **3** No internal heat generation in the cladding material.

Properties For uranium nuclear fuel element we are given: k = 35 W/m·K, ρ = 19070 kg/m^3 and c_p = 116 J/kg·K. For stainless steel we are given: k = 15 W/m·K, ρ = 8055 kg/m^3 and c_p = 480 J/kg·K.

Analysis For the given geometry, nodes 0 and 7 are the boundary nodes while node 5 is the interface node. Use the finite difference formulation for the internal node as given by Eq. (5-48) for nodes 1, 2, 3, 4 and 6. Due to thermal symmetry about the centerline at node 0 it can be also treated as an internal node.



Node 0: $2T_1^{i+1} - 2T_0^{i+1} + \dfrac{\dot{e}_0 \Delta x^2}{k_1} = \dfrac{T_0^{i+1} - T_0^i}{\tau_1}$

Node 1: $T_0^{i+1} - 2T_1^{i+1} + T_2^{i+1} + \dfrac{\dot{e}_1 \Delta x^2}{k_1} = \dfrac{T_1^{i+1} - T_1^i}{\tau_1}$

Node 2: $T_1^{i+1} - 2T_2^{i+1} + T_3^{i+1} + \dfrac{\dot{e}_2 \Delta x^2}{k_1} = \dfrac{T_2^{i+1} - T_2^i}{\tau_1}$

Node 3: $T_2^{i+1} - 2T_3^{i+1} + T_4^{i+1} + \dfrac{\dot{e}_3 \Delta x^2}{k_1} = \dfrac{T_3^{i+1} - T_3^i}{\tau_1}$

Node 4: $T_3^{i+1} - 2T_4^{i+1} + T_5^{i+1} + \dfrac{\dot{e}_4 \Delta x^2}{k_1} = \dfrac{T_4^{i+1} - T_4^i}{\tau_1}$

Node 5: (Interface node) $k_1 \dfrac{\left(T_4^{i+1} - T_5^{i+1}\right)}{\Delta x} + k_2 \dfrac{\left(T_6^{i+1} - T_5^{i+1}\right)}{\Delta x} + \dot{e}_5 \dfrac{\Delta x}{2} = \dfrac{\Delta x}{2\Delta t}\left(\rho_1 c_{p,1} + \rho_2 c_{p,2}\right)\left(T_5^{i+1} - T_5^i\right)$

Node 6: $T_5^{i+1} - 2T_6^{i+1} + T_7^{i+1} = \dfrac{T_6^{i+1} - T_6^i}{\tau_2}$

Node 7: (Right boundary node) $2\left(T_6^{i+1} - T_7^{i+1}\right) + 2\dfrac{h\Delta x}{k_2}\left(T_\infty - T_7^{i+1}\right) = \dfrac{T_7^{i+1} - T_7^i}{\tau_2}$

The EES code used to solve the implicit finite difference equations is as follows

```
"Given data"
k_1 = 35 [W/mK] " Thermal conductivity of nuclear rod"
k_2 = 15 [W/mK] " Thermal conductivity of caldding material"
h = 1000 [W/m^2K] "Convective heat transfer coefficient"
T_infi = 50 [C]  " Temperature of cooling water"
DELTAx = 0.02 [m] "mesh size"
DELTAt = 60 [s]  " time step"
rho_1 = 19070 [kg/m^3] "Density of nuclear rod"
rho_2 = 8055 [kg/m^3] "Density of cladding material"
c_1 = 116 [J/kgK] "Specific heat of uranium rod"
c_2 = 480 [J/kgK] "Specific heat of cladding material"
e_gen = 4e5 [W/m^3] " Volumetric heat generation"

alpha_1 = k_1/(rho_1*c_1) " Thermal diffusivity of uranium rod"
alpha_2 = k_2/(rho_2*c_2) " Thermal diffusivity of cladding material"

tau_1 = (alpha_1*DELTAt)/DELTAx^2  " Mesh Fourier number"
```

tau_2 = (alpha_2*DELTAt)/DELTAx^2

"The technique is to store the temperatures in the parametric table and recover them (as old temperatures) using the variable ROW. The first row contains the initial values so Solve Table must begin at row 2. Use the DUPLICATE statement to reduce the number of equations that need to be typed. Column 1 contains the time, column 2 the value of T[1], column 3, the value of T[2], etc., and column 9 the Row.

To start the Solve Table at 2 go to 'Calculate' and select 'Solve table' (or hit F3) and make the 'First Run Number' as 2. The initial temperatures and the initial time ‘0’ can be set manually in the parametric table"

```
Row = TableRun#
Time = TableValue('Table 1', Row-1, #Time)+DELTAt/60

        Duplicate i = 0,7
                T_old[i] = TableValue('Table 1', Row-1, #T[i])
        End
" Finite difference equations"

"Node 0" 2*T[1]-2*T[0]+e_gen*DELTAx^2/k_1 = (T[0] - T_old[0])/tau_1
"Node 1" T[0]-2*T[1]+T[2]+e_gen*DELTAx^2/k_1 = (T[1] - T_old[1])/tau_1
"Node 2" T[1]-2*T[2]+T[3]+e_gen*DELTAx^2/k_1 = (T[2] - T_old[2])/tau_1
"Node 3" T[2]-2*T[3]+T[4]+e_gen*DELTAx^2/k_1 = (T[3] - T_old[3])/tau_1
"Node 4" T[3]-2*T[4]+T[5]+e_gen*DELTAx^2/k_1 = (T[4] - T_old[4])/tau_1
"Node 5" k_1*(T[4]-T[5])/DELTAx + k_2*(T[6]-T[5])/DELTAx+e_gen*(DELTAx/2) =
1/2*(((rho_1*c_1*DELTAx)/DELTAt)+(((rho_2*c_2*DELTAx)/DELTAt)))*(T[5]-T_old[5])
"Node 6" T[5]-2*T[6]+T[7] = (T[6] - T_old[6])/tau_2
"Node 7" 2*(T[6]-T[7])+2*DELTAx*h/k_2*(T_infi-T[7]) = (T[7]-T_old[7])/tau_2
```

Temperature distribution in the fuel rod and cladding at different times

Node temperature (^{o}C)	Time = 10 min	Time = 20 min	Time = 30 min
	Node temperature (^{o}C)		
T_0	468.3	389.8	339.1
T_1	464.1	386.1	335.9
T_2	451.4	375.2	326.5
T_3	430.3	357.1	310.9
T_4	400.8	331.9	289.1
T_5	362.9	299.8	261.3
T_6	257.2	213	187
T_7	140	120.4	109

Discussion In most of the practical cases, during the cooling of nuclear reactor rods, the water used for convective cooling undergoes a phase change process that enhances the rate of heat removal from the nuclear rods. Determination of the exact heat transfer rates to the cooling water and the estimation of pressure drop due to two- phase flow of water (as the water undergoes phase change) are quite challenging.

5-148 Starting with an energy balance on a volume element, the two-dimensional transient *explicit* finite difference equation for a general interior node in rectangular coordinates for $T(x, y, t)$ for the case of constant thermal conductivity k and uniform heat generation $\dot{e}_0$ is to be obtained.

Analysis (See Figure 5-24 in the text)**.** We consider a rectangular region in which heat conduction is significant in the x and y directions, and consider a unit depth of $\Delta z = 1$ in the z direction. There is uniform heat generation in the medium, and the thermal conductivity k of the medium is constant. Now we divide the x-y plane of the region into a *rectangular mesh* of nodal points which are spaced Δx and Δy apart in the x and y directions, respectively, and consider a general interior node (m, n) whose coordinates are $x = m\Delta x$ and $y = n\Delta y$. Noting that the volume element centered about the general interior node (m, n) involves heat conduction from four sides (right, left, top, and bottom) and expressing them at previous time step i, the transient explicit finite difference formulation for a general interior node can be expressed as

$$k(\Delta y \times 1)\frac{T^i_{m-1,n} - T^i_{m,n}}{\Delta x} + k(\Delta x \times 1)\frac{T^i_{m,n+1} - T^i_{m,n}}{\Delta y} + k(\Delta y \times 1)\frac{T^i_{m+1,n} - T^i_{m,n}}{\Delta x} + k(\Delta x \times 1)\frac{T^i_{m,n-1} - T^i_{m,n}}{\Delta y}$$

$$+\dot{e}_0(\Delta x \times \Delta y \times 1) = \rho(\Delta x \times \Delta y \times 1)c_p\frac{T^{i+1}_{m,n} - T^i_{m,n}}{\Delta t}$$

Taking a square mesh ($\Delta x = \Delta y = l$) and dividing each term by k gives, after simplifying,

$$T^i_{m-1,n} + T^i_{m+1,n} + T^i_{m,n+1} + T^i_{m,n-1} - 4T^i_{m,n} + \frac{\dot{e}_0 l^2}{k} = \frac{T^{i+1}_{m,n} - T^i_{m,n}}{\tau}$$

where $\alpha = k/\rho c_p$ is the thermal diffusivity of the material and $\tau = \alpha\Delta t/l^2$ is the dimensionless mesh Fourier number. It can also be expressed in terms of the temperatures at the neighboring nodes in the following easy-to-remember form:

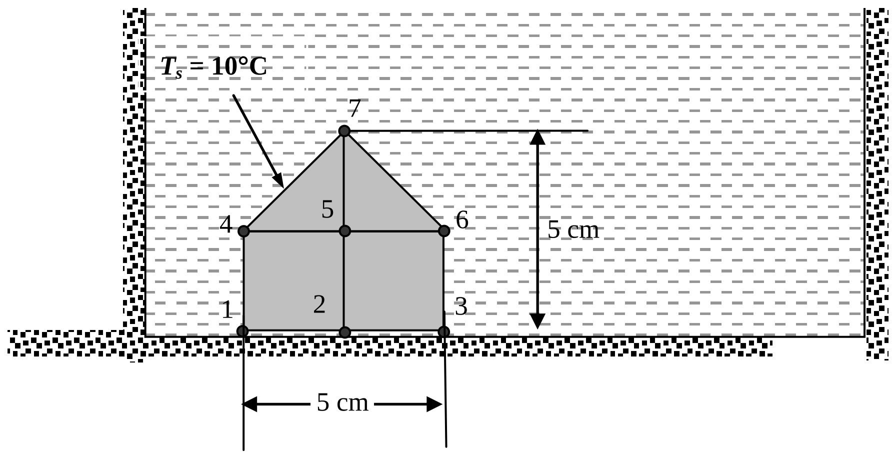
$$T^i_{\text{left}} + T^i_{\text{top}} + T^i_{\text{right}} + T^i_{\text{bottom}} - 4T^i_{\text{node}} + \frac{\dot{e}_0 l^2}{k} = \frac{T^{i+1}_{\text{node}} - T^i_{\text{node}}}{\tau}$$

Discussion We note that setting $T^{i+1}_{\text{node}} = T^i_{\text{node}}$ gives the steady finite difference formulation.

5-149 A two-dimensional long steel bar shown in the figure is considered. The finite difference equations for the unknown temperatures in the grid using the explicit method is to be written and dimensionless parameters are to be identified. Also, the range of time steps for stability condition and the temperature field at certain times are to be determined.

Assumptions **1** Heat transfer through the body is transient and two-dimensional. **2** All surafces of the bar except the bottom surface are maintained at a constant temperature. **3** Thermal conductivity is constant. **4** There is no heat generation.

Analysis (a) the finite difference equations for the unknown temperatures in the grid using the explicit method are



Node	T (10 s)	T (20 s)
1	10	10
2	443.3	234.4
3	10	10
4	10	10
5	315	168.6
6	10	10
7	10	10

Node 5:

$$k\frac{T_7^i-T_5^i}{\Delta x}\Delta x+k\frac{T_6^i-T_5^i}{\Delta x}\Delta x+k\frac{T_2^i-T_5^i}{\Delta x}\Delta x+k\frac{T_4^i-T_5^i}{\Delta x}\Delta x=\rho c_p\Delta x^2\frac{T_5^{i+1}-T_5^i}{\Delta t}$$

$$T_7^i+T_6^i+T_2^i+T_4^i-4T_5^i=\frac{\rho c\Delta x^2}{k\Delta t}(T_5^{i+1}-T_5^i) \qquad (1)$$

$$T_5^{i+1}=T_5^i(1-4\text{Fo})+\text{Fo}T_2^i+\text{Fo}\times 30$$

where $\quad \text{Fo}=\dfrac{k\Delta t}{\rho c_p\Delta x^2}$

Node 2:

$$k\frac{T_1^i-T_2^i}{\Delta x}\frac{\Delta x}{2}+k\frac{T_5^i-T_2^i}{\Delta x}\Delta x+k\frac{T_3^i-T_2^i}{\Delta x}\frac{\Delta x}{2}=\rho c_p\frac{\Delta x^2}{2}\frac{T_2^{i+1}-T_2^i}{\Delta t}$$

$$T_1^i+T_5^i+T_3^i-4T_2^i=\frac{-\rho c_p\Delta x^2}{k\Delta t}(T_2^{i+1}-T_2^i) \qquad (2)$$

$$T_2^{i+1}=T_2^i(1-4\text{Fo})+2\text{Fo}T_5^i+\text{Fo}\times 20$$

(*b*) For both steps, stability condition is

$$1 - 4\text{Fo} \geq 0 \longrightarrow \text{Fo} \leq \frac{1}{4} \longrightarrow \frac{k\Delta t}{\rho c_p \Delta x^2} \leq \frac{1}{4}$$

$$\Delta t \leq \frac{\rho c_p \Delta x^2}{4k} = \frac{(8000)(430)(0.025)^2}{4(40)} = \mathbf{13.44\ s}$$

(*c*) For Δt = 10 s,

$$\text{Fo} = \frac{k\Delta t}{\rho c_p \Delta x^2} = \frac{(40)(10)}{(8000)(430)(0.025)^2} = 0.186$$

Then, Eq. (1) and (2) become

$$T_5^{i+1} = 0.256T_5^i + 0.186T_2^i + 5.58$$
$$T_2^{i+1} = 0.256T_2^i + 0.372T_5^i + 3.72$$

Substituting at Δt = 10 s,

$$T_5^1 = 0.256(700) + 0.186(700) + 5.58 = \mathbf{315°C}$$
$$T_2^1 = 0.256(700) + 0.372(700) + 3.72 = \mathbf{443.3°C}$$

Substituting at Δt = 20 s,

$$T_5^2 = 0.256(315) + 0.186(443) + 5.58 = \mathbf{168.6°C}$$
$$T_2^1 = 0.256(443.3) + 0.372(315) + 3.72 = \mathbf{234.4°C}$$

Fundamentals of Engineering (FE) Exam Problems

5-150 The unsteady forward-difference heat conduction for a constant area, *A*, pin fin with perimeter, *p*, exposed to air whose temperature is T_0 with a convection heat transfer coefficient of *h* is

$$T_m^{*+1} = \frac{k}{\rho c_p \Delta x^2}\left[T_{m-1}^* + T_{m+1}^* + \frac{hp\Delta x^2}{A}T_0\right] - \left[1 - \frac{2k}{\rho c_p \Delta x^2} - \frac{hp}{\rho c_p A}\right]T_m^*$$

In order for this equation to produce a stable solution, the quantity $\frac{2k}{\rho c_p \Delta x^2} + \frac{hp}{\rho c_p A}$ must be

(a) Negative (b) zero (c) Positive (d) Greater than 1 (e) Less than 1

Answer (d) Greater than 1

5-151 Air at T_0 acts on top surface of the rectangular solid shown in Fig. P5-151 with a convection heat transfer coefficient of *h*. The correct steady-state finite-difference heat conduction equation for node 3 of this solid is

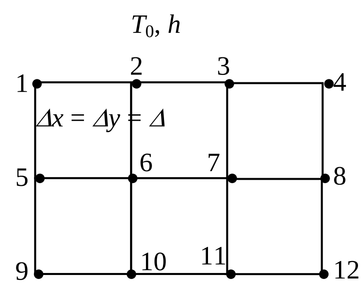
(a) $T_3 = [(k/2\Delta)(T_2 + T_4 + T_7) + hT_0] / [(k/\Delta) + h]$

(b) $T_3 = [(k/2\Delta)(T_2 + T_4 + 2T_7) + hT_0] / [(2k/\Delta) + h]$

(c) $T_3 = [(k/\Delta)(T_2 + T_4) + hT_0] / [(2k/\Delta) + h]$

(d) $T_3 = [(k/\Delta)(T_2 + T_4 + T_7) + hT_0] / [(k/\Delta) + h]$

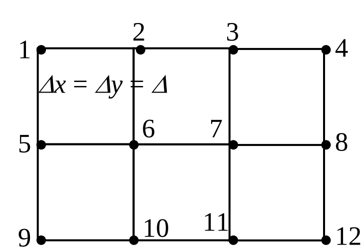
(e) $T_3 = [(k/\Delta)(2T_2 + 2T_4 + T_7) + hT_0] / [(k/\Delta) + h]$



Answer (b)

5-152 What is the correct unsteady forward-difference heat conduction equation of node 6 of the rectangular solid shown in Fig. P5-152 if its temperature at the previous time (Δt) is T_6^* ?

$$(a)\ T_6^{i+1} = \left[k\Delta t/(\rho c_p \Delta^2)\right](T_5^* + T_2^* + T_7^* + T_{10}^*) + \left[1 - 4k\Delta t/(\rho c_p \Delta^2)\right]T_6^*$$
$$(b)\ T_6^{i+1} = \left[k\Delta t/(\rho c_p \Delta^2)\right](T_5^* + T_2^* + T_7^* + T_{10}^*) + \left[1 - k\Delta t/(\rho c_p \Delta^2)\right]T_6^*$$
$$(c)\ T_6^{i+1} = \left[k\Delta t/(\rho c_p \Delta^2)\right](T_5^* + T_2^* + T_7^* + T_{10}^*) + \left[2k\Delta t/(\rho c_p \Delta^2)\right]T_6^*$$
$$(d)\ T_6^{i+1} = \left[2k\Delta t/(\rho c_p \Delta^2)\right](T_5^* + T_2^* + T_7^* + T_{10}^*) + \left[1 - 2k\Delta t/(\rho c_p \Delta^2)\right]T_6^*$$
$$(e)\ T_6^{i+1} = \left[2k\Delta t/(\rho c_p \Delta^2)\right](T_5^* + T_2^* + T_7^* + T_{10}^*) + \left[1 - 4k\Delta t/(\rho c_p \Delta^2)\right]T_6^*$$



Answer (a)

5-153 What is the correct steady-state finite-difference heat conduction equation of node 6 of the rectangular solid shown in Fig. P5-153?

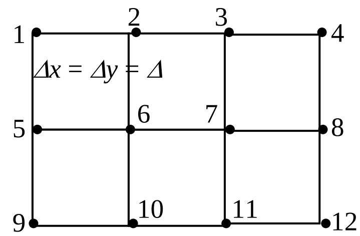
(a) $T_6 = (T_1 + T_3 + T_9 + T_{11}) / 2$

(b) $T_6 = (T_5 + T_7 + T_2 + T_{10}) / 2$

(c) $T_6 = (T_1 + T_3 + T_9 + T_{11}) / 4$

(d) $T_6 = (T_2 + T_5 + T_7 + T_{10}) / 4$

(e) $T_6 = (T_1 + T_2 + T_9 + T_{10}) / 4$



Answer (d)

5-154 The height of the cells for a finite-difference solution of the temperature in the rectangular solid shown in Fig. P5-154 is one-half the cell width to improve the accuracy of the solution. The correct steady-state finite-difference heat conduction equation for cell 6 is

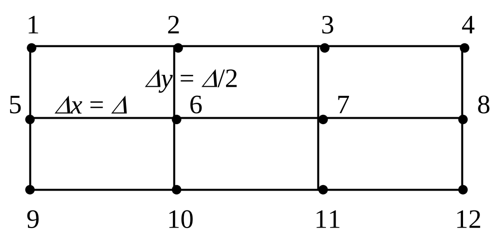
(a) $T_6 = 0.1(T_5 + T_7) + 0.4(T_2 + T_{10})$

(b) $T_6 = 0.25(T_5 + T_7) + 0.25(T_2 + T_{10})$

(c) $T_6 = 0.5(T_5 + T_7) + 0.5(T_2 + T_{10})$

(d) $T_6 = 0.4(T_5 + T_7) + 0.1(T_2 + T_{10})$

(e) $T_6 = 0.5(T_5 + T_7) + 0.5(T_2 + T_{10})$



Answer (a)

5-155 The height of the cells for a finite-difference solution of the temperature in the rectangular solid shown in Fig. P5-155 is one-half the cell width to improve the accuracy of the solution. If the left surface is exposed to air at T_0 with a heat transfer coefficient of h, the correct finite-difference heat conduction energy balance for node 5 is

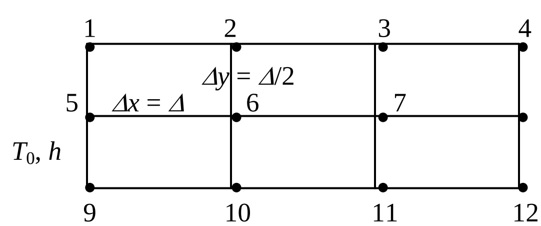
(a) $2T_1 + 2T_9 + T_6 - T_5 + h\Delta/k\,(T_0 - T_5) = 0$

(b) $2T_1 + 2T_9 + T_6 - 2T_5 + h\Delta/k\,(T_0 - T_5) = 0$

(c) $2T_1 + 2T_9 + T_6 - 3T_5 + h\Delta/k\,(T_0 - T_5) = 0$

(d) $2T_1 + 2T_9 + T_6 - 4T_5 + h\Delta/k\,(T_0 - T_5) = 0$

(e) $2T_1 + 2T_9 + T_6 - 5T_5 + h\Delta/k\,(T_0 - T_5) = 0$



Answer (e)

5-156 ….. 5-159 Design and Essay Problems