CHAPTER 19

19.1 A table of integrals can be consulted to determine

$$\int \tanh\, dx = \frac{1}{a}\ln\cosh ax$$

Therefore,

$$\int_0^t \sqrt{\frac{gm}{c_d}}\tanh\left(\sqrt{\frac{gc_d}{m}}t\right)dt = \sqrt{\frac{gm}{c_d}}\sqrt{\frac{m}{gc_d}}\left[\ln\cosh\left(\sqrt{\frac{gc_d}{m}}t\right)\right]_0^t$$

$$= \sqrt{\frac{gm^2}{gc_d^2}}\left[\ln\cosh\left(\sqrt{\frac{gc_d}{m}}t\right) - \ln\cosh(0)\right]$$

Since cosh(0) = 1 and ln(1) = 0, this reduces to

$$\frac{m}{c_d}\ln\cosh\left(\sqrt{\frac{gc_d}{m}}t\right)$$

19.2 **(a)** The analytical solution can be evaluated as

$$\int_0^4 (1-e^{-x})\,dx = \left[x+e^{-x}\right]_0^4 = 4+e^{-4}-0-e^{-0} = 3.018316$$

(b) single application of the trapezoidal rule:

$$(4-0)\frac{0+0.981684}{2} = 1.963369 \qquad (\varepsilon_t = 34.95\%)$$

(c) composite trapezoidal rule:

$\underline{n = 2:}$

$$(4-0)\frac{0+2(0.864665)+0.981684}{4} = 2.711014 \qquad (\varepsilon_t = 10.18\%)$$

$\underline{n = 4:}$

$$(4-0)\frac{0+2(0.632121+0.864665+0.950213)+0.981684}{8} = 2.93784 \qquad (\varepsilon_t = 2.67\%)$$

(d) single application of Simpson's 1/3 rule:

$$(4-0)\frac{0+4(0.864665)+0.981684}{6}=2.960229 \quad (\varepsilon_t=1.92\%)$$

(e) composite Simpson's 1/3 rule ($n = 4$):

$$(4-0)\frac{0+4(0.632121+0.950213)+2(0.864665)+0.981684}{12}=3.013449 \quad (\varepsilon_t=0.16\%)$$

(f) Simpson's 3/8 rule:

$$(4-0)\frac{0+3(0.736403+0.930517)+0.981684}{8}=2.991221 \quad (\varepsilon_t=0.9\%)$$

(g) Simpson's rules ($n = 5$):

$$\begin{aligned} I &= (1.6-0)\frac{0+4(0.550671)+0.798103}{6} \\ &\quad +(4-1.6)\frac{0.798103+3(0.909282+0.959238)+0.981684}{8} \\ &= 0.80021+2.215604=3.015814 \qquad \varepsilon_t=0.08\% \end{aligned}$$

19.3 (a) Analytical solution:

$$\int_0^{\pi/2}(8+4\cos x)\,dx=\left[8x+4\sin x\right]_0^{\pi/2}=8(\pi/2)+4\sin(\pi/2)-0=16.56637$$

(b) Trapezoidal rule ($n = 1$):

$$I=(1.570796-0)\frac{12+8}{2}=15.70796 \qquad \varepsilon_t=\left|\frac{16.56637-15.70796}{16.56637}\right|\times 100\%=5.182\%$$

(c) Trapezoidal rule ($n = 2$):

$$I=(1.570796-0)\frac{12+2(10.82843)+8}{4}=16.35861 \quad \varepsilon_t=1.254\%$$

Trapezoidal rule ($n = 4$):

$$I = (1.570796 - 0)\frac{12 + 2(11.69552 + 10.82843 + 9.530734) + 8}{8} = 16.51483 \qquad \varepsilon_t = 0.311\%$$

(d) Simpson's 1/3 rule:

$$I = (1.570796 - 0)\frac{12 + 4(10.82843) + 8}{6} = 16.57549 \quad \varepsilon_t = 0.055\%$$

(e) Simpson's rule ($n = 4$):

$$I = (1.570796 - 0)\frac{12 + 4(11.69552 + 9.530734) + 2(10.82843) + 8}{12} = 16.56691 \qquad \varepsilon_t = 0.0032\%$$

(f) Simpson's 3/8 rule:

$$I = (1.570796 - 0)\frac{12 + 3(11.4641 + 10) + 8}{8} = 16.57039 \quad \varepsilon_t = 0.024\%$$

(g) Simpson's rules ($n = 5$):

$$\begin{aligned} I &= (0.628319 - 0)\frac{12 + 4(11.80423) + 11.2307}{6} \\ &\quad + (1.570796 - 0.628319)\frac{11.2307 + 3(10.35114 + 9.236068) + 8}{8} \\ &= 7.377818 + 9.188887 = 16.5667 \qquad (\varepsilon_t = 0.002\%) \end{aligned}$$

19.4 **(a)** The analytical solution can be evaluated as

$$\begin{aligned} \int_{-2}^{4} (1 - x - 4x^3 + 2x^5)\, dx &= \left[x - \frac{x^2}{2} - x^4 + \frac{x^6}{3} \right]_{-2}^{4} \\ &= 4 - \frac{4^2}{2} - 4^4 + \frac{4^6}{3} - (-2) + \frac{(-2)^2}{2} + (-2)^4 - \frac{(-2)^6}{3} = 1104 \end{aligned}$$

(b) single application of the trapezoidal rule:

$$(4 - (-2))\frac{-29 + 1789}{2} = 5280 \qquad (\varepsilon_t = 378.3\%)$$

(c) composite trapezoidal rule:

<u>$n = 2$:</u>

$$(4-(-2))\frac{-29+2(-2)+1789}{4} = 2634 \qquad (\varepsilon_t = 138.6\%)$$

<u>$n = 4$:</u>

$$(4-(-2))\frac{-29+2(1.9375+(-2)+131.3125)+1789}{8} = 1516.875 \qquad (\varepsilon_t = 37.4\%)$$

(d) single application of Simpson's 1/3 rule:

$$(4-(-2))\frac{-29+4(-2)+1789}{6} = 1752 \qquad (\varepsilon_t = 58.7\%)$$

(c) Simpson's 3/8 rule:

$$(4-(-2))\frac{-29+3(1+31)+1789}{8} = 1392 \qquad (\varepsilon_t = 26.09\%)$$

(f) Boole's rule:

$$(4-(-2))\frac{7(-29)+32(1.9375)+12(-2)+32(131.3125)+7(1789)}{90} = 1104 \qquad (\varepsilon_t = 0\%)$$

19.5 (a) The analytical solution can be evaluated as

$$\int_0^{1.2} e^{-x}\,dx = \left[-e^{-x}\right]_0^{1.2} = -e^{-1.2} - (-e^0) = 0.69880579$$

(b) Trapezoidal rule:

$$(0.1-0)\frac{1+0.9048}{2} + (0.3-0.1)\frac{0.9048+0.7408}{2} + (0.5-0.3)\frac{0.7408+0.6065}{2}$$
$$+(0.7-0.5)\frac{0.6065+0.4966}{2} + (0.95-0.7)\frac{0.4966+0.3867}{2} + (1.2-0.95)\frac{0.3867+0.3012}{2}$$
$$= 0.0952+0.1646+0.1347+0.1103+0.1104+0.0860 = 0.7012 \quad (\varepsilon_t = 0.35\%)$$

(c) Trapezoidal and Simpson's Rules:

$$(0.1-0)\frac{1+0.9048}{2} + (0.7-0.3)\frac{0.9048+3(0.7408+0.6065)+0.4966}{8}$$
$$+(1.2-0.7)\frac{0.4966+4(0.3867)+0.3012}{6} = 0.0952+0.4082+0.1954 = 0.6989 \quad (|\varepsilon_t| = 0.01\%)$$

19.6 **(a)** The integral can be evaluated analytically as,

$$\int_{-2}^{2}\left[\frac{x^3}{3}-3y^2x+y^3\frac{x^2}{2}\right]_0^4 dy$$

$$\int_{-2}^{2}\left[\frac{(4)^3}{3}-3y^2(4)+y^3\frac{(4)^2}{2}\right] dy$$

$$\int_{-2}^{2}\left[21.33333-12y^2+8y^3\right] dy$$

$$\left[21.33333y-4y^3+2y^4\right]_{-2}^{2}$$

$$21.33333(2)-4(2)^3+2(2)^4-21.33333(-2)+4(-2)^3-2(-2)^4=21.33333$$

(b) The composite trapezoidal rule with $n = 2$ can be used to evaluate the inner integral at the three equispaced values of y,

$$y=-2: \quad (4-0)\frac{-12+2(-24)-28}{4}=-88$$
$$y=0: \quad (4-0)\frac{0+2(4)+16}{4}=24$$
$$y=2: \quad (4-0)\frac{-12+2(8)+36}{4}=40$$

These results can then be integrated in y to yield

$$(2-(-2))\frac{-88+2(24)+40}{4}=0$$

which represents a percent relative error of

$$\varepsilon_t=\left|\frac{21.33333-0}{21.33333}\right|\times 100\%=100\%$$

which is not very good.

(c) Single applications of Simpson's $1/3$ rule can be used the evaluate the inner integral at the three equispaced values of y,

$$y = -2: \quad (4-0)\frac{-12+4(-24)-28}{6} = -90.66667$$

$$y = 0: \quad (4-0)\frac{0+4(4)+16}{6} = 21.33333$$

$$y = 2: \quad (4-0)\frac{-12+4(8)+36}{6} = 37.33333$$

These results can then be integrated in y to yield

$$(2-(-2))\frac{-90.66667+4(21.33333)+37.33333}{6} = 21.33333$$

which represents a percent relative error of

$$\varepsilon_t = \left|\frac{21.33333-21.33333}{21.33333}\right| \times 100\% = 0\%$$

which is perfect

(d)

```
clear,clc
format compact
fun = @(x,y) x.^2-3*y.^2+x.*y.^3
q = integral2(fun,0,4,-2,2)

q =
   21.3333
```

19.7 (a) The integral can be evaluated analytically as,

$$\int_{-4}^{4}\int_{0}^{6}\left[\frac{x^4}{4}-2yzx\right]_{-1}^{3} dy\, dz = \int_{-4}^{4}\int_{0}^{6}\left[20-8yz\right] dy\, dz$$

$$\int_{-4}^{4}\int_{0}^{6}(20-8yz)\, dy\, dz = \int_{-4}^{4}\left[20y-4zy^2\right]_0^6 dz = \int_{-4}^{4}(120-144z)\, dz$$

$$\int_{-4}^{4}(120-144z)\, dz = \left[120z-72z^2\right]_{-4}^{4} = 120(4)-72(4)^2-120(-4)+72(-4)^2 = 960$$

(b) Single applications of Simpson's 1/3 rule can be used the evaluate the inner integral at the three equispaced values of y for each value of z,

<u>$z = -4$:</u>

$y = 0$: $\quad (3-(-1))\dfrac{-1+4(1)+27}{6} = 20$

$y = 3$: $\quad (3-(-1))\dfrac{23+4(25)+51}{6} = 116$

$y = 6$: $\quad (3-(-1))\dfrac{47+4(49)+75}{6} = 212$

These results can then be integrated in y to yield

$$(6-0)\frac{20+4(116)+212}{6} = 696$$

<u>$z = 0$:</u>

$y = 0$: $\quad (3-(-1))\dfrac{-1+4(1)+27}{6} = 20$

$y = 3$: $\quad (3-(-1))\dfrac{-1+4(1)+27}{6} = 20$

$y = 6$: $\quad (3-(-1))\dfrac{-1+4(1)+27}{6} = 20$

These results can then be integrated in y to yield

$$(6-0)\frac{20+4(20)+20}{6} = 120$$

<u>$z = 4$:</u>

$y = 0$: $\quad (3-(-1))\dfrac{-1+4(1)+27}{6} = 20$

$y = 3$: $\quad (3-(-1))\dfrac{-25+4(-23)+3}{6} = -76$

$y = 6$: $\quad (3-(-1))\dfrac{-49+4(-47)-21}{6} = -172$

These results can then be integrated in y to yield

$$(6-0)\frac{20+4(-76)-172}{6} = -456$$

The results of the integrations in y can then be integrated in z to yield a perfect result:

$$(4-(-4))\frac{696+4(120)-456}{6} = 960$$

(c)

```
clear,clc
format compact
fun = @(x,y,z) x.^3-2*y.*z;
q = integral3(fun,-1,3,0,6,-4,4)

q =
   960
```

19.8 **(a)** The trapezoidal rule can be implemented as,

$$I = (2-1)\frac{5+6}{2} + (3.25-2)\frac{6+5.5}{2} + \ldots = 60.125\,\frac{\text{m}\cdot\text{min}}{\text{s}} \times \frac{60\text{ s}}{\text{min}} = 3{,}607.5\text{ m}$$

$$\bar{v} = \frac{3{,}607.5\text{ m}}{10\text{ min}} \times \frac{\text{min}}{60\text{ s}} = 6.0125\,\frac{\text{m}}{\text{s}}$$

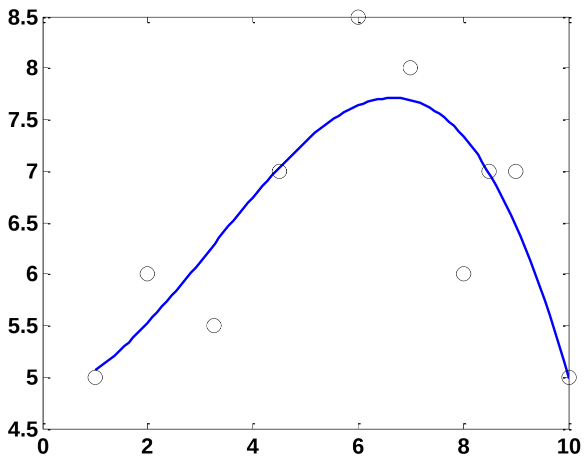
(b) The polynomial can be fit as,

```
>> format long
>> t = [1 2 3.25 4.5 6 7 8 8.5 9 10];
>> v = [5 6 5.5 7 8.5 8 6 7 7 5];
>> p = polyfit(t,v,3)

p =
  -0.01800093017330   0.17531524519646   0.06028850825860
4.85065783294403
```

The cubic can be plotted along with the data,

```
>> tt = linspace(1,10);
>> vv = polyval(p,tt);
>> plot(tt,vv,t,v,'o')
```



This equation can be integrated to yield

$$x = \int_1^{10} (-0.018t^3 + 0.1753t^2 + 0.0629t + 4.8507)\, dt$$

$$= \left[-0.0045t^4 + 0.05844t^3 + 0.030144t^2 + 4.8507t\right]_1^{10} = 60.02235 \frac{\text{m} \cdot \text{min}}{\text{s}} \times \frac{60 \text{ s}}{\text{min}} = 3{,}601.341 \text{ m}$$

19.9

z	$w(z)$	$\rho g w(z)(75 - z)$	$\rho g z w(z)(75 - z)$
75	200	0.0000E+00	0.0000E+00
62.5	190	2.3299E+07	1.4562E+09
50	175	4.2919E+07	2.1459E+09
37.5	160	5.8860E+07	2.2073E+09
25	135	6.6218E+07	1.6554E+09
12.5	130	7.9706E+07	9.9633E+08
0	122	8.9762E+07	0.0000E+00

$$f_t = 75 \frac{8.9762 + 4(7.9706 + 5.8860 + 2.3299) + 2(6.6218 + 4.2919) + 0}{3(6)} \times 10^7 = 3.9812 \times 10^9$$

$$\int_0^D \rho g z w(z)(D - z)\, dz = 75 \frac{0 + 4(0.99633 + 2.2073 + 1.4562) + 2(1.6554 + 2.1459) + 0}{3(6)} \times 10^9 = 1.0934 \times 10^{11}$$

$$d = \frac{1.0934 \times 10^{11}}{3.9812 \times 10^9} = 27.464$$

19.10 (a) Trapezoidal rule:

$$f = 30 \frac{0 + 2(71.653 + 68.456 + 55.182 + 42.176 + 31.479) + 23.200}{2(6)} = 1402.728$$

$$d = \frac{30 \dfrac{0 + 2(358.266 + 684.556 + 827.729 + 843.511 + 786.982) + 696.010}{2(6)}}{1402.728} = \frac{19{,}245.24}{1402.728} = 13.720 \text{ m}$$

(b) Simpson's 1/3 rule:

$$f = 30 \frac{0 + 4(71.653 + 55.182 + 31.479) + 2(68.456 + 42.176) + 23.200}{3(6)} = 1462.867$$

$$d = \frac{30\dfrac{0+4(358.266+827.729+786.982)+2(684.556+843.511)+696.010}{3(6)}}{1462.867} = \frac{19{,}406.75}{1462.867} = 13.266 \text{ m}$$

19.11 The values needed to perform the evaluation can be tabulated:

Height *l*, m	**Force, *F*(*l*), N/m**	$l \times F(l)$
0	0	0
30	340	10200
60	1200	72000
90	1550	139500
120	2700	324000
150	3100	465000
180	3200	576000
210	3500	735000
240	3750	900000

Because there are an even number of equally-spaced segments, we can evaluate the integrals with the multi-segment Simpson's 1/3 rule.

$$F = (240-0)\frac{0+4(340+1550+3100+3500)+2(1200+2700+3200)+3750}{24} = 519{,}100$$

$$I = (240-0)\frac{0+4(10200+139500+465000+735000)+2(72000+324000+576000)+900000}{24}$$
$$= 82{,}428{,}000$$

The line of action can therefore be computed as

$$d = \frac{82{,}428{,}000}{519{,}100} = 158.7902$$

19.12 (a) Analytical solution:

$$M = \int_0^{11} (5+0.25x^2)\,dx = \left[5x + 0.083333x^3\right]_0^{11} = 165.9167$$

(b) composite trapezoidal rule:

$$I = (1-0)\frac{5+5.25}{2} + (2-1)\frac{5.25+6}{2} + \ldots = 166.375$$

(c) composite Simpson's rule:

$$I = (2-0)\frac{5+4(5.25)+6}{6} + (4-2)\frac{6+4(7.25)+9}{6} + ... = 165.9167$$

19.13 We can set up a table that contains the values comprising the integrand

x, cm	ρ, g/cm^3	A_c, cm^2	$\rho \times A_c$, g/cm
0	4	100	400
400	3.95	103	406.85
600	3.89	106	412.34
800	3.8	110	418
1200	3.6	120	432
1600	3.41	133	453.53
2000	3.3	150	495

We can integrate this data using a combination of the trapezoidal and Simpson's rules,

$$I = (400-0)\frac{400+406.85}{2} + (800-400)\frac{406.85+4(412.34)+418}{6}$$
$$+(2000-800)\frac{418+3(432+453.53)+495}{8} = 861{,}755.8 \text{ g} = 861.7558 \text{ kg}$$

19.14 We can set up a table that contains the values comprising the integrand

t, hr	t, d	rate (cars/4 min)	rate (cars/d)
7:30	0.312500	18	6480
7:45	0.322917	23	8280
8:00	0.333333	14	5040
8:15	0.343750	24	8640
8:45	0.364583	20	7200
9:15	0.385417	9	3240

We can integrate this data using a combination of Simpson's 3/8 and 1/3 rules. This yields the number of cars that go through the intersection between 7:30 and 9:15 (1.75 hrs),

$$I = (0.34375-0.3125)\frac{6480+3(8280+5040)+8640}{8}$$
$$+(0.385417-0.34375)\frac{8640+4(7200)+3240}{6}$$
$$= 215.1563 + 282.5 = 497.6563 \text{ cars}$$

The number of cars going through the intersection per minute can be computed as

$$\frac{497.6563 \text{ cars}}{1.75 \text{ hr}}\frac{\text{hr}}{60 \text{ min}} = 4.7396\frac{\text{cars}}{\text{min}}$$

19.15 We can use Simpson's 1/3 rule to integrate across the *y* dimension,

<u>$x = 0$:</u>

$$I = (4-0)\frac{-2+4(-4)-8}{6} = -17.3333$$

<u>$x = 4$:</u>

$$I = (4-0)\frac{-1+4(-3)-8}{6} = -14$$

<u>$x = 8$:</u>

$$I = (4-0)\frac{4+4(1.5)-6}{6} = 2.6667$$

<u>$x = 12$:</u>

$$I = (4-0)\frac{10+4(6.5)+4}{6} = 26.6667$$

These values can then be integrated along the *x* dimension with Simpson's 3/8 rule:

$$I = (12-0)\frac{-17.3333+3(-14+2.6667)+26.6667}{8} = -37$$

19.16

```
>> t=[0 10 20 30 35 40 45 50];
>> Q=[4 4.8 5.2 5.0 4.6 4.3 4.3 5.0];
>> c=[10 35 55 52 40 37 32 34];
>> Qc=Q.*c;
>> M=trapz(t,Qc)

M =
  9.5185e+003
```

The problem can also be solved with a combination of Simpson's 3/8 and composite Simpson's rules:

```
>> M=(30-0)*(Qc(1)+3*(Qc(2)+Qc(3))+Qc(4))/8;
>> M=M+(50-30)*(Qc(4)+4*(Qc(5)+Qc(7))+2*Qc(6)+Qc(8))/12

M =
  9.6235e+003
```

Thus, the answers are 9.5185 g (trapezoidal rule) and 9.6235 g (Simpson's rules).

19.17 A table can be set up to hold the values that are to be integrated:

***y*, m**	***H*, m**	***U*, m/s**	***UH*, m^2/s**
0	0.5	0.03	0.015
2	1.3	0.06	0.078
4	1.25	0.05	0.0625
5	1.8	0.13	0.234
6	1	0.11	0.11
9	0.25	0.02	0.005

The cross-sectional area can be evaluated using a combination of Simpson's 1/3 rule and the trapezoidal rule:

$$A_c = (4-0)\frac{0.5+4(1.3)+1.25}{6} + (6-4)\frac{1.25+4(1.8)+1}{6} + (9-6)\frac{1+0.25}{2}$$
$$= 4.633333 + 3.15 + 1.875 = 9.658333 \text{ m}^2$$

The flow can be evaluated in a similar fashion:

$$Q = (4-0)\frac{0.015+4(0.078)+0.0625}{6} + (6-4)\frac{0.0625+4(0.234)+0.11}{6} + (9-6)\frac{0.11+0.005}{2}$$
$$= 0.259667 + 0.3695 + 0.1725 = 0.801667 \ \frac{\text{m}^3}{\text{s}}$$

19.18 First, we can estimate the areas by numerically differentiating the volume data. Because the values are equally spaced, we can use the second-order difference formulas from Figs. 21.3-5 to compute the derivatives at each depth. For example, at the first depth, we can use the forward difference to compute

$$A_s(0) = -\frac{dV}{dz}(0) = -\frac{-1{,}963{,}500 + 4(5{,}105{,}100) - 3(9{,}817{,}500)}{8} = 1{,}374{,}450 \text{ m}^2$$

For the interior points, second-order centered differences can be used. For example, at the second point at ($z = 4$ m),

$$A_s(4) = -\frac{dV}{dz}(4) = -\frac{1{,}963{,}500 - 9{,}817{,}500}{8} = 981{,}750 \text{ m}^2$$

The other interior points can be determined in a similar fashion

$$A_s(8) = -\frac{dV}{dz}(8) = -\frac{392{,}700 - 5{,}105{,}100}{8} = 589{,}050 \text{ m}^2$$

$$A_s(12) = -\frac{dV}{dz}(12) = -\frac{0 - 1{,}963{,}500}{8} = 245{,}437.5\ \text{m}^2$$

For the last point, the second-order backward formula yields

$$A_s(16) = -\frac{dV}{dz}(16) = -\frac{3(0) - 4(392{,}700) + 1{,}963{,}500}{8} = -49{,}087.5\ \text{m}^2$$

Since this is clearly a physically unrealistic result, we will assume that the bottom area is 0. The results are summarized in the following table along with the other quantities needed to determine the average concentration.

z, m	V, m^3	c, g/m^3	A_s, m^2	$c \times A_s$
0	9817500	10.2	1374450.0	14019390
4	5105100	8.5	981750.0	8344875
8	1963500	7.4	589050.0	4358970
12	392700	5.2	245437.5	1276275
16	0	4.1	0	0

The necessary integrals can then be evaluated with the multi-segment Simpson's 1/3 rule,

$$\int_0^z A_s(z)\,dz = (16-0)\frac{1{,}374{,}450 + 4(981{,}750 + 245{,}437.5) + 2(589{,}050) + 0}{12} = 9{,}948{,}400\ \text{m}^3$$

$$\int_0^z c(z)A_s(z)\,dz = (16-0)\frac{14{,}019{,}390 + 4(8{,}344{,}875 + 1{,}276{,}275) + 2(4{,}358{,}970) + 0}{12} = 81{,}629{,}240\ \text{g}$$

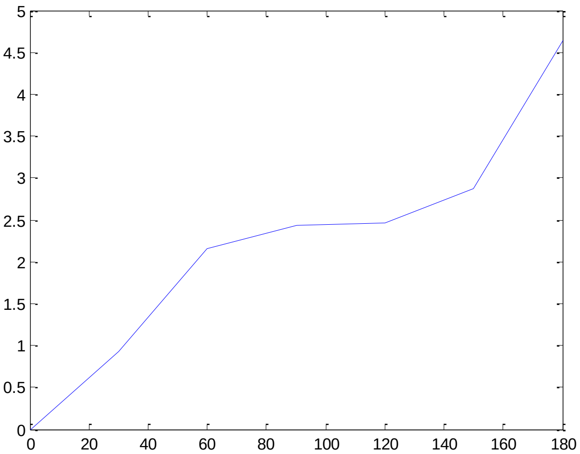
The average concentration can then be computed as

$$\bar{c} = \frac{\int_0^Z c(z)A_s(z)\,dz}{\int_0^Z A_s(z)\,dz} = \frac{81{,}629{,}240}{9{,}948{,}400} = 8.205263\frac{\text{g}}{\text{m}^3}$$

19.19 The following script can be written to solve this problem:

```
format short g
x=[0 1 2.8 3.9 3.8 3.2 1.3];
th=[0 30 60 90 120 150 180];
F=cos(th*pi/180);
W=cumtrapz(x,F)
plot(th,W)
```

```
W =
         0    0.93301      2.1624      2.4374      2.4624      2.8722
    4.645
```



19.20 The work can be computed as

$$W = \int_0^{30} (1.6x - 0.045x^2)\cos(-0.00055x^3 + 0.0123x^2 + 0.13x)\,dx$$

For the 4-segment trapezoidal rule, we can compute values of the integrand at equally-spaced values of x with $h = 7.5$. The results are summarized in the following table,

x	$F(x)$	$\theta(x)$	$F(x)\cos\theta(x)$	Trap Rule
0	0	0	0	
7.5	9.46875	1.434844	1.283339	4.812522
15	13.875	2.86125	–13.333330	–45.187464
22.5	13.21875	2.887031	–12.792760	–97.972837
30	7.5	0.12	7.446065	–20.050109
			Sum →	–158.39789

The finer-segment versions can be generated in a similar fashion. The results are summarized below:

Segments	W
4	–158.398
8	–159.472
16	–157.713

The computation can be also implemented with a tool like MATLAB's `trapz` function,

```
>> F=@(x) (1.6*x-0.045*x.^2).*cos(-
0.00055*x.^3+0.0123*x.^2+0.13*x);
>> W=quad(F,0,30)
W =
 -157.0871

>> format short g
>> x=[0:0.1:30];
>> Fx=1.6*x-.045*x.^2;
>> th=-0.00055*x.^3+0.0123*x.^2+0.13*x;
>> Fcos=Fx.*cos(th);
>> W=trapz(x,Fcos)
W =
      -157.09
```

19.21 The mass can be computed as

$$m = \int_0^r \rho(r) A_s(r)\, dr$$

The surface area of a sphere, $A_s(r) = 4\pi r^2$, can be substituted to give

$$m = \int_0^r \rho(r) 4\pi r^2 \, dr$$

The average density is equal to the mass per volume, where the volume of a sphere is

$$V = \frac{4}{3}\pi R^3$$

where R = the sphere's radius. For this problem, $V = 0.0041888$ cm^3. The integral can be evaluated by a combination of trapezoidal and Simpson's rules as outlined in the following table

r, mm	**r, cm**	**ρ (g/cm^3)**	**A_s (cm^2)**	**$\rho \times A_s$**	**Integrals**	**Method**
0	0	6	0	0		
0.12	0.012	5.81	0.00181	0.010514		
0.24	0.024	5.14	0.007238	0.037204		
0.36	0.036	4.29	0.016286	0.069867	0.000959	← Simpson's 3/8 rule
0.49	0.049	3.39	0.030172	0.102283		
0.62	0.062	2.7	0.048305	0.130424	0.002641	← Simpson's 1/3 rule
0.79	0.079	2.19	0.078427	0.171755	0.002569	← Trapezoidal rule
0.86	0.086	2.1	0.092941	0.195176		
0.93	0.093	2.04	0.108687	0.221721		
1	0.1	2	0.125664	0.251327	0.004394	← Simpson's 3/8 rule
				mass →	0.010562	

Therefore, the mass is 0.010562 g and the average density is $\bar{\rho} = 0.010562 / 0.0041888 = 2.521393 \text{ g/cm}^3$.

An alternative would be to use the trapezoidal rule. This can be done using MATLAB and the `trapz` function,

```
format short g
r=[0 0.12 0.24 0.36 0.49 0.62 0.79 0.86 0.93 1];
rho=[6 5.81 5.14 4.29 3.39 2.7 2.19 2.1 2.04 2];
r=r*0.1;  %convert radius to cm
integrand=rho*4*pi.*r.^2;
m=trapz(r,integrand)
V=4/3*pi*max(r).^3;
density=m./V
```

When this script is run the results are:

```
mass =
     0.010591
density =
       2.5284
```

19.22 The mass can be computed as

$$m = \int_0^r \rho(r) A_s(r)\, dr$$

The surface area of a sphere, $A_s(r) = 4\pi r^2$, can be substituted to give

$$m = \int_0^r \rho(r) 4\pi r^2 \; dr$$

The average density is equal to the mass per volume, where the volume of a sphere is

$$V = \frac{4}{3}\pi R^3$$

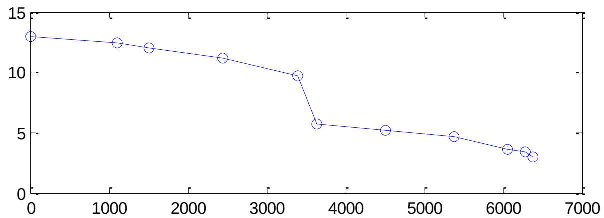
where R = the sphere's radius. For this problem, $V = 1.0878\times10^{21}$ m^3. The integral can be evaluated by a combination of trapezoidal and Simpson's rules as outlined in the following table. The following script can be used to implement these equations and solve this problem with MATLAB.

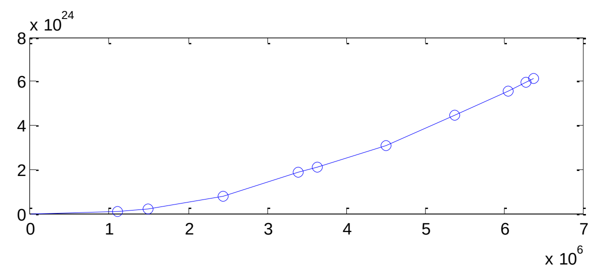
```
clear,clc,clf
format short g
r=[0 1100 1500 2450 3400 3630 4500 5380 6060 6280 6380];
rho=[13 12.4 12 11.2 9.7 5.7 5.2 4.7 3.6 3.4 3];
subplot(2,1,1)
plot(r,rho,'o-')
r=r*1e3;  %convert km to m
rho=rho*1e6/1e3;  %convert g/cm3 to kg/m3
integrand=rho*4*pi.*r.^2;
```

```
mass=cumtrapz(r,integrand);
EarthMass=max(mass)
EarthVolume=4/3*pi*max(r).^3
EarthDensity=EarthMass/EarthVolume/1e3
subplot(2,1,2)
plot(r,mass,'o-')
```

The results are

```
EarthMass =
  6.1087e+024
EarthVolume =
  1.0878e+021
EarthDensity =
        5.6156
```





19.23 The volume flowing out of the tank can be computed by integration with conversions to express the result in liters.

$$V = \frac{1 \text{ hr}}{3600 \text{ s}} \times \frac{10^3 \text{L}}{\text{m}^3} \times \int_0^{7500} Q\, dt$$

The volume of a partially filled spherical tank

$$V = \frac{\pi}{3} H^2 (3r - H)$$

Thus, if we know V and r, determining H can be expressed as a roots problem

$$f(H)=\frac{\pi}{3}H^2(3r-H)-V$$

Here is the script that solves the problem

```
clear,clc
format short g, format compact
t=[0  500 1000  1500  2200  2900  3600  4300  5200  6500  7000
7500];
Q=[10.55  9.576 9.072 8.640 8.100 7.560 7.020 6.480 5.688 4.752
3.348 1.404];
V_liters=1e3/3600*trapz(t,Q)
r=1.5;
V_m3=V_liters/1e3
Vfull=4/3*pi*r^3
f=@(H) pi/3*H^2*(3*r-H)-V_m3;
Hinitial=fzero(f,2*r)

V_liters =
        14101
V_m3 =
       14.101
Vfull =
       14.137
Hinitial =
       2.9117
```

19.24

```
clear,clc,format compact
x=[0 1 2 3 4 5 6 7 8 9 10 11 12 13];
y=0.078125*x.^3-1.25*x.^2+5.75*x;
EquispacedSimp(x,y)

function intout = EquispacedSimp(x,y)
n = length(x);
if length(y)~=n, error('x and y must be same length'), end
if any(diff(diff(x))~=0), error('unequal spacing'), end
if any(diff(x)<=0), error('not in ascending order'), end
if n==1
  error('Vectors must have at least 2 values')
elseif n==2
  intout=(x(2)-x(1))*(y(1)+y(2))/2;
elseif n==3
  intout=(x(3)-x(1))*(y(1)+4*y(2)+y(3))/6;
elseif n==4
  intout=(x(4)-x(1))*(y(1)+3*(y(2)+y(3))+y(4))/8;
elseif mod(n,2)
  intout=(x(n)-x(1))*simp13(x,y,n)/(3*(n-1));
else
  intout=simp13_38(x,y);
end

function s = simp13(x,y,n)
```

```
s=y(1);
for i = 2:2:n-2
  s=s+4*y(i)+2*y(i+1);
end
s=s+4*y(n-1)+y(n);
end

function iout = simp13_38(x,y)
n=length(x);
s1=(x(n-3)-x(1))*simp13(x,y,n-3)/(3*(n-4));
s2=(x(n)-x(n-3))*(y(n-3)+3*(y(n-2)+y(n-1))+y(n))/8;
iout=s1+s2;
end
```

Results:

```
ans =
  128.2904
```

19.25 The net force and the height at which it would be applied (the "line of action") can be determined by

$$F = \int_0^z f(z)\,dz$$

$$d = \frac{\int_0^z zf(z)\,dz}{\int_0^z f(z)\,dz}$$

The data from the plot can be tabulated, along with a row for the force times height

z, m	0	50	100	150	225	300	375	450	600
F, kN/m	0	30	40	40	50	50	60	80	100
$F\times z$, kN	0	1500	4000	6000	11250	15000	22500	36000	60000

$$\begin{aligned} F &= (150-0)\frac{0+3(30+40)+40}{8} + (300-150)\frac{40+4(50)+50}{6} \\ &\quad + (450-300)\frac{50+4(60)+80}{6} + (600-450)\frac{80+100}{2} \\ &= 4687.5 + 7250 + 9250 + 13500 = 34687.5 \text{ kN} \end{aligned}$$

$$\begin{aligned} \int_0^z zf(z)\,dz &= (150-0)\frac{0+3(1500+4000)+6000}{8} + (300-150)\frac{6000+4(11250)+15000}{6} \\ &\quad + (450-300)\frac{15000+4(22500)+36000}{6} + (600-450)\frac{36000+60000}{2} \\ &= 421875 + 1650000 + 3525000 + 7200000 = 12{,}796{,}875 \text{ kN m} \end{aligned}$$

$$d = \frac{12{,}796{,}875}{34{,}687.5} = 368.92 \text{ m}$$

19.26 (a)

$$I = (8-0)\frac{0+4(18)+31}{6} + (16-8)\frac{31+4(42)+50}{6} + (28-16)\frac{50+3(56+61)+65}{8} + (30-28)\frac{65+70}{2}$$
$$= 137.3333 + 332 + 699 + 135 = 1303.33 \text{ m}$$

(b)

$$v = \frac{1303.33 \text{ m}}{30 \text{ s}} = 43.4444 \frac{\text{m}}{\text{s}}$$

19.27 As shown below compute the product of density and area for each value of x.

x (m)	ρ (g/cm^3)	A_c (cm^2)	$\rho \times A_c$
0	4.00	100	400
2	3.95	103	406.85
3	3.89	106	412.34
4	3.80	110	418
6	3.60	120	432
8	3.41	133	453.53
10	3.30	150	495

Then because of the spacing of the points use trapezoidal rule ($x = 0$ to 2), Simpson's 1/3 rule ($x = 2$ to 4) and Simpson's 3/8 rule ($x = 4$ to 10).

$$m = (2-0)\frac{400+406.85}{2} + (4-2)\frac{406.85+4(412.34)+418}{6}$$
$$+ (10-4)\frac{418+3(432+453.53)+495}{8}$$
$$= 806.85 + 824.7367 + 2{,}677.193 = 4{,}308.779 \frac{\text{g}}{\text{cm}} \text{m} \times \frac{100 \text{ cm}}{\text{m}} = 430{,}878 \text{ g}$$

19.28

```
clear, clc, format compact
n=1.3;
Pinit=2550;
Pfinal=210;
Vfinal=.75;
Vinit=(Pfinal*(Vfinal^n)/Pinit)^(1/n);
c=Pfinal*Vfinal^n;
v=Vinit:.001:Vfinal;
p=c./(v.^n);
Work=trapz(v,p)
```

Results:

```
Work =
  409.0666
```

19.29 Script:

```
clear, clc, format compact
a=0.01; b=0.001; pinit=100; vinit=1; vfin=2;
k=(pinit+(a/(vinit^2))*(vinit-b));
v=vinit:.001:vfin;
p=(k./(v-b))-(a./(v.^2));
WorkDone=trapz(v,p);
str = ['The work done in expanding from ', num2str(vinit), ...
  ' cubic meter to ', num2str(vfin), ' cubic meter is ', ...
  num2str(WorkDone), ' kW.'];
disp(str);
```

Results:

```
The work done in expanding from 1 cubic meter to 2 cubic meter
is 69.3667 kW.
```