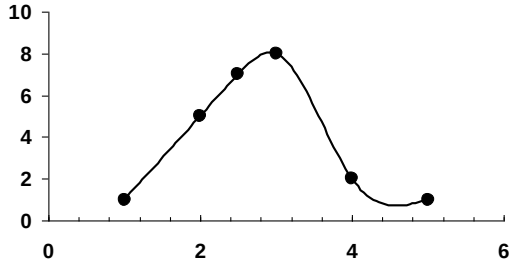
CHAPTER 18

18.1 **(a)** The simultaneous equations for the natural spline can be set up as

$$\begin{bmatrix} 1 & & & & & & \\ 1 & 3 & 0.5 & & & & \\ & 0.5 & 2 & 0.5 & & & \\ & & 0.5 & 3 & 1 & & \\ & & & 1 & 4 & 1 & \\ & & & & & 1 & \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -6 \\ -24 \\ 15 \\ 0 \end{Bmatrix}$$

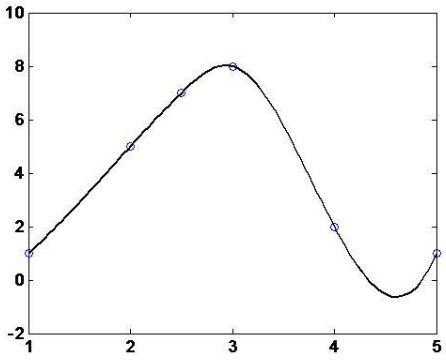
These equations can be solved for the *c*'s and then Eqs. (18.21) and (18.18) can be used to solve for the *b*'s and the *d*'s. The coefficients for the intervals can be tabulated and can be used to generate the following plot of the natural spline:

interval	*a*	*b*	*c*	*d*
1	1	3.970954	0	0.029046
2	5	4.058091	0.087137	–0.40664
3	7	3.840249	–0.52282	–6.31535
4	8	–1.41909	–9.99585	5.414938
5	2	–5.16598	6.248963	–2.08299



(b) The not-a-knot spline and its plot can be generated with the following MATLAB script:

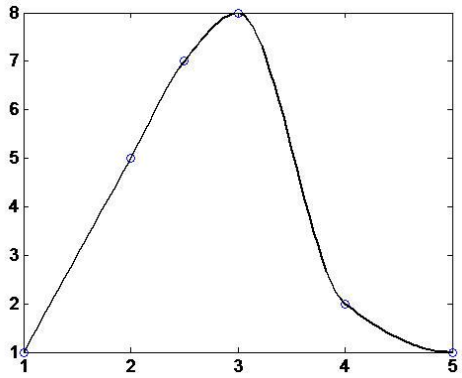
```
x = [1 2 2.5 3 4 5]; y = [1 5 7 8 2 1];
xx = linspace(1,5); yy = spline(x,y,xx);
plot(x,y,'o',xx,yy)
```



Notice how the not-a-knot version exhibits much more curvature, particularly between the last points.

(c) The piecewise cubic Hermite polynomial and its plot can be generated with MATLAB as

```
x = [1 2 2.5 3 4 5]; y = [1 5 7 8 2 1];
xx = linspace(1,5);
yy = interp1(x,y,xx,'pchip');
plot(x,y,'o',xx,yy)
```



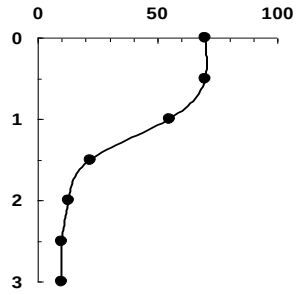
18.2 The simultaneous equations for the clamped spline with zero end slopes can be set up as

$$\begin{bmatrix} 1 & 0.5 & & & & & \\ 0.5 & 2 & 0.5 & & & & \\ & 0.5 & 2 & 0.5 & & & \\ & & 0.5 & 2 & 0.5 & & \\ & & & 0.5 & 2 & 0.5 & \\ & & & & 0.5 & 2 & 0.5 \\ & & & & & 0.5 & 1 \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \\ c_7 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -90 \\ -108 \\ 144 \\ 36 \\ 18 \\ 0 \end{Bmatrix}$$

These equations can be solved for the c's and then Eqs. (18.21) and (18.18) can be used to solve for the b's and the d's. The coefficients for the intervals can be summarized as

interval	a	b	c	d
1	70	0	15.87692	–31.7538
2	70	–7.93846	–31.7538	–24.7385
3	55	–58.2462	–68.8615	106.7077
4	22	–47.0769	91.2	–66.0923
5	13	–5.44615	–7.93846	13.66154
6	10	–3.13846	12.55385	–12.5538

The fit can be displayed in graphical form. Note that we are plotting the points as depth versus temperature so that the graph depicts how the temperature changes down through the tank.



Inspection of the plot indicates that the inflection point occurs in the 3rd interval. The cubic equation for this interval is

$$T_3(x) = 55 - 58.2462(x-1) - 68.861(x-1)^2 + 106.7077(x-1)^3$$

where T = temperature and x = depth. This equation can be differentiated to yield the first derivative

$$\frac{dT_3(x)}{dx} = -58.2462 - 137.723(x-1) + 320.1231(x-1)^2$$

and differentiated again to yield the second derivative

$$\frac{d^2T_3(x)}{dx^2} = -137.723 + 640.2462(x-1)$$

This can be set equal to zero and solved for the depth of the thermocline as $x = 1.21511$ m. The value of the gradient at that depth is

$$\frac{dT_3(x)}{dx} = -58.2462 - 137.723(1.21511-1) + 320.1231(1.21511-1)^2 = -73.059\frac{^\circ\text{C}}{\text{m}}$$

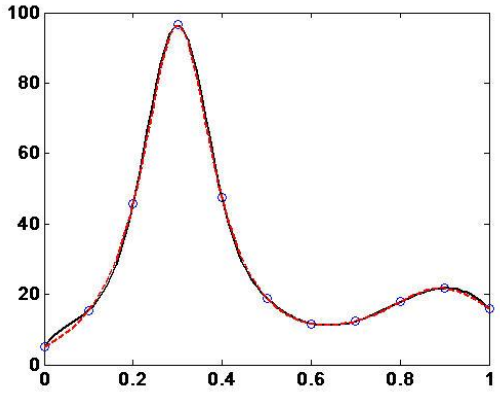
The heat flux can then be computed with Fick's law as

$$J = -0.01\frac{\text{cal}}{\text{s}\cdot\text{cm}\cdot^\circ\text{C}} \times \left(-73.059\frac{^\circ\text{C}}{\text{m}}\right) \times \frac{\text{m}}{100\ \text{cm}} = 0.007306\frac{\text{cal}}{\text{cm}^2\cdot\text{s}}$$

where the positive flux indicates that heat is transferred downward.

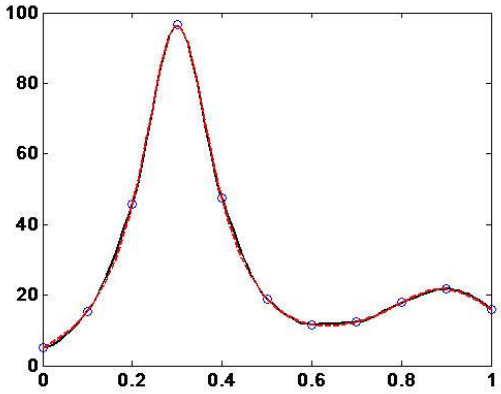
18.3 **(a)** The not-a-knot fit can be set up with the following script:

```
x = linspace(0,1,11); y = 1./((x-0.3).^2+0.01)+1./((x-
0.9).^2+0.04)-6;
xx = linspace(0,1);
yy = spline(x,y,xx);
yh = 1./((xx-0.3).^2+0.01)+1./((xx-0.9).^2+0.04)-6;
plot(x,y,'o',xx,yy,xx,yh,'--')
```



(b) The piecewise cubic Hermite polynomial fit can be set up with the following script:

```
x = linspace(0,1,11); y = 1./((x-0.3).^2+0.01)+1./((x-
0.9).^2+0.04)-6;
xx = linspace(0,1); yy = interp1(x,y,xx,'pchip');
yh = 1./((xx-0.3).^2+0.01)+1./((xx-0.9).^2+0.04)-6;
plot(x,y,'o',xx,yy,xx,yh,'--')
```



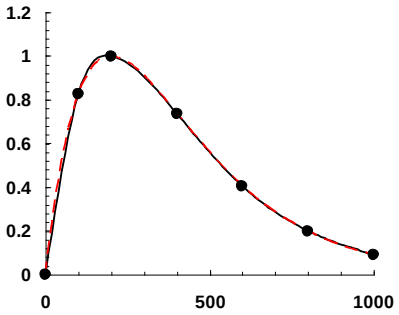
18.4 The simultaneous equations for the clamped spline with zero end slopes can be set up as

$$\begin{bmatrix} 1 & & & & & & \\ 100 & 400 & 100 & & & & \\ & 100 & 600 & 200 & & & \\ & & 200 & 800 & 200 & & \\ & & & 200 & 800 & 200 & \\ & & & & 200 & 800 & 200 \\ & & & & & & 1 \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \\ c_7 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -0.01946 \\ -0.00923 \\ -0.00098 \\ 0.001843 \\ 0.001489 \\ 0 \end{Bmatrix}$$

These equations can be solved for the c's and then Eqs. (18.21) and (18.18) can be used to solve for the b's and the d's. The coefficients for the intervals can be summarized as

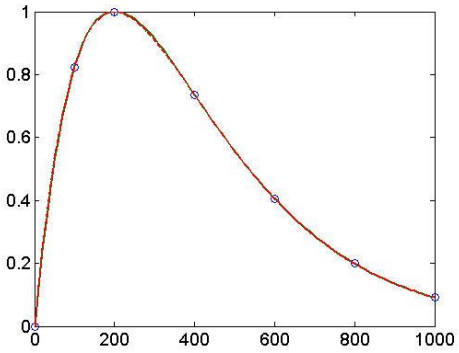
interval	*a*	*b*	*c*	*d*
1	0	0.009801	0	–1.6E–07
2	0.824361	0.005128	–4.7E–05	1.3E–07
3	1	–0.00031	–7.7E–06	1.31E–08
4	0.735759	–0.0018	2.13E–07	2.82E–09
5	0.406006	–0.00138	1.9E–06	–8.7E–10
6	0.199148	–0.00072	1.39E–06	–2.3E–09

The fit can be displayed in graphical form as



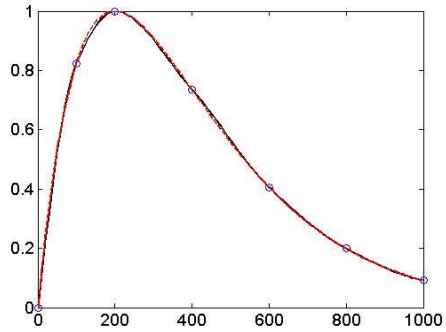
(b) The not-a-knot fit can be set up with the following script:

```
x = [0 100 200 400 600 800 1000]; y = x/200.*exp(-x/200+1);
xx = linspace(0,1000);
yc = xx/200.*exp(-xx/200+1);
yy = spline(x,y,xx);
plot(x,y,'o',xx,yy,xx,yc,'--')
```



(c) The piecewise cubic Hermite polynomial fit can be set up with the following script:

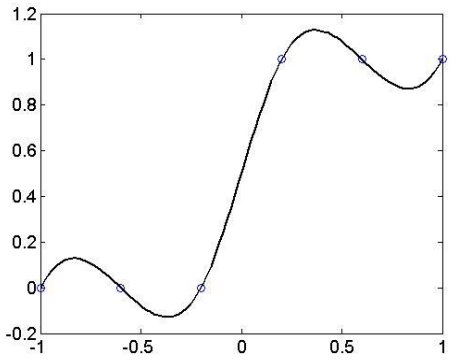
```
x = [0 100 200 400 600 800 1000]; y = x/200.*exp(-x/200+1);
xx = linspace(0,1000);
yc = xx/200.*exp(-xx/200+1);
yy = interp1(x,y,xx,'pchip');
plot(x,y,'o',xx,yy,xx,yc,'--')
```



Summary: For this case, the not-a-knot fit is the best.

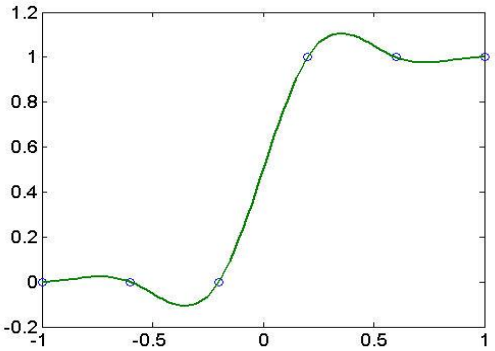
18.5 **(a)** The not-a-knot fit can be set up with the following script:

```
x = [-1 -0.6 -0.2 0.2 0.6 1]; y = [0 0 0 1 1 1];
xx = linspace(-1,1);
yy = spline(x,y,xx);
plot(x,y,'o',xx,yy)
```



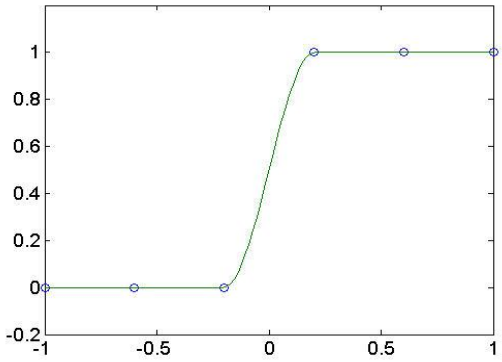
(b) The clamped spline with zero end slopes can be set up with the following script:

```
x = [-1 -0.6 -0.2 0.2 0.6 1]; y = [0 0 0 1 1 1];
ys = [0 y 0];
xx = linspace(-1,1); yy = spline(x,ys,xx);
plot(x,y,'o',xx,yy)
```



(c) The piecewise cubic Hermite polynomial fit can be set up with the following script:

```
x = [-1 -0.6 -0.2 0.2 0.6 1]; y = [0 0 0 1 1 1];
xx = linspace(-1,1); yy = interp1(x,y,xx,'pchip');
plot(x,y,'o',xx,yy)
```



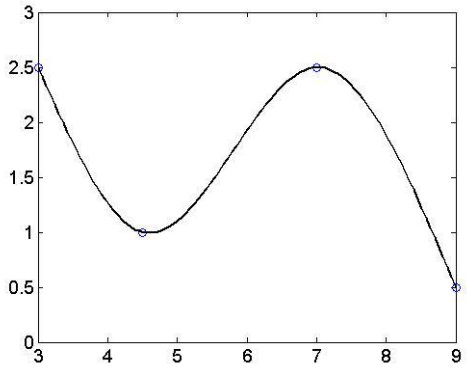
18.6 An M-file function to implement the natural spline can be written as

```
function yy = natspline(x,y,xx)
% natspline(x,y,xx):
%   uses a natural cubic spline interpolation to find yy, the
values
%   of the underlying function y at the points in the vector xx.
%   The vector x specifies the points at which the data y is
given.
n = length(x); m = length(xx);
aa(1,1) = 1; aa(n,n) = 1; bb(1) = 0; bb(n) = 0;
for i = 2:n-1
  aa(i,i-1) = h(x, i - 1);
  aa(i,i) = 2 * (h(x, i - 1) + h(x, i));
  aa(i,i+1) = h(x, i);
  bb(i) = 3 * (fd(i + 1, i, x, y) - fd(i, i - 1, x, y));
end
c = aa\bb';
for i = 1:n - 1
```

```
  a(i) = y(i);
  b(i) = fd(i + 1, i, x, y) - h(x, i) / 3 * (2 * c(i) + c(i +
1));
  d(i) = (c(i + 1) - c(i)) / 3 / h(x, i);
end
for i = 1:m
  yy(i) = SplineInterp(x, n, a, b, c, d, xx(i));
end

function hh = h(x, i)
hh = x(i + 1) - x(i);

function fdd = fd(i, j, x, y)
fdd = (y(i) - y(j)) / (x(i) - x(j));

function yyy = SplineInterp(x, n, a, b, c, d, xi)
for ii = 1:n - 1
  if xi >= x(ii) - 0.000001 & xi <= x(ii + 1) + 0.000001
    yyy=a(ii)+b(ii)*(xi-x(ii))+c(ii)*(xi-x(ii))^2+d(ii)*(xi-x(ii))^3;
    break
  end
end
```

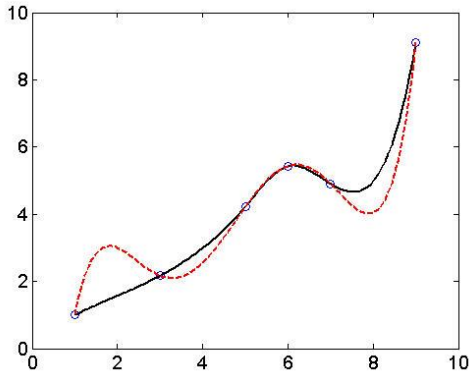
The following script uses the function to duplicate Example 18.3:

```
x = [3 4.5 7 9]; y = [2.5 1 2.5 .5];
xx = linspace(3,9); yy = natspline(x,y,xx);
plot(x,y,'o',xx,yy)
```



18.7 **(a)** The not-a-knot fit can be set up with the following script:

```
x = [1 3 5 6 7 9];
y = 0.0185*x.^5-0.444*x.^4+3.9125*x.^3-15.456*x.^2+27.069*x-14.1;
xx=linspace(1,9);
yy=spline(x,y,xx);
yc = 0.0185*xx.^5-0.444*xx.^4+3.9125*xx.^3-15.456*xx.^2+27.069*xx-
14.1;
plot(x,y,'o',xx,yy,xx,yc,'--')
```

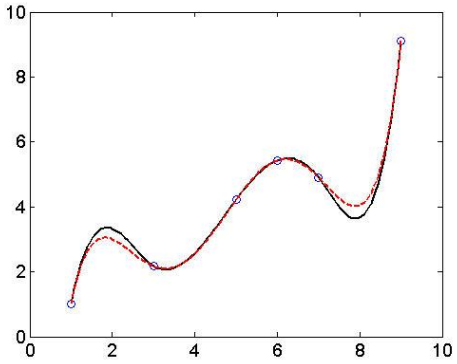


(b) The function can be differentiated to give

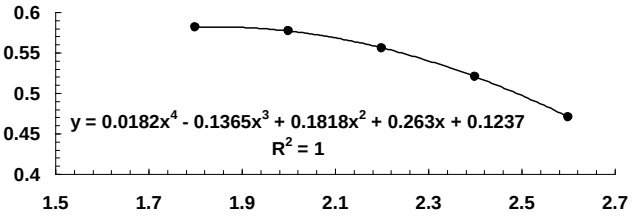
$$f'(x) = 0.0925x^4 - 1.776x^3 + 11.7375x^2 - 30.912x + 27.069$$

This function can be evaluated at the end nodes to give $f'(1) = 6.211$ and $f'(9) = 11.787$. These values can then be added to the y vector and the `spline` function invoked to develop the clamped fit:

```
yd = [6.211 y 11.787];
yy=spline(x,yd,xx);
plot(x,y,'o',xx,yy,xx,yc,'--')
```



18.8 (a) A 4th-order interpolating polynomial can be generated as



The polynomial can be used to compute

$$J_1(2.1) = 0.018229(2.1)^4 - 0.13646(2.1)^3 + 0.181771(2.1)^2 + 0.262958(2.1) + 0.1237 = 0.568304$$

The relative error is

$$\varepsilon_t = \left|\frac{0.568292 - 0.568304}{0.568292}\right| \times 100\% = 0.0021\%$$

Thus, the interpolating polynomial yields an excellent result.

(b) A program can be developed to fit natural cubic splines through data based on Fig. 18.12. If this program is run with the data for this problem, the interpolation at 2.1 is 0.56846 which has a relative error of $\varepsilon_t = 0.0295\%$.

A spline can also be fit with MATLAB. It should be noted that MATLAB does not use a natural spline. Rather, it uses a so-called "not-a-knot" spline. Thus, as shown below, although it also yields a very good prediction, the result differs from the one generated with the natural spline,

```
format long
x=[1.8 2 2.2 2.4 2.6]; y=[0.5815 0.5767 0.556 0.5202 0.4708];
spline(x,y,2.1)

ans =
   0.56829843750000
```

This result has a relative error of $\varepsilon_t = 0.0011\%$.

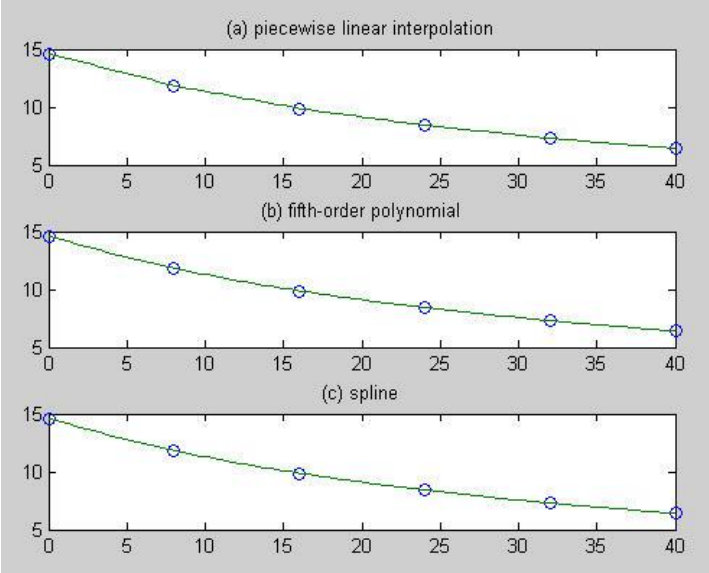
18.9 The following script generates the solution along with the plots:

```
otrue=7.986;
%(a) piecewise linear interpolation:
T=[0 8 16 24 32 40];
o=[14.621 11.843 9.870 8.418 7.305 6.413];
TT=linspace(0,40);
ol=interp1(T,o,TT);
subplot(3,1,1);plot(T,o,'o',TT,ol)
title('(a) piecewise linear interpolation')
opred=interp1(T,o,27);
et=abs((otrue-opred)/otrue)*100;
fprintf('o(27)-linear = %8.4g et = %8.4g pct\n', opred,et)
%(b) fifth-order polynomial:
a=polyfit(T,o,5);
op=polyval(a,TT);
subplot(3,1,2);plot(T,o,'o',TT,op)
title('(b) fifth-order polynomial')
```

```
opred=polyval(a,27);
et=abs((otrue-opred)/otrue)*100;
fprintf('o(27)-polyfit = %8.4g et = %8.4g pct\n', opred,et)
%(c) spline:
os=interp1(T,o,TT,'spline');
subplot(3,1,3);plot(T,o,'o',TT,os)
title('(c) spline')
opred=interp1(T,o,27,'spline');
et=abs((otrue-opred)/otrue)*100;
fprintf('o(27)-spline = %8.4g et = %8.4g pct\n', opred,et)
```

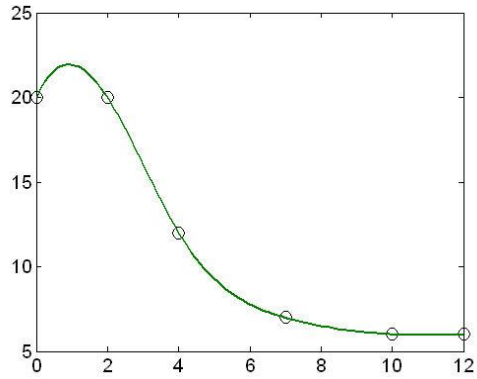
When it is run, the following results are generated:

```
o(27)-linear =    8.001 et =   0.1831 pct
o(27)-polyfit =    7.968 et =   0.2224 pct
o(27)-spline =    7.968 et =   0.2268 pct
```



18.10 (a) Here is a MATLAB script to develop the spline and plot it along with the data:

```
x=[0 2 4 7 10 12]; y=[20 20 12 7 6 6];
xx=linspace(0,12);
yy=spline(x,y,xx);
plot(x,y,'o',xx,yy)
```

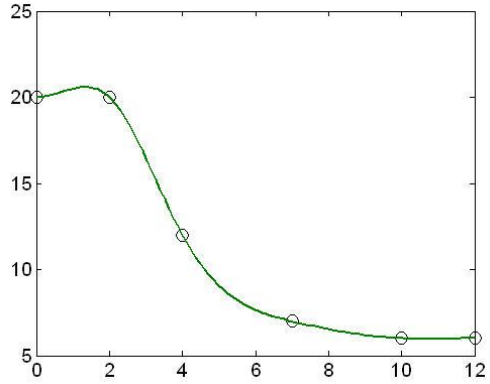


Here is the command to make the prediction:

```
>> yp=spline(x,y,1.5)

yp =
   21.3344
```

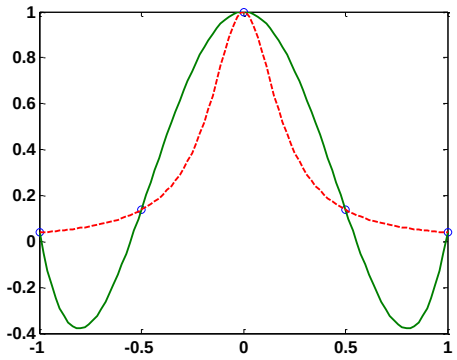
(b) To prescribe zero first derivatives at the end knots, the *y* vector is modified so that the first and last elements are set to the desired values of zero. The plot and the prediction both indicate that there is less overshoot between the first two points because of the prescribed zero slopes.

```
>> yd=[0 y 0];
>> yy=spline(x,yd,xx);
>> plot(x,y,'o',xx,yy)
>> yy=spline(x,yd,1.5)

yy =
   20.5701
```



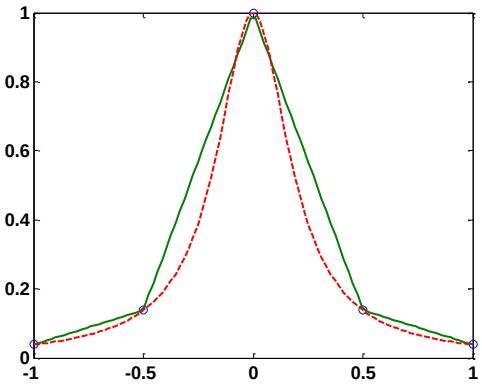
18.11 (a) Fourth-order polynomial:

```
clear,clc,clf
x=linspace(-1,1,5); y=1./(1+25*x.^2);
p=polyfit(x,y,4);
xx=linspace(-1,1); yy=polyval(p,xx);
yr=1./(1+25*xx.^2);
plot(x,y,'o',xx,yy,xx,yr,'--')
```



(b) linear spline:

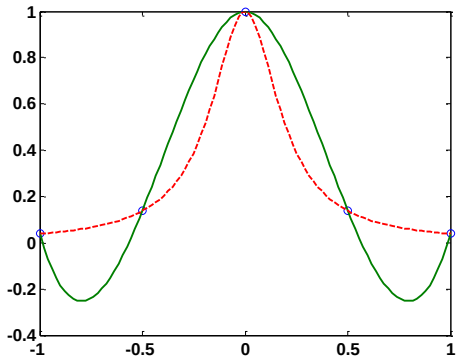
```
clf
x=linspace(-1,1,5); y=1./(1+25*x.^2);
xx=linspace(-1,1); yy=interp1(x,y,xx);
yr=1./(1+25*xx.^2);
plot(x,y,'o',xx,yy,xx,yr,'--')
```



(c) cubic spline:

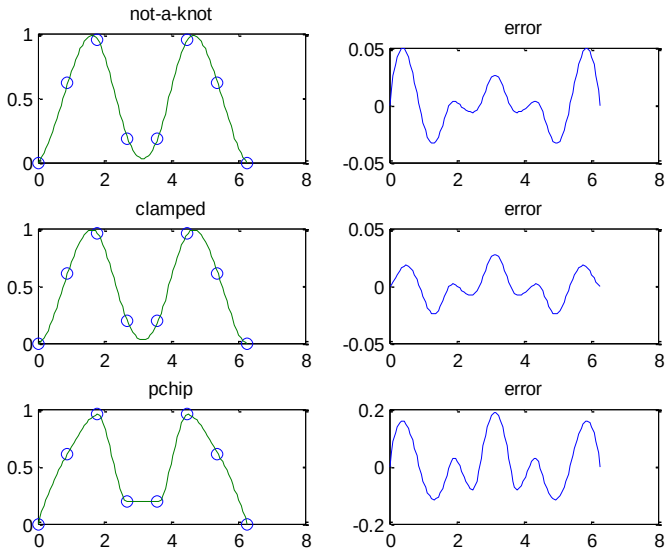
```
clf
x=linspace(-1,1,5); y=1./(1+25*x.^2);
xx=linspace(-1,1); yy=spline(x,y,xx);
```

```
yr=1./(1+25*xx.^2);
plot(x,y,'o',xx,yy,xx,yr,'--')
```



18.12 The following script generates the solution along with the plots:

```
clear,clc,clf
t=linspace(0,2*pi,8); f=sin(t).^2;
tt=linspace(0,2*pi); ftrue=sin(tt).^2;
%not-a-knot spline
f1=spline(t,f,tt);
Et1=f1-ftrue;
%cubic spline with zero derivatives at the end knots
fp=[0 f 0];
f2=spline(t,fp,tt);
Et2=f2-ftrue;
%piecewise cubic hermite interpolation
f3=pchip(t,f,tt);
subplot(3,1,3);plot(t,f,'o',tt,f3)
Et3=f3-ftrue;
subplot(3,2,1);plot(t,f,'o',tt,f1)
title('not-a-knot')
subplot(3,2,3);plot(t,f,'o',tt,f2)
title('clamped')
subplot(3,2,5);plot(t,f,'o',tt,f3)
title('pchip')
subplot(3,2,2);plot(tt,Et1)
title('error')
subplot(3,2,4);plot(tt,Et2)
title('error')
subplot(3,2,6);plot(tt,Et3)
title('error')
```



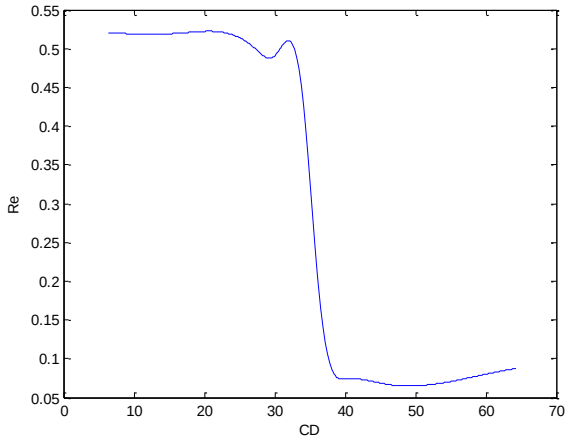
18.13 Function:

```
function CDout = Drag(ReCD,ReIn)
x=ReCD(1,:);y=ReCD(2,:);
CDout=spline(x,y,ReIn);
```

Script:

```
clear,clf,clc
ReCD=[2 5.8 16.8 27.2 29.9 33.9 36.3 40 46 60 100 200 400;
0.52 0.52 0.52 0.5 0.49 0.44 0.18 0.074 0.067 0.08 0.12 0.16
0.19];
x=ReCD(1,:);y=ReCD(2,:);
V=[4:0.1:40]; n=length(V);
rho=1.3;D=22/100;mu=1.78e-5;
ReNu=rho*V*D/mu/1e4;
CD=zeros(n,1);
for i = 1:n
  CD(i,1)= Drag(ReCD,ReNu(i));
end
plot(ReNu,CD);ylabel('Re');xlabel('CD')
```

Results:

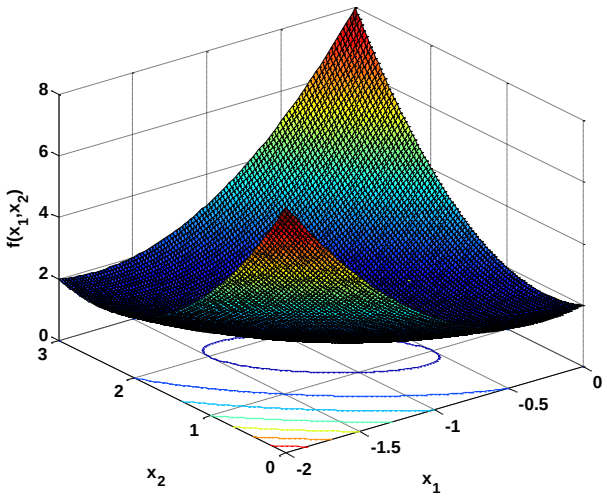


18.14 Script:

```
clear,clc,clf,format short g
x=linspace(-2,0,100);y=linspace(0,3,100);
[X,Y] = meshgrid(x,y);
Z=2+X-Y+2*X.^2+2*X.*Y+Y.^2;
cs=surfc(X,Y,Z);
zmin=floor(min(Z));
zmax=ceil(max(Z));
xlabel('x_1');ylabel('x_2');zlabel('f(x_1,x_2)');
x=linspace(-2,0,9);y=linspace(0,3,9);
[X,Y] = meshgrid(x,y);
Z=2+X-Y+2*X.^2+2*X.*Y+Y.^2;
xunk=-1.63;yunk=1.627;
ztrue=2+xunk-yunk+2*xunk.^2+2*xunk.*yunk+yunk.^2
zlinear=interp2(X,Y,Z,xunk,yunk)
et=abs((ztrue-zlinear)/ztrue)*100
zspline=interp2(X,Y,Z,xunk,yunk,'spline')
et=abs((ztrue-zspline)/ztrue)*100
```

Results:

```
ztrue =
       1.3999
zlinear =
       1.4626
et =
       4.4786
zspline =
       1.3999
et =
  6.3445e-014
```

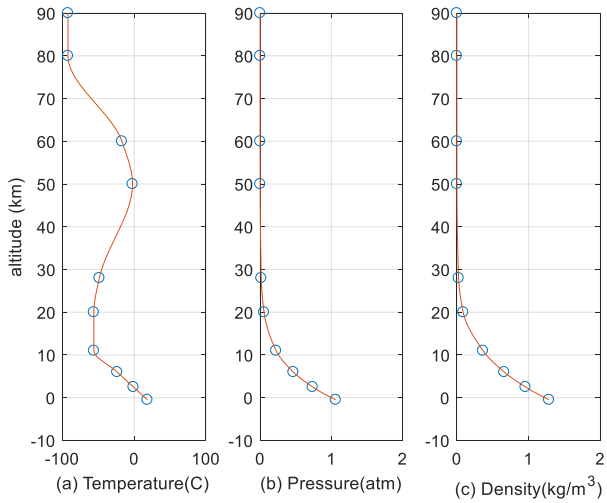


18.15

```
% Script to generate a plot of temperature, pressure and density
% for the U.S. Standard Atmosphere
clear,clc,clf,format short g,format compact
z=[-0.5 2.5 6 11 20 28 50 60 80 90];
T=[18.4 -1.1 -23.8 -56.2 -56.3 -48.5 -2.3 -17.2 -92.3 -92.3];
p=[1.0607 0.73702 0.46589 0.22394 0.054557 0.015946 ...
  7.8721e-4 2.2165e-4 1.02275e-05 1.6216e-06];
rho=[1.285025 0.95697 0.6601525 0.364805 0.0889105...
    0.02507575 0.001026918 0.000305883 0.000019992 3.1703e-06];
zint=[-0.5:0.1:90];
for i=1:length(zint)
  [Tint(i),pint(i),rint(i)]=StdAtm(z,T,p,rho,zint(i));
end
subplot(1,3,1)
plot(T,z,'o',Tint,zint),grid,xlabel('(a) Temperature(C)')
ylabel('altitude (km)')
subplot(1,3,2)
plot(p,z,'o',pint,zint)
grid,xlabel('(b) Pressure(atm)')
subplot(1,3,3)
plot(rho,z,'o',rint,zint),grid,xlabel('(c) Density(kg/m^3)')
Te=StdAtm(z,T,p,rho,-1000);

function [Tint,pint,rint] = StdAtm(z,T,p,rho,zint)
if zint<z(1) | zint>z(length(z))
  error('Outside range')
else
  Tint=interp1(z,T,zint,'pchip');
  pint=interp1(z,p,zint,'pchip');
  rint=interp1(z,rho,zint,'pchip');
end
end
```

Output:

```
Error using StdAtm (line 6)
Outside range

Error in StdAtmScript (line 22)
Te=StdAtm(z,T,p,rho,-1000);
```



18.16 The following script creates the plot:

```
clear,clf,clc
format compact
zz=[-1 0 1 2 3 4 5 6 7 8 9 10 15 20 25 30 40 50 60 70 80];
rho=[1.347 1.225 1.112 1.007 0.9093 0.8194 0.7364...
0.6601 0.5900 0.5258 0.4671 0.4135 0.1948 0.08891...
0.04008 0.01841 0.003996 0.001027 0.0003097 8.283e-5 1.846e-5];
A=0.55;Cd=1.1;m=80;
z=[0:0.5:40];
g=9.806412-0.003039734*z;
rho=spline(zz,rho,z);
cd=0.5*rho*A*Cd;
vterminal=sqrt(g*m./cd)*3600*3.281/5280;
plot(z,vterminal)
```