CHAPTER 16

16.1 (a) 12.8

(b) $\sqrt{4^2+3^2}=5$

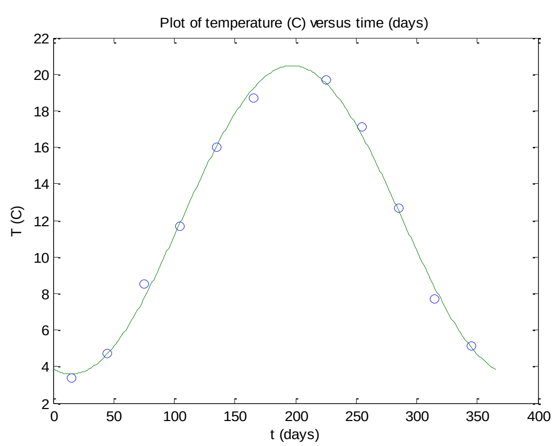
(c) 365

16.2 The following script can be developed to fit the data with linear least squares regression:

```
clear,clc,clf
format short g, format compact
w0=2*pi/365;
t=[15 45  75  105 135 165 225 255 285 315 345]';
T=[3.4  4.7 8.5 11.7  16  18.7  19.7  17.1  12.7  7.7 5.1]';
Z=[ones(size(t)) cos(w0*t) sin(w0*t)];
a=(Z'*Z)\(Z'*T)
theta = atan2(-a(3),a(2))*365/(2*pi)
Amplitude = sqrt(a(2)^2+a(3)^2)
time_Max_T = 365-theta
tp=[0:365];
Tp=a(1)+a(2)*cos(w0*tp)+a(3)*sin(w0*tp);
plot(t,T,'o',tp,Tp)
title('Plot of temperature (C) versus time (days)')
xlabel('t (days)'),ylabel('T (C)')
```

The resulting output is:

```
a =
       12.034
      -8.1724
      -2.1156
theta =
       167.78
Amplitude =
       8.4418
time_Max_T =
       197.22
```



16.3 The angular frequency can be computed as $\omega_0 = 2\pi/24 = 0.261799$. The various summations required for the normal equations can be set up as

t	y	$\cos(\omega_0 t)$	$\sin(\omega_0 t)$	$\sin(\omega_0 t)\cos(\omega_0 t)$	$\cos^2(\omega_0 t)$	$\sin^2(\omega_0 t)$	$y\cos(\omega_0 t)$	$y\sin(\omega_0 t)$
0	7.6	1.00000	0.00000	0.00000	1.00000	0.00000	7.60000	0.00000
2	7.2	0.86603	0.50000	0.43301	0.75000	0.25000	6.23538	3.60000
4	7	0.50000	0.86603	0.43301	0.25000	0.75000	3.50000	6.06218
5	6.5	0.25882	0.96593	0.25000	0.06699	0.93301	1.68232	6.27852
7	7.5	–0.25882	0.96593	–0.25000	0.06699	0.93301	–1.94114	7.24444
9	7.2	–0.70711	0.70711	–0.50000	0.50000	0.50000	–5.09117	5.09117
12	8.9	–1.00000	0.00000	0.00000	1.00000	0.00000	–8.90000	0.00000
15	9.1	–0.70711	–0.70711	0.50000	0.50000	0.50000	–6.43467	–6.43467
20	8.9	0.50000	–0.86603	–0.43301	0.25000	0.75000	4.45000	–7.70763
22	7.9	0.86603	–0.50000	–0.43301	0.75000	0.25000	6.84160	–3.95000
24	7	1.00000	0.00000	0.00000	1.00000	0.00000	7.00000	0.00000
sum→	**84.8**	**2.31784**	**1.93185**	**0.00000**	**6.13397**	**4.86603**	**14.94232**	**10.18401**

The normal equations can be assembled as

$$\begin{bmatrix} 11 & 2.317837 & 1.931852 \\ 2.317837 & 6.13397 & 0 \\ 1.931852 & 0 & 4.86603 \end{bmatrix} \begin{Bmatrix} A_0 \\ A_1 \\ B_1 \end{Bmatrix} = \begin{Bmatrix} 84.8 \\ 14.9423 \\ 10.184 \end{Bmatrix}$$

This system can be solved for $A_0 = 8.0270$, $A_1 = -0.59717$, and $B_1 = -1.09392$. Therefore, the best-fit sinusoid is

$$pH = 8.0270 - 0.59717\cos(\omega_0 t) - 1.09392\sin(\omega_0 t)$$

The result can also be expressed in the alternate form of Eq. 16.2 by computing the amplitude and the phase shift,

$$C_1 = \sqrt{(-0.59717)^2 + (-1.09392)^2} = 1.2463$$

$$\theta = \arctan\left(-\frac{-1.09392}{-0.59717}\right) + \pi = 2.0705 \times \frac{24 \text{ hr}}{2\pi} = 7.9087 \text{ hr}$$

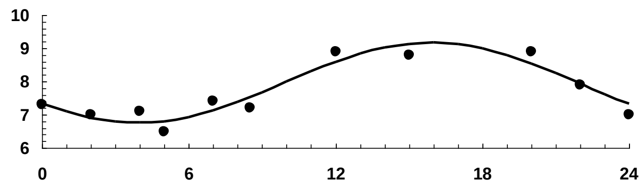
Therefore, the fit can also be expressed as

$$pH = 8.0270 + 1.2463\cos\left[\omega_0 (t + 7.9087)\right]$$

Consequently, the mean is 8.0270 and the amplitude is 1.2463. To determine the time of the maximum, inspection of Fig. 16.3 indicates that a positive phase shift represents the time prior to midnight that the peak occurs. Therefore the time of the maximum can be computed as

$$t_{max} = 24 - 7.9087 = 16.0913 \text{ hrs}$$

This is equal to about 16:05:29 or 4:05:29 PM. The data and the model can be plotted as

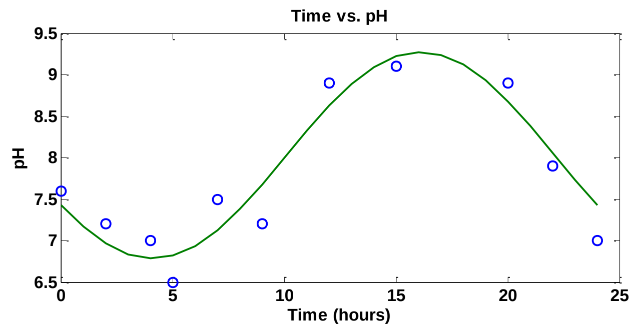


The same calculation can be implemented with MATLAB. The coefficients can also be determined with the following script:

```
clear,clc,clf,format compact
w0 = 2*pi/24;
t = [0 2 4 5 7 9 12 15 20 22 24]';
pH = [7.6 7.2 7 6.5 7.5 7.2 8.9 9.1 8.9 7.9 7]';
Z = [ones(size(t)) cos(w0*t) sin(w0*t)];
a = (Z'*Z)\(Z'*pH)
Sr = sum((pH - Z*a).^2);
syx = sqrt(Sr/(length(t)-length(a)))
tp = [0:24]; pHp = a(1)+a(2)*cos(w0*tp)+a(3)*sin(w0*tp);
plot(t,pH,'o',tp,pHp)
title('Time vs. pH'),xlabel('Time (hours)'),ylabel('pH')
mean=a(1),theta = atan2(-a(3),a(2))*24/(2*pi)
Amplitude = sqrt(a(2)^2+a(3)^2)
time_Max_pH = 24-theta
```

Output:

```
a =
    8.0270
   -0.5972
   -1.0939
syx =
    0.3447
mean =
    8.0270
theta =
    7.9087
Amplitude =
    1.2463
time_Max_pH =
   16.0913
```



16.4 The angular frequency can be computed as $\omega_0 = 2\pi/360 = 0.017453$. Because the data are equispaced, the coefficients can be determined with Eqs. 16.14-16.16. The various summations required to set up the model can be determined as

t	Radiation	$\cos(\omega_0 t)$	$\sin(\omega_0 t)$	$y\cos(\omega_0 t)$	$y\sin(\omega_0 t)$
15	144	0.96593	0.25882	139.093	37.270
45	188	0.70711	0.70711	132.936	132.936
75	245	0.25882	0.96593	63.411	236.652
105	311	−0.25882	0.96593	−80.493	300.403
135	351	−0.70711	0.70711	−248.194	248.194
165	359	−0.96593	0.25882	−346.767	92.916
195	308	−0.96593	−0.25882	−297.505	−79.716
225	287	−0.70711	−0.70711	−202.940	−202.940
255	260	−0.25882	−0.96593	−67.293	−251.141
285	211	0.25882	−0.96593	54.611	−203.810
315	159	0.70711	−0.70711	112.430	−112.430
345	131	0.96593	−0.25882	126.536	−33.905
sum→	**2954**			**−614.175**	**164.429**

The coefficients can be determined as

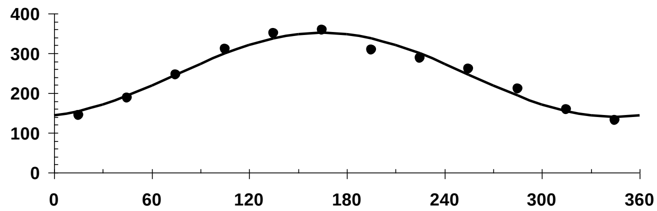
$$A_0 = \frac{\Sigma y}{N} = \frac{2954}{12} = 246.1667 \qquad A_1 = \frac{2}{N}\Sigma y\cos(\omega_0 t) = \frac{2}{12}(-614.175) = -102.363$$

$$B_1 = \frac{2}{N}\Sigma y\sin(\omega_0 t) = \frac{2}{12}(164.429) = 27.4048$$

Therefore, the best-fit sinusoid is

$$R = 246.1667 - 102.363\cos(0.017453t) + 27.4048\sin(0.017453t)$$

The data and the model can be plotted as



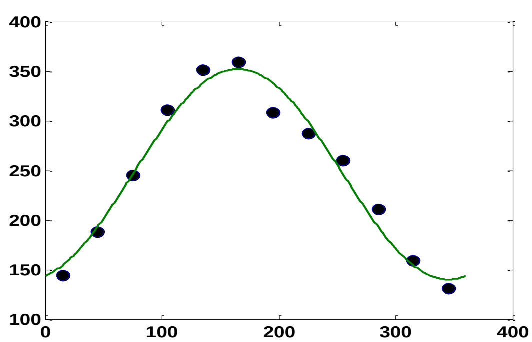
The value for mid-August can be computed as

$$R = 246.1667 - 102.363\cos(0.017453(225)) + 27.4048\sin(0.017453(225)) = 299.1698$$

The same calculation can be implemented with MATLAB. The coefficients can be determined with the following script:

```
clear,clc,clf
w0=2*pi/360;
t=[15 45 75 105 135 165 195 225 255 285 315 345]';
R=[144 188 245 311 351 359 308 287 260 211 159 131]';
Z=[ones(size(t)) cos(w0*t) sin(w0*t)];
a=(Z'*Z)\(Z'*R)
theta = atan2(-a(3),a(2))*360/(2*pi)
Amplitude = sqrt(a(2)^2+a(3)^2)
RadAug = a(1)+a(2)*cos(w0*225)+a(3)*sin(w0*225)
tp=[0:360];
Rp=a(1)+a(2)*cos(w0*tp)+a(3)*sin(w0*tp);
plot(t,R,'o',tp,Rp)

a =
  246.1667
 -102.3625
   27.4048
theta =
 -165.0121
Amplitude =
  105.9675
RadAug =
  299.1698
```



16.5 In the following equations, $\omega_0 = 2\pi/T$

$$\frac{\int_0^T \sin(\omega_0 t)dt}{T} = \frac{-\omega_0\left[\cos(\omega_0 t)\right]_0^T}{T} = \frac{-\omega_0}{T}(\cos 2\pi - \cos 0) = 0$$

$$\frac{\int_0^T \cos(\omega_0 t)dt}{T} = \frac{\omega_0\left[\sin(\omega_0 t)\right]_0^T}{T} = \frac{\omega_0}{T}(\sin 2\pi - \sin 0) = 0$$

$$\frac{\int_0^T \sin^2(\omega_0 t)dt}{T} = \frac{\left[\frac{t}{2} - \frac{\sin(2\omega_0 t)}{4\omega_0}\right]_0^T}{T} = \frac{\frac{T}{2} - \frac{\sin 4\pi}{4\omega_0} - 0 + 0}{T} = \frac{1}{2}$$

$$\frac{\int_0^T \cos^2(\omega_0 t)dt}{T} = \frac{\left[\frac{t}{2} + \frac{\sin(2\omega_0 t)}{4\omega_0}\right]_0^T}{T} = \frac{\frac{T}{2} + \frac{\sin 4\pi}{4\omega_0} - 0 - 0}{T} = \frac{1}{2}$$

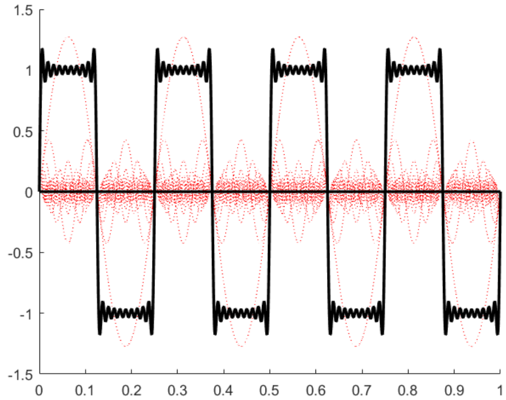
$$\frac{\int_0^T \cos(\omega_0 t)\sin(\omega_0 t)dt}{T} = \left[\frac{\sin^2(\omega_0 t)}{2T\omega_0}\right]_0^T = \frac{\sin^2 2\pi}{2\omega_0 T} - 0 = 0$$

16.6 The function is:

```
function [t,f] = FourierSquare(A0,T,n)
t=[0:T/256:4*T];
nn=length(t);
f=zeros(n,nn);
s=zeros(nn);
for ii = 1:n
  for j = 1:nn
    f(ii,j)=f(ii,j)+4*A0/(2*ii-1)/pi*sin(2*pi*(2*ii-1)*t(j)/T);
    s(j)=s(j)+f(ii,j);
  end
end
hold on
for ii=1:n
  plot(t,f(ii,:),'r:')
end
plot(t,s,'k-','linewidth',2)
hold off
end
```

The function can be run (with $n = 10$) to generate the resulting plot

```
clear,clf,clc
[t,f] = FourierSquare(1,0.25,10);
```



16.7 $a_0 = 0$

$$a_k = \frac{2}{T}\int_{-T/2}^{T/2} -2t\cos(k\omega_0 t)\, dt$$
$$= -\frac{4}{T}\left[\frac{1}{(k\omega_0)^2}\cos(k\omega_0 t) + \frac{t}{k\omega_0}\sin(k\omega_0 t)\right]_{-T/2}^{T/2}$$

$$b_k = \frac{2}{T}\int_{-T/2}^{T/2} -2t\sin(k\omega_0 t)\, dt$$
$$= -\frac{4}{T}\left[\frac{1}{(k\omega_0)^2}\sin(k\omega_0 t) - \frac{t}{k\omega_0}\cos(k\omega_0 t)\right]_{-T/2}^{T/2}$$

On the basis of these, all a's = 0.

For k = odd: $\quad b_k = -\dfrac{2}{k\pi}$

For k = even: $\quad b_k = \dfrac{2}{k\pi}$

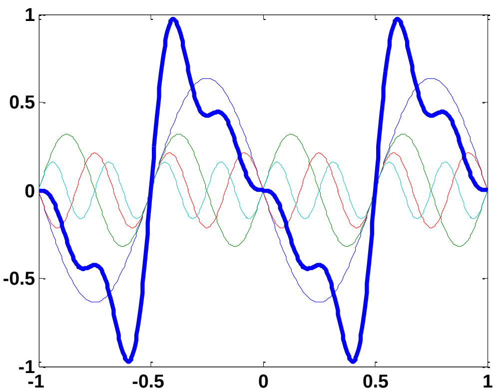
Therefore, the series is

$$f(t) = -\frac{2}{\pi}\sin(\omega_0 t) + \frac{1}{\pi}\sin(2\omega_0 t) - \frac{2}{3\pi}\sin(3\omega_0 t) + \frac{1}{2\pi}\sin(4\omega_0 t) + \cdots$$

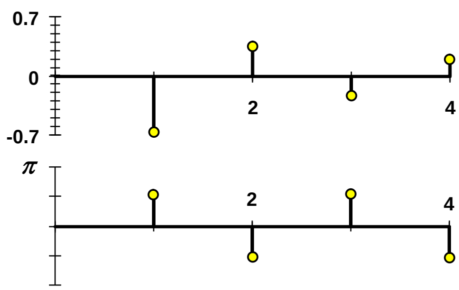
The following script plots the first 4 terms:

```
clear,clc,clf
T=1;w0=2*pi/T;tp=[-1:1/256:1];
term1=-2/pi*sin(w0*tp);
```

```
term2=1/pi*sin(2*w0*tp);
term3=-2/(3*pi)*sin(3*w0*tp);
term4=1/(2*pi)*sin(4*w0*tp);
summ=term1+term2+term3+term4;
plot(tp,term1,tp,term2,tp,term3,tp,term4)
hold on
plot(tp,summ,'linewidth',2)
hold off
```



Here are the amplitude and phase spectra:



16.8 $a_0 = 0.5$

$$a_k = \frac{2}{2}\left[\int_{-1}^{0} -t\cos(k\pi t)\,dt + \int_{0}^{1} t\cos(k\pi t)\,dt\right]$$

$$= 1\left\{\left[-\frac{\cos(k\pi t)}{(k\pi)^2} - \frac{t\sin(k\pi t)}{k\pi}\right]_{-1}^{0} + \left[\frac{\cos(k\pi t)}{(k\pi)^2} + \frac{t\sin(k\pi t)}{k\pi}\right]_{0}^{1}\right\} = \frac{2}{(k\pi)^2}(\cos k\pi - 1)$$

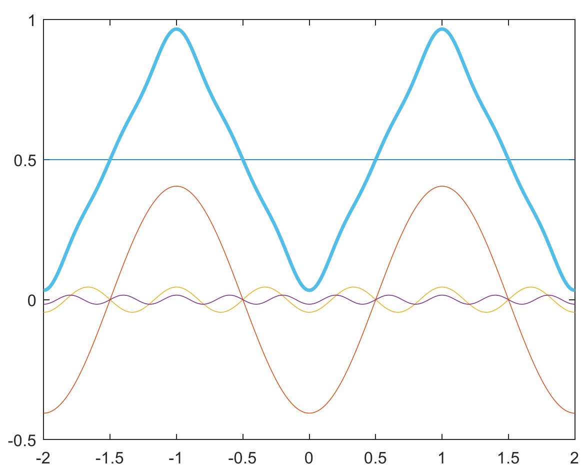
$b_k = 0$

Substituting these coefficients into Eq. (16.17) gives

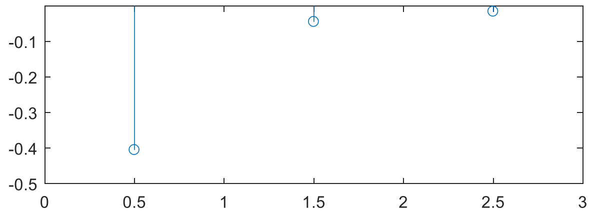
$$f(t) = \frac{1}{2} - \frac{4}{\pi^2}\cos(\pi t) - \frac{4}{9\pi^2}\cos(3\pi t) - \frac{4}{25\pi^2}\cos(5\pi t) + \cdots$$

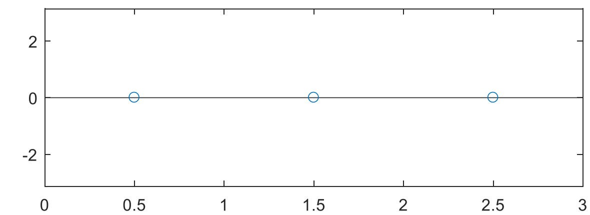
The following script generates a plot of the first 4 terms along with the summation as well as the amplitude and phase spectra:

```
clear,clc,clf
T=2;w0=2*pi/T;
tp=[-2:1/256:2];
term1=0.5*[1;1];
term2=-4/pi^2*cos(pi*tp);
term3=-4/(3*pi)^2*cos(3*pi*tp);
term4=-4/(5*pi)^2*cos(5*pi*tp);
summ=term1+term2+term3+term4;
plot([-2;2],term1,tp,term2,tp,term3,tp,term4)
hold on
plot(tp,summ,'linewidth',2)
hold off
pause
clf
f=[w0/(2*pi) 3*w0/(2*pi) 5*w0/(2*pi)];
y=[-4/pi^2 -4/(3*pi)^2 -4/(5*pi)^2];
subplot(2,1,1)
stem(f,y),xlim([0 3]),ylim([-0.5 0])
y=[0 0 0];
subplot(2,1,2)
stem(f,y),xlim([0 3]),ylim([-pi pi])
```



Here are the amplitude and phase spectra:





16.9 The Maclaurin series expansions are

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \cdots$$

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

Euler's formula with positive exponent is

$$e^{ix} = \cos x + i \sin x$$

Substitute series into the formula

$$1 + ix + \frac{(ix)^2}{2} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \frac{(ix)^6}{6!} + \frac{(ix)^7}{7!} + \cdots = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right)$$

Understanding that $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, $i^5 = i$, $i^6 = -1$, …:

$$1 + ix - \frac{x^2}{2} - \frac{x^3}{3!}i + \frac{x^4}{4!} + \frac{x^5}{5!}i - \frac{x^6}{6!} - \frac{x^7}{7!}i + \cdots = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + ix - i\frac{x^3}{3!} + i\frac{x^5}{5!} - i\frac{x^7}{7!} + \cdots$$

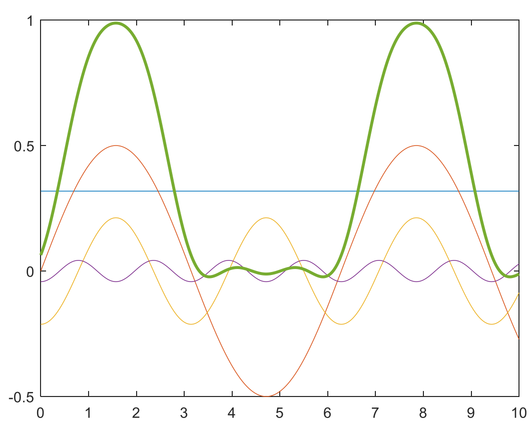
Collecting imaginary and real parts on the left-hand side yields

$$\left(1-\frac{x^2}{2}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots\right)+\left(ix-\frac{x^3}{3!}i+\frac{x^5}{5!}i-\frac{x^7}{7!}i+\cdots\right)=1-\frac{x^2}{2}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots+ix-i\frac{x^3}{3!}+i\frac{x^5}{5!}-i\frac{x^7}{7!}+\cdots$$

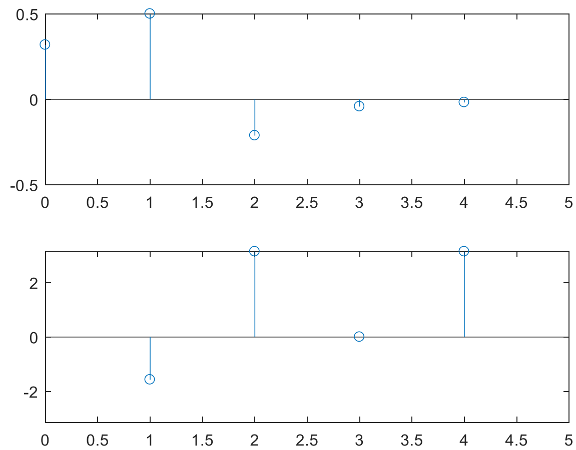
Thus, we can see that the left and right-hand sides are equivalent.

16.10 Here is a script to generate the plot of the first 4 terms along with the spectra:

```
clear,clc,clf
t=0:.01:10;
term1=1/pi*ones(1,length(t));
term2=0.5*sin(t);
term3=-2/(3*pi)*cos(2*t);
term4=-2/(15*pi)*cos(4*t);
summ=term1+term2+term3+term4;
plot(t,term1,t,term2,t,term3,t,term4)
hold on
plot(t,summ,'linewidth',2)
hold off
pause
clf
f=[0 1 2 3 4];
y=[1/pi 1/2 -2/(3*pi) -2/(15*pi) -2/(35*pi)];
subplot(2,1,1)
stem(f,y);xlim([0 5]),ylim([-0.5 0.5])
f=[1 2 3 4];
y=[-pi/2 pi 0 pi];
subplot(2,1,2)
stem(f,y);xlim([0 5]);ylim([-pi pi])
```

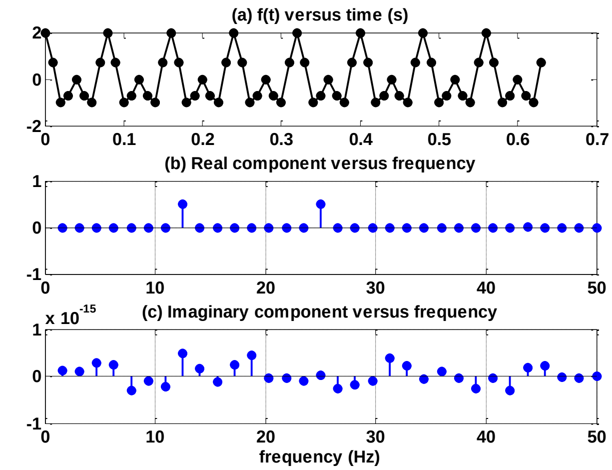


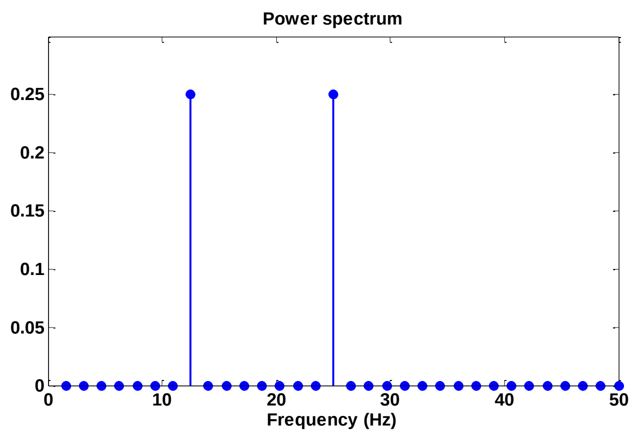
(b) The amplitude and phase line spectra can be developed as:



16.11

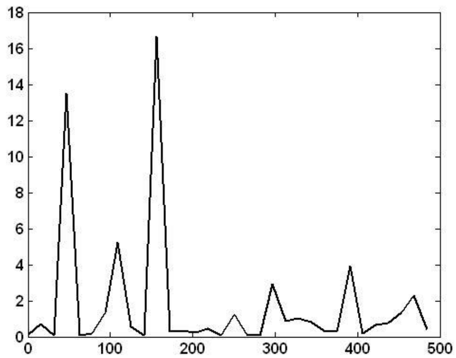
```
clear,clc,clf
n=64; dt=0.01; fs=1/dt;T=n/fs;tspan=(0:n-1)/fs;
y=cos(2*pi*12.5*tspan)+cos(2*pi*25*tspan);
subplot(3,1,1);
plot(tspan,y,'-ok','linewidth',2,'MarkerFaceColor','black');
title('(a) f(t) versus time (s)');
Y=fft(y)/n;
nyquist=fs/2;fmin=1/T;
f = linspace(fmin,nyquist,n/2);
Y(1)=[];YP=Y(1:n/2);
subplot(3,1,2)
stem(f,real(YP),'linewidth',2,'MarkerFaceColor','blue')
grid;title('(b) Real component versus frequency')
subplot(3,1,3)
stem(f,imag(YP),'linewidth',2,'MarkerFaceColor','blue')
grid;title('(c) Imaginary component versus frequency')
xlabel('frequency (Hz)')
pause
% compute and display the power spectrum
clf
Pyy = abs(Y(1:n/2)).^2;
stem(f,Pyy,'linewidth',2,'MarkerFaceColor','blue')
title('Power spectrum')
xlabel('Frequency (Hz)');ylim([0 0.3])
```





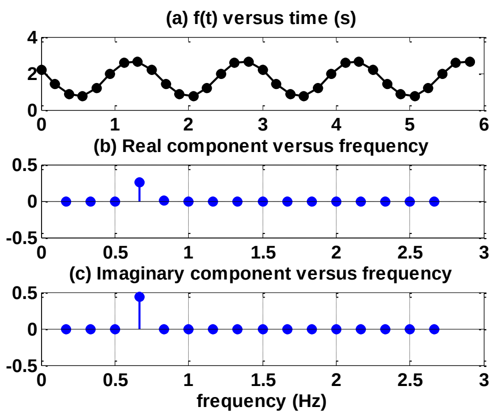
16.12 The following MATLAB session develops the `fft` along with a plot of the power spectral density versus frequency.

```
>> t=0:63;
>> y=cos(10*2*pi*t/63)+sin(3*2*pi*t/63)+randn(size(t));
>> Y=fft(y,64);
>> Pyy=Y.*conj(Y)/64;
>> f=1000*(0:31)/64;
>> plot(f,Pyy(1:32))
```



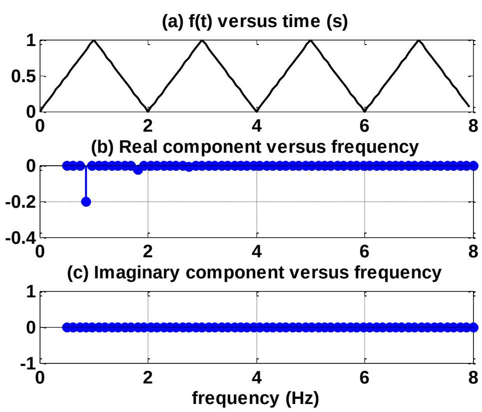
16.13

```
clear,clc,clf
T=6;n=32;dt=T/n;fs=1/dt;
tspan=linspace(0,6-dt,32);
w0=2*pi/1.5;y=1.7+cos(w0*tspan+pi/3);
subplot(3,1,1);
plot(tspan,y,'-ok','linewidth',2,'MarkerFaceColor','black');
title('(a) f(t) versus time (s)');
Y=fft(y)/n;
nyquist=fs/2;fmin=1/T;
f = linspace(fmin,nyquist,n/2);
Y(1)=[];YP=Y(1:n/2);
subplot(3,1,2)
stem(f,real(YP),'linewidth',2,'MarkerFaceColor','blue')
grid;title('(b) Real component versus frequency')
subplot(3,1,3)
stem(f,imag(YP),'linewidth',2,'MarkerFaceColor','blue')
grid;title('(c) Imaginary component versus frequency')
xlabel('frequency (Hz)')
```



16.14

```
clear,clc,clf
Tp=2;n=128;dt=8/n;fs=1/dt;
tspan=linspace(0,4*Tp-4*Tp/n,n);
y=abs(2*(tspan/Tp-floor(tspan/Tp+1/2)));
subplot(3,1,1);
plot(tspan,y,'-k','linewidth',2);
title('(a) f(t) versus time (s)');
Y=fft(y)/n;
nyquist=fs/2;fmin=1/Tp;
f = linspace(fmin,nyquist,n/2);
Y(1)=[];YP=Y(1:n/2);
subplot(3,1,2)
stem(f,real(YP),'linewidth',2,'MarkerFaceColor','blue')
grid;title('(b) Real component versus frequency')
subplot(3,1,3)
stem(f,imag(YP),'linewidth',2,'MarkerFaceColor','blue')
grid;title('(c) Imaginary component versus frequency')
xlabel('frequency (Hz)')
ylim([-1 1])
```



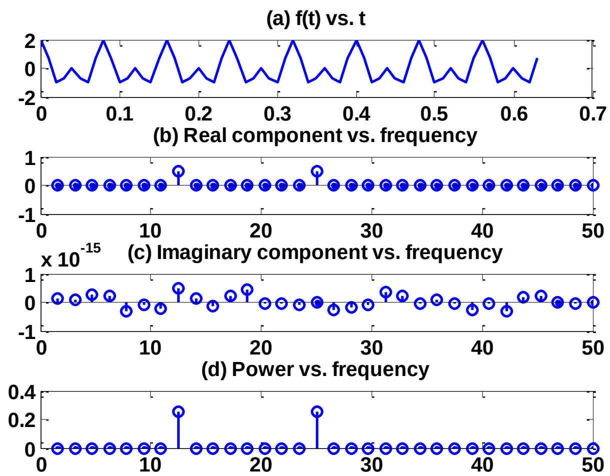
16.15 Here is the function:

```
function Y=fftmakerNew(t,y)
subplot(4,1,1)
plot(t,y),title('(a) f(t) vs. t')
n=length(t);dt=(max(t)-min(t))/(n-1);
T = t(n)- t(1) + dt;
fs=1/dt;nyquist=fs/2;fmin=1/T;fmax=.5*fs;df=fs/n;
Y=fft(y)/n;Y(1)=[];
f=[fmin:df:fmax];
YP=Y(1:length(f));
subplot(4,1,2)
stem(f,real(YP)),title('(b) Real component vs. frequency')
subplot(4,1,3)
stem(f,imag(YP)),title('(c) Imaginary component vs. frequency')
P=abs(YP.^2);
subplot(4,1,4)
stem(f,P),title('(d) Power vs. frequency')
```

Here is a script that uses the function to solve Prob. 16.11:

```
clear,clc,clf,format compact,format short g
n=64; dt=0.01; fs=1/dt;
tspan=(0:n-1)/fs;
y=cos(2*pi*12.5*tspan)+cos(2*pi*25*tspan);
Y=fftmakerNew(tspan,y);
```

When this script is run, the result is:



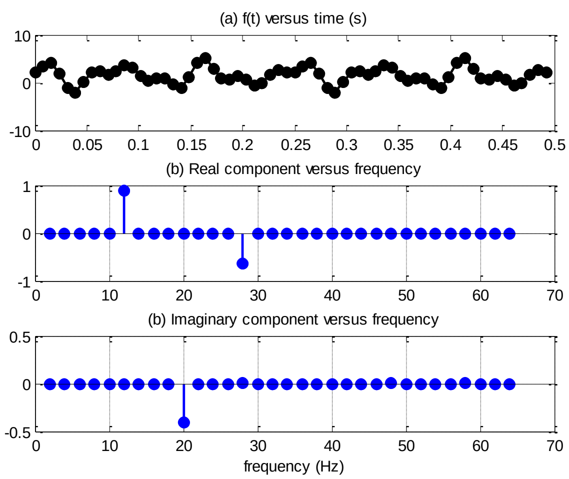
16.16 Script:

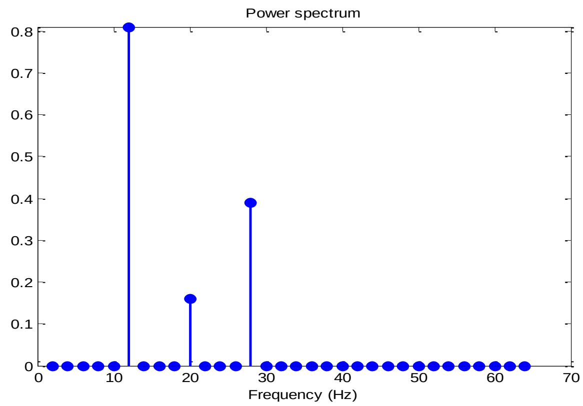
```
clear,clc,clf
format compact, format short g
n=64, dt=1/128, fs=1/dt, tn = n/fs, df=fs/n
tspan=(0:n-1)/fs;
nyquist=fs/2,fmin=1/tn,fmax=0.5*fs
y=1.5+1.8*cos(2*pi*12*tspan)+0.8*sin(2*pi*20*tspan)-
1.25*cos(2*pi*28*tspan);
subplot(3,1,1);
plot(tspan,y,'-ok','linewidth',2,'MarkerFaceColor','black');
title('(a) f(t) versus time (s)');
pause
% Compute and display the DFT
Y=fft(y)/n;
Y';
f = linspace(fmin,nyquist,n/2);
Y(1)=[];YP=Y(1:n/2);
subplot(3,1,2)
stem(f,real(YP),'linewidth',2,'MarkerFaceColor','blue')
grid;title('(b) Real component versus frequency')
subplot(3,1,3)
stem(f,imag(YP),'linewidth',2,'MarkerFaceColor','blue')
grid;title('(b) Imaginary component versus frequency')
xlabel('frequency (Hz)')
pause
```

```
% compute and display the power spectrum
Pyy = abs(Y(1:n/2)).^2;
stem(f,Pyy,'linewidth',2,'MarkerFaceColor','blue')
title('Power spectrum')
xlabel('Frequency (Hz)');ylim([0 max(Pyy)])
```

Output:

```
n =
    64
dt =
    0.0078125
fs =
   128
tn =
          0.5
df =
     2
nyquist =
    64
fmin =
     2
fmax =
    64
```





16.17

(a) The sample frequency, f_s (sample/s)

$$f_s = \frac{n}{t_n} = \frac{128 \text{ samples}}{0.4 \text{ s}} = 320 \frac{\text{samples}}{\text{s}}$$

(b) The sample interval, Δt (s/sample)

$$\Delta t = \frac{1}{f_s} = \frac{1}{320 \text{ samples/s}} = 0.003125 \frac{\text{s}}{\text{sample}}$$

(c) The Nyquist frequency, $f_{\max}$ (Hz)

$$f_{\max} = 0.5 f_s = 0.5(320) = 160 \text{ Hz}$$

(d) The minimum frequency, $f_{\min}$ (Hz)

$$f_{\min} = \frac{1}{t_n} = \frac{1}{0.4} = 2.5 \text{ Hz}$$