CHAPTER 10

10.1 The flop counts for *LU* decomposition can be determined in a similar fashion as was done for Gauss elimination. The major difference is that the elimination is only implemented for the left-hand side coefficients. Thus, for every iteration of the inner loop, there are n multiplications/divisions and $n - 1$ addition/subtractions. The computations can be summarized as

Outer Loop k	**Inner Loop** i	**Addition/Subtraction flops**	**Multiplication/Division flops**
1	$2, n$	$(n-1)(n-1)$	$(n-1)n$
2	$3, n$	$(n-2)(n-2)$	$(n-2)(n-1)$
.	.		
.	.		
.	.		
k	$k+1, n$	$(n-k)(n-k)$	$(n-k)(n+1-k)$
.	.		
.	.		
.	.		
$n-1$	n, n	$(1)(1)$	$(1)(2)$

Therefore, the total addition/subtraction flops for elimination can be computed as

$$\sum_{k=1}^{n-1}(n-k)(n-k) = \sum_{k=1}^{n-1}\left[n^2 - 2nk + k^2\right]$$

Applying some of the relationships from Eq. (9.14) yields

$$\sum_{k=1}^{n-1}\left[n^2 - 2nk + k^2\right] = \frac{n^3}{3} - \frac{n^2}{2} + \frac{n}{6}$$

A similar analysis for the multiplication/division flops yields

$$\sum_{k=1}^{n-1}(n-k)(n+1-k) = \frac{n^3}{3} - \frac{n}{3}$$

$$\left[n^3 + O(n^2)\right] - \left[n^3 + O(n)\right] + \left[\frac{1}{3}n^3 + O(n^2)\right] = \frac{n^3}{3} + O(n^2)$$

Summing these results gives

$$\frac{2n^3}{3}-\frac{n^2}{2}-\frac{n}{6}$$

For forward substitution, the numbers of multiplications and subtractions are the same and equal to

$$\sum_{i=1}^{n-1} i = \frac{(n-1)n}{2} = \frac{n^2}{2} - \frac{n}{2}$$

Back substitution is the same as for Gauss elimination: $n^2/2 - n/2$ subtractions and $n^2/2 + n/2$ multiplications/divisions. The entire number of flops can be summarized as

	Mult/Div	**Add/Subtr**	**Total**
Forward elimination	$\frac{n^3}{3}-\frac{n}{3}$	$\frac{n^3}{3}-\frac{n^2}{2}+\frac{n}{6}$	$\frac{2n^3}{3}-\frac{n^2}{2}-\frac{n}{6}$
Forward substitution	$\frac{n^2}{2}-\frac{n}{2}$	$\frac{n^2}{2}-\frac{n}{2}$	n^2-n
Back substitution	$\frac{n^2}{2}+\frac{n}{2}$	$\frac{n^2}{2}-\frac{n}{2}$	n^2
Total	$\frac{n^3}{3}+n^2-\frac{n}{3}$	$\frac{n^3}{3}+\frac{n^2}{2}-\frac{5n}{6}$	$\frac{2n^3}{3}+\frac{3n^2}{2}-\frac{7n}{6}$

The total number of flops is identical to that obtained with standard Gauss elimination.

10.2 Equation (10.6) is

$$[L]\ [U]\{x\}-\{d\}\ =[A]\{x\}-\{b\} \tag{10.6}$$

Matrix multiplication is distributive, so the left-hand side can be rewritten as

$$[L][U]\{x\}-[L]\{d\}=[A]\{x\}-\{b\}$$

Equating the terms that are multiplied by $\{x\}$ yields,

$$[L][U]\{x\}=[A]\{x\}$$

and, therefore, Eq. (10.7) follows

$$[L][U]=[A] \tag{10.7}$$

Equating the constant terms yields Eq. (10.8)

$$[L]\{d\} = \{b\} \tag{10.8}$$

10.3 (a) The coefficient a_{21} is eliminated by multiplying row 1 by $f_{21} = -0.3$ and subtracting the result from row 2. a_{31} is eliminated by multiplying row 1 by $f_{31} = 0.1$ and subtracting the result from row 3. The factors f_{21} and f_{31} can be stored in a_{21} and a_{31}.

$$\begin{bmatrix} 10 & 2 & -1 \\ -0.3 & -5.4 & 1.7 \\ 0.1 & 0.8 & 5.1 \end{bmatrix}$$

a_{32} is eliminated by multiplying row 2 by $f_{32} = -0.14815$ and subtracting the result from row 3. The factor f_{32} can be stored in a_{32}.

$$\begin{bmatrix} 10 & 2 & -1 \\ -0.3 & -5.4 & 1.7 \\ 0.1 & -0.14815 & 5.3519 \end{bmatrix}$$

Therefore, the *LU* decomposition is

$$[L] = \begin{bmatrix} 1 & 0 & 0 \\ -0.3 & 1 & 0 \\ 0.1 & -0.14815 & 1 \end{bmatrix} \qquad [U] = \begin{bmatrix} 10 & 2 & -1 \\ 0 & -5.4 & 1.7 \\ 0 & 0 & 5.3519 \end{bmatrix}$$

These two matrices can be multiplied to yield the original system. For example, using MATLAB to perform the multiplication gives

```
>> L =[1 0 0;-0.3 1 0;0.1 -0.14815 1]
>> U =[10 2 -1;0 -5.4 1.7;0 0 5.3519]
>> A=L*U

A =
   10.0000    2.0000   -1.0000
   -3.0000   -6.0000    2.0000
    1.0000    1.0000    5.0000
```

10.4 (a) Forward substitution: $[L]\{D\} = \{B\}$

$$\begin{bmatrix} 1 & 0 & 0 \\ -0.3 & 1 & 0 \\ 0.1 & -0.14815 & 1 \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \end{Bmatrix} = \begin{Bmatrix} 27 \\ -61.5 \\ -21.5 \end{Bmatrix}$$

Solving yields $d_1 = 27$, $d_2 = -53.4$, and $d_3 = -32.11111$.

Back substitution:

$$\begin{bmatrix} 10 & 2 & -1 \\ 0 & -5.4 & 1.7 \\ 0 & 0 & 5.351852 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 27 \\ -53.4 \\ -32.11111 \end{Bmatrix}$$

$$x_3 = \frac{-32.11111}{5.351852} = -6$$

$$x_2 = \frac{-53.4 - 1.7(-6)}{-5.4} = 8$$

$$x_1 = \frac{27 + 1(-6) - 2(8)}{10} = 0.5$$

(b) Forward substitution: $[L]\{D\} = \{B\}$

$$\begin{bmatrix} 1 & 0 & 0 \\ -0.3 & 1 & 0 \\ 0.1 & -0.14815 & 1 \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \end{Bmatrix} = \begin{Bmatrix} 12 \\ 18 \\ -6 \end{Bmatrix}$$

Solving yields $d_1 = 12$, $d_2 = 21.6$, and $d_3 = -4$.

Back substitution:

$$\begin{bmatrix} 10 & 2 & -1 \\ 0 & -5.4 & 1.7 \\ 0 & 0 & 5.351852 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 12 \\ 21.6 \\ -4 \end{Bmatrix}$$

$$x_3 = \frac{-4}{5.351852} = -0.7474$$

$$x_2 = \frac{-53.4 - 1.7(-0.7474)}{-5.4} = -4.23529$$

$$x_1 = \frac{27 + 1(-0.7474) - 2(-4.23529)}{10} = 1.972318$$

10.5 The system can be written in matrix form as

$$[A] = \begin{bmatrix} 2 & -6 & -1 \\ -3 & -1 & 7 \\ -8 & 1 & -2 \end{bmatrix} \qquad \{b\} = \begin{Bmatrix} -38 \\ -34 \\ -40 \end{Bmatrix} \qquad [\mathrm{P}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Partial pivot:

$$[A] = \begin{bmatrix} -8 & 1 & -2 \\ -3 & -1 & 7 \\ 2 & -6 & -1 \end{bmatrix} \qquad [\mathrm{P}] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Compute factors:

$$f_{21} = -3/-8 = 0.375 \qquad f_{31} = 2/(-8) = -0.25$$

Forward eliminate and store factors in zeros:

$$[A] = \begin{bmatrix} -8 & 1 & -2 \\ 0.375 & -1.375 & 7.75 \\ -0.25 & -5.75 & -1.5 \end{bmatrix}$$

Pivot again

$$[A] = \begin{bmatrix} -8 & 1 & -2 \\ -0.25 & -5.75 & -1.5 \\ 0.375 & -1.375 & 7.75 \end{bmatrix} \qquad [\mathrm{P}] = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Compute factors:

$$f_{32} = -1.375/(-5.75) = 0.23913$$

Forward eliminate and store factor in zero:

$$[LU] = \begin{bmatrix} -8 & 1 & -2 \\ -0.25 & -5.75 & -1.5 \\ 0.375 & 0.23913 & 8.1087 \end{bmatrix}$$

Therefore, the *LU* decomposition is

$$[L] = \begin{bmatrix} 1 & 0 & 0 \\ -0.25 & 1 & 0 \\ 0.375 & 0.23913 & 1 \end{bmatrix} \qquad [U] = \begin{bmatrix} -8 & 1 & -2 \\ 0 & -5.75 & -1.5 \\ 0 & 0 & 8.1087 \end{bmatrix}$$

Forward substitution. First pre-multiply right-hand side vector $\{b\}$ by $[P]$ to give

$$[P]\{b\} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{Bmatrix} -38 \\ -34 \\ -40 \end{Bmatrix} = \begin{Bmatrix} -40 \\ -38 \\ -34 \end{Bmatrix}$$

Therefore,

$$\begin{bmatrix} 1 & 0 & 0 \\ -0.25 & 1 & 0 \\ 0.375 & 0.23913 & 1 \end{bmatrix} \{d\} = \begin{Bmatrix} -40 \\ -38 \\ -34 \end{Bmatrix}$$

which can be solved for

$$\begin{aligned} d_1 &= -40 \\ d_2 &= -38 - 0.25(-40) = -48 \\ d_3 &= -34 - 0.375(-40) - 0.23913(-48) = -7.52174 \end{aligned}$$

Back substitution:

$$\begin{bmatrix} -8 & 1 & -2 \\ 0 & -5.75 & -1.5 \\ 0 & 0 & 8.1087 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} -40 \\ -48 \\ -7.52174 \end{Bmatrix}$$

$$x_3 = \frac{-7.52174}{8.1087} = -0.92761$$
$$x_2 = \frac{-48 + 1.5(-0.92761)}{-5.75} = 8.589812$$
$$x_1 = \frac{-40 + 2(-0.92761) - 1(8.589812)}{8} = 6.30563$$

10.6 Here is an M-file to generate the *LU* decomposition without pivoting

```
function [L, U] = LUNaive(A)
% LUNaive(A):
%   LU decomposition without pivoting.
% input:
%   A = coefficient matrix
% output:
%   L = lower triangular matrix
%   U = upper triangular matrix

[m,n] = size(A);
if m~=n, error('Matrix A must be square'); end
L = eye(n);
U = A;
% forward elimination
for k = 1:n-1
  for i = k+1:n
    L(i,k) = U(i,k)/U(k,k);
    U(i,k) = 0;
    U(i,k+1:n) = U(i,k+1:n)-L(i,k)*U(k,k+1:n);
  end
end
```

Test with Prob. 10.3

```
>> A = [10 2 -1;-3 -6 2;1 1 5];
>> [L,U] = LUNaive(A)

L =
    1.0000         0         0
   -0.3000    1.0000         0
    0.1000   -0.1481    1.0000
U =
   10.0000    2.0000   -1.0000
         0   -5.4000    1.7000
         0         0    5.3519
```

Verification that $[L][U] = [A]$.

```
>> L*U
ans =
   10.0000    2.0000   -1.0000
   -3.0000   -6.0000    2.0000
    1.0000    1.0000    5.0000
```

Check using the `lu` function,

```
>> [L,U]=lu(A)
L =
    1.0000         0         0
   -0.3000    1.0000         0
    0.1000   -0.1481    1.0000
U =
   10.0000    2.0000   -1.0000
         0   -5.4000    1.7000
         0         0    5.3519
```

10.7 The result of Example 10.5 can be substituted into Eq. (10.14) to give

$$[A]=[U]^T[U]=\begin{bmatrix} 2.44949 & & \\ 6.123724 & 4.1833 & \\ 22.45366 & 20.9165 & 6.110101 \end{bmatrix}\begin{bmatrix} 2.44949 & 6.123724 & 22.45366 \\ & 4.1833 & 20.9165 \\ & & 6.110101 \end{bmatrix}$$

The multiplication can be implemented as in

$$a_{11}=2.44949^2=6.000001$$
$$a_{12}=6.123724\times 2.44949=15$$
$$a_{13}=22.45366\times 2.44949=55.00002$$
$$a_{21}=2.44949\times 6.123724=15$$
$$a_{22}=6.123724^2+4.1833^2=54.99999$$
$$a_{23}=22.45366\times\ 6.1237724+20.9165\times 4.1833=225$$
$$a_{31}=2.44949\times 22.45366=55.00002$$
$$a_{32}=6.123724\times 22.45366+4.1833\times 20.9165=225$$
$$a_{33}=22.45366^2+20.9165^2+6.110101^2=979.0002$$

The product yields the original matrix [A]. Thus, the Cholesky factorization of Example 10.5 is valid.

10.8 (a) For the first row (i = 1), Eq. (10.15) is employed to compute

$$u_{11}=\sqrt{a_{11}}=\sqrt{8}=2.828427$$

Then, Eq. (10.16) can be used to determine

$$u_{12}=\frac{a_{12}}{u_{11}}=\frac{20}{2.828427}=7.071068$$

$$u_{13} = \frac{a_{13}}{u_{11}} = \frac{15}{2.8288427} = 5.303301$$

For the second row ($i = 2$),

$$u_{22} = \sqrt{a_{22} - u_{12}^2} = \sqrt{80 - (7.071068)^2} = 5.477226$$

$$u_{23} = \frac{a_{23} - u_{12}u_{13}}{u_{22}} = \frac{50 - 7.071068(5.303301)}{5.477226} = 2.282177$$

For the third row ($i = 3$),

$$u_{33} = \sqrt{a_{33} - u_{13}^2 - u_{23}^2} = \sqrt{60 - 5.303301^2 - 2.282177^2} = 5.163978$$

Thus, the Cholesky decomposition yields

$$[U] = \begin{bmatrix} 2.828427 & 7.071068 & 5.303301 \\ & 5.477226 & 2.282177 \\ & & 5.163978 \end{bmatrix}$$

The validity of this decomposition can be verified by substituting it and its transpose into Eq. (10.14) to see if their product yields the original matrix $[A]$.

```
>> U =[2.828427 7.071068 5.303301; 0 5.477226 2.282177;0 0
5.163978];
>> A=U'*U

A =
    8.0000   20.0000   15.0000
   20.0000   80.0000   50.0000
   15.0000   50.0000   60.0000
```

(b)

```
>> A = [8 20 15;20 80 50;15 50 60];
>> U = chol(A)

U =
    2.8284    7.0711    5.3033
         0    5.4772    2.2822
         0         0    5.1640
```

(c) The solution can be obtained by hand or by MATLAB. Using MATLAB:

```
>> b = [50;250;100];
>> d=U'\b
```

```
>> x=U\d

d =
   17.6777
   22.8218
   -8.8756
x =
   -2.7344
    4.8828
   -1.7187
```

10.9 Here is an M-file to generate the Cholesky decomposition without pivoting

```
function U = cholesky(A)
% cholesky(A):
%   cholesky decomposition without pivoting.
% input:
%   A = coefficient matrix
% output:
%   U = upper triangular matrix
[m,n] = size(A);
if m~=n, error('Matrix A must be square'); end
for i = 1:n
  s = 0;
  for k = 1:i-1
    s = s + U(k, i) ^ 2;
  end
  U(i, i) = sqrt(A(i, i) - s);
  for j = i + 1:n
    s = 0;
    for k = 1:i-1
      s = s + U(k, i) * U(k, j);
    end
    U(i, j) = (A(i, j) - s) / U(i, i);
  end
end
```

Script to test with Prob. 10.8

```
clear,clc
format compact
A = [8 20 15;20 80 50;15 50 60];
cholesky(A)

ans =
    2.8284    7.0711    5.3033
         0    5.4772    2.2822
         0         0    5.1640
```

Check with the `chol` function

```
>> U = chol(A)

U =
    2.8284    7.0711    5.3033
         0    5.4772    2.2822
         0         0    5.1640
```

10.10 The system can be written in matrix form as

$$[A]=\begin{bmatrix}3 & -2 & 1\\ 2 & 6 & -4\\ -8 & -2 & 5\end{bmatrix} \qquad \{b\}=\begin{Bmatrix}-10\\ 44\\ -26\end{Bmatrix} \qquad [P]=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

Partial pivot:

$$[A]=\begin{bmatrix}-8 & 1 & -2\\ 2 & -6 & -1\\ 3 & -2 & 1\end{bmatrix} \qquad [P]=\begin{bmatrix}0 & 0 & 1\\ 0 & 1 & 0\\ 1 & 0 & 0\end{bmatrix}$$

Compute factors:

$$f_{21}=2/(-8)=-0.25 \qquad f_{31}=3/(-8)=-0.375$$

Forward eliminate and store factors in zeros:

$$[LU]=\begin{bmatrix}-8 & -2 & 5\\ -0.25 & 5.5 & -2.75\\ -0.375 & -2.75 & 2.875\end{bmatrix}$$

Pivot again

$$[LU]=\begin{bmatrix}-8 & -2 & 5\\ -0.375 & -2.75 & 2.875\\ -0.25 & 5.5 & -2.75\end{bmatrix} \qquad [P]=\begin{bmatrix}0 & 0 & 1\\ 1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}$$

Compute factors:

$$f_{32}=5.5/(-2.75)=-2$$

Forward eliminate and store factor in zero:

$$[LU] = \begin{bmatrix} -8 & -2 & 5 \\ -0.375 & -2.75 & 2.875 \\ -0.25 & -2 & 3 \end{bmatrix}$$

Therefore, the *LU* decomposition is

$$[L]\{U\} = \begin{bmatrix} 1 & 0 & 0 \\ -0.375 & 1 & 0 \\ -0.25 & -2 & 1 \end{bmatrix} \begin{bmatrix} -8 & -2 & 5 \\ 0 & -2.75 & 2.875 \\ 0 & 0 & 3 \end{bmatrix}$$

Forward substitution. First multiply right-hand side vector $\{b\}$ by $[P]$ to give

$$[P]\{b\} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{Bmatrix} -10 \\ 44 \\ -26 \end{Bmatrix} = \begin{Bmatrix} -26 \\ -10 \\ 44 \end{Bmatrix}$$

Therefore,

$$\begin{bmatrix} 1 & 0 & 0 \\ -0.375 & 1 & 0 \\ -0.25 & -2 & 1 \end{bmatrix} \{d\} = \begin{Bmatrix} -26 \\ -10 \\ 44 \end{Bmatrix}$$

$$d_1 = -26$$

$$d_2 = -10 - (-0.375)(-26) = -19.75$$

$$d_3 = 44 - (-0.25)(-26) - (-2)(-19.75) = -2$$

Back substitution:

$$\begin{bmatrix} -8 & -2 & 5 \\ 0 & -2.75 & 2.875 \\ 0 & 0 & 3 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} -26 \\ -19.75 \\ -2 \end{Bmatrix}$$

$$x_3 = \frac{-2}{3} = -0.66667$$

$$x_2 = \frac{-19.75 - 2.875(-0.66667)}{-2.75} = 6.484848$$

$$x_1 = \frac{-26 - 5(0.666667) + 2(6.484848)}{-8} = 1.212121$$

10.11 (a) Multiply first row by $f_{21} = 3/8 = 0.375$ and subtract the result from the second row and multiply first row by $f_{31} = 2/8 = 0.25$ and subtract the result from the third row to give

$$\begin{bmatrix} 8 & 2 & 1 \\ 0 & 6.25 & 1.625 \\ 0 & 2.5 & 8.75 \end{bmatrix}$$

Multiply second row by $f_{32} = 2.5/6.25 = 0.4$ and subtract the result from the third row to give

$$[U] = \begin{bmatrix} 8 & 2 & 1 \\ 0 & 6.25 & 1.625 \\ 0 & 0 & 8.1 \end{bmatrix}$$

As indicated, this is the U matrix. The L matrix is simply constructed from the f's as

$$[L] = \begin{bmatrix} 1 & 0 & 0 \\ 0.375 & 1 & 0 \\ 0.25 & 0.4 & 1 \end{bmatrix}$$

Merely multiply $[L][U]$ to yield the original matrix

$$[L][U] = \begin{bmatrix} 1 & 0 & 0 \\ 0.375 & 1 & 0 \\ 0.25 & 0.4 & 1 \end{bmatrix} \begin{bmatrix} 8 & 2 & 1 \\ 0 & 6.25 & 1.625 \\ 0 & 0 & 8.1 \end{bmatrix} = \begin{bmatrix} 8 & 2 & 1 \\ 3 & 7 & 2 \\ 2 & 3 & 9 \end{bmatrix}$$

(b) The determinant is equal to the product of the diagonal elements of $[U]$:

$$D = 8 \times 6.25 \times 8.1 = 405$$

(c) Solution with MATLAB:

```
clear, clc
format compact
```

```
A=[8 2 1;3 7 2;2 3 9];
[L,U]=lu(A)
A = L*U
D = det(A)

L =
    1.0000         0         0
    0.3750    1.0000         0
    0.2500    0.4000    1.0000
U =
    8.0000    2.0000    1.0000
         0    6.2500    1.6250
         0         0    8.1000
A =
     8     2     1
     3     7     2
     2     3     9
D =
   405
```

10.12 **(a)** The determinant is equal to the product of the diagonal elements of $[U]$:

$$D = 3 \times 7.3333 \times 3.6364 = 80$$

(b) Forward substitution:

```
clear,clc, format compact
L=[1 0 0;0.6667 1 0;-0.3333 -0.3636 1];
U=[3 -2 1;0 -7.3333 -4.6667;0 0 3.6364];
b=[-10 44 -26]';
d=L\b

d =
  -10.0000
   50.6670
  -10.9105
```

Back substitution:

```
x=U\d

x =
   -5.6664
   -4.9998
   -3.0004
```

10.13 Using MATLAB:

```
>> A=[2 -1 0;-1 2 -1;0 -1 2];
>> U=chol(A)
```

```
U =
    1.4142   -0.7071         0
         0    1.2247   -0.8165
         0         0    1.1547
```

The result can be validated by

```
>> U'*U

ans =
    2.0000   -1.0000         0
   -1.0000    2.0000   -1.0000
         0   -1.0000    2.0000
```

10.14 Using MATLAB:

```
clear,clc, format compact
A=[9 0 0;0 25 0;0 0 4];
U=chol(A)

U =
     3     0     0
     0     5     0
     0     0     2
```

Thus, the factorization of this diagonal matrix consists of another diagonal matrix where the elements are the square root of the original. This is consistent with Eqs. (10.15) and (10.16), which for a diagonal matrix reduce to

$$u_{ii} = \sqrt{a_{ii}}$$

$$u_{ij} = 0 \quad \text{for } i \neq j$$