Chapter 8 • Potential Flow and Computational Fluid Dynamics

8.1 Prove that the streamlines $\psi(r, \theta)$ in polar coordinates, from Eq. (8.10), are orthogonal to the potential lines $\phi(r, \theta)$.

Solution: The streamline slope is represented by

$$\left.\frac{dr}{r\,d\theta}\right|_{\text{streamline}} = \frac{v_r}{v_\theta} = \frac{\partial\phi/\partial r}{(1/r)(\partial\phi/\partial\theta)} = \frac{\mathbf{-1}}{\left(\dfrac{dr}{r\,d\theta}\right)_{\text{potential line}}}$$

Since the ψ – slope = $\mathbf{-1}$/(ϕ – slope), the two sets of lines are **orthogonal**. *Ans.*

8.2 The steady plane flow in the figure has the polar velocity components $v_\theta = \Omega r$ and $v_r = 0$. Determine the circulation Γ around the path shown.

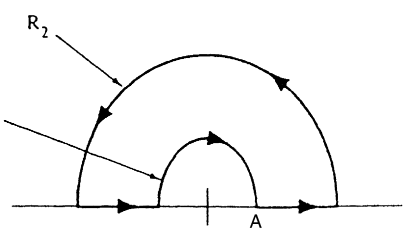


Fig. P8.2

Solution: Start at the inside right corner, point A, and go around the complete path:

$$\Gamma = \oint \boldsymbol{V}\cdot d\boldsymbol{s} = 0(R_2 - R_1) + \Omega R_2(\pi R_2) + 0(R_1 - R_2) + \Omega R_1(-\pi R_1)$$

$$\text{or:}\quad \boldsymbol{\Gamma = \pi\Omega\left(R_2^2 - R_1^2\right)} \quad \textit{Ans.}$$

8.3 Using cartesian coordinates, show that each velocity component (u, v, w) of a potential flow satisfies Laplace's equation separately if $\nabla^2\phi = 0$.

Solution: This is true because the order of integration may be changed in each case:

$$\text{Example:}\quad \nabla^2 u = \nabla^2\left(\frac{\partial\phi}{\partial x}\right) = \frac{\partial}{\partial x}(\nabla^2\phi) = \frac{\partial}{\partial x}(0) = 0 \quad \textit{Ans.}$$

8.4 Is the function *1/r* a legitimate velocity potential in plane polar coordinates? If so, what is the associated stream function $\psi(r, \theta)$?

Solution: Evaluation of the laplacian of (*1/r*) shows that it is *not* legitimate:

$$\nabla^2\left(\frac{1}{r}\right)=\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial}{\partial r}\left(\frac{1}{r}\right)\right]=\frac{1}{r}\frac{\partial}{\partial r}\left[r\left(-\frac{1}{r^2}\right)\right]=\frac{\mathbf{1}}{\mathbf{r^3}}\neq\mathbf{0}\quad \textbf{Illegitimate}\quad \textit{Ans.}$$

8.5 Consider the two-dimensional velocity distribution u = –By, v = +Bx, where B is a constant. If this flow possesses a stream function, find its form. If it has a velocity potential, find that also. Compute the local angular velocity of the flow, if any, and describe what the flow might represent.

Solution: It <u>does</u> has a stream function, because it satisfies continuity:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0+0=0\text{ (OK);}\quad \text{Thus }\ u=-By=\frac{\partial\psi}{\partial y}\quad\text{and}\quad v=Bx=-\frac{\partial\psi}{\partial x}$$

$$\text{Solve for}\quad \psi=-\frac{\mathbf{B}}{\mathbf{2}}(\mathbf{x^2}+\mathbf{y^2})+\mathbf{const}\quad \textit{Ans.}$$

It does <u>not</u> have a velocity potential, because it has a non-zero curl:

$$2\omega=\text{curl }\mathbf{V}=\mathbf{k}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)=\mathbf{k}[B-(-B)]=2B\mathbf{k}\neq 0\quad\text{thus }\ \phi\ \textbf{does not exist}\quad \textit{Ans.}$$

The flow represents <u>solid-body rotation</u> at uniform clockwise angular velocity *B*.

8.6 If the velocity potential of a realistic two-dimensional flow is $\phi = C\ln(x^2 + y^2)^{1/2}$, where C is a constant, find the form of the stream function $\psi(x, y)$. *Hint*: Try polar coordinates.

Solution: Using polar coordinates is certainly an excellent hint! Then the velocity potential translates simply to $\phi = C\ \ln(r)$, which is a line source. Equation (8.12*b*) also shows that,

$$\text{Eq. (8.12}b\text{):}\quad \psi=C\theta=\mathbf{C}\,\mathbf{tan^{-1}}\left(\frac{\mathbf{y}}{\mathbf{x}}\right)\quad \textit{Ans.}$$

8.7 Consider a flow with constant density and viscosity. If the flow possesses a velocity potential as defined by Eq. (8.1), show that it exactly satisfies the full Navier-Stokes equation (4.38). If this is so, why do we back away from the full Navier-Stokes equation in solving potential flows?

Solution: If $\mathbf{V} = \nabla\phi$, the full Navier-Stokes equation is satisfied identically:

$$\rho\frac{d\mathbf{V}}{dt} = -\nabla p + \rho\mathbf{g} + \mu\nabla^2\mathbf{V} \quad \text{becomes}$$

$$\rho\left[\nabla\left(\frac{\partial\phi}{\partial t}\right) + \nabla\left(\frac{V^2}{2}\right)\right] = -\nabla p - \nabla(\rho g z) + \mu\nabla(\nabla^2\phi), \text{ where the last term is } \textbf{zero}.$$

The viscous (final) term drops out identically for potential flow, and what remains is

$$\frac{\partial\phi}{\partial t} + \frac{V^2}{2} + \frac{p}{\rho} + gz = \text{constant} \quad (\textbf{Bernoulli's equation})$$

The Bernoulli relation is an exact solution of Navier-Stokes for potential flow. We don't exactly "back away," we need also to solve $\nabla^2\phi = 0$ in order to find the velocity potential.

8.8 For the velocity distribution of Prob. 8.5, u = –By, v = +Bx, evaluate the circulation Γ around the rectangular closed curve defined by (x, y) = (1, 1), (3, 1), (3, 2), and (1, 2).

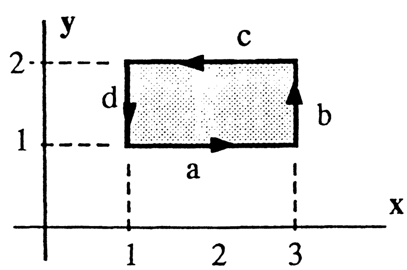


Fig. P8.8

Solution: Given $\Gamma = \int \mathbf{V}\cdot d\mathbf{s}$ around the curve, divide the rectangle into (a, b, c, d) pieces as shown:

$$\Gamma = \int_a u\,ds + \int_b v\,ds + \int_c u\,ds + \int_d v\,ds = (-B)(2) + (3B)(1) + (2B)(2) + (-B)(1)$$

$$\text{or} \quad \Gamma = \mathbf{+4B} \quad \textit{Ans.}$$

Since, from Prob. 8.5, $|\text{curl}\,\mathbf{V}| = 2B$, also $\Gamma = |\text{curl}\,\mathbf{V}|A_{\text{region}} = (2B)(2) = 4B$. (Check)

8.9 Consider the two-dimensional flow u = –Ax, v = +Ay, where A is a constant. Evaluate the circulation Γ around the rectangular closed curve defined by (x, y) = (1, 1), (4, 1), (4, 3), and (1, 3). Interpret your result especially *vis-a-vis* the velocity potential.

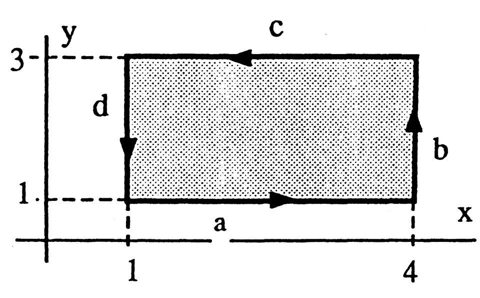


Fig. P8.9

Solution: Given $\Gamma = \int \mathbf{V \cdot ds}$ around the curve, divide the rectangle into (a, b, c, d) pieces as shown:

$$\Gamma = \int_a u\,dx + \int_b v\,dy + \int_c u\,dx + \int_d v\,dy$$

$$= \int_1^4 (-Ax)\,dx + \int_1^3 Ay\,dy + \int_1^4 Ax\,dx + \int_1^3 (-Ay)\,dy = \mathbf{0} \quad \textit{Ans.}$$

The circulation is <u>zero</u> because the flow is **irrotational**: curl $\mathbf{V} \equiv \mathbf{0}$, $\Gamma = \int \mathbf{d\phi} \equiv \mathbf{0}$.

8.10 A mathematical relation sometimes used in fluid mechanics is the theorem of Stokes [1]

$$\oint_C \mathbf{V} \cdot d\boldsymbol{s} = \iint_A (\nabla \times \mathbf{V}) \cdot \mathbf{n}\, dA$$

where A is any surface and C is the curve enclosing that surface. The vector ds is the differential arc length along C, and **n** is the unit outward normal vector to A. How does this relation simplify for irrotational flow, and how does the resulting line integral relate to velocity potential?

Solution: If $\mathbf{V} = \nabla\phi$, we obtain

$$\int_C \nabla\phi \cdot d\boldsymbol{s} = \iint_A (\nabla \times \nabla\phi) \cdot \mathbf{n}\, dA, \quad \text{or:} \quad \int_C d\phi = 0 = \iint_A 0\, dA \equiv 0$$

8.11 A power-plant discharges cooling water through the manifold in Fig. P8.11, which is 55 cm in diameter and 8 m high and is perforated with 25,000 holes 1 cm in diameter. Does this manifold simulate a line source? If so, what is the equivalent source strength m?

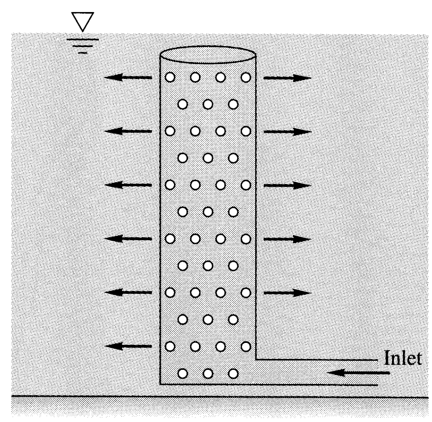


Fig. P8.11

Solution: With that many small holes, equally distributed and presumably with equal flow rates, the manifold **does indeed** simulate a line source of strength

$$m = \frac{\mathbf{Q}}{\mathbf{2\pi b}}, \quad \text{where } b = 8 \text{ m and } Q = \sum_{i=1}^{25000} Q_{1\,\text{hole}} \quad \textit{Ans.}$$

8.12 Consider the flow due to a vortex of strength K at the origin. Evaluate the circulation from Eq. (8.15) about the clockwise path from (a, 0) to (2a, 0) to (2a, $3\pi/2$) to (a, $3\pi/2$) and back to (a, 0). Interpret your result.

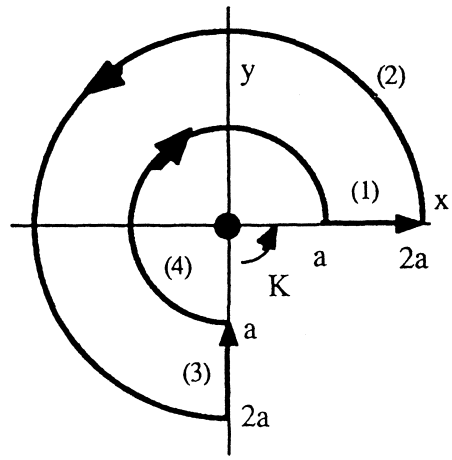


Fig. P8.12

Solution: Break the path up into (1, 2, 3, 4) as shown. Then

$$\Gamma = \int_{\text{path}} \mathbf{V}\cdot\mathbf{ds}$$

$$= \int_{(1)} u\,ds + \int_{(2)} v_\theta\,ds + \int_{(3)} v\,ds + \int_{(4)} v_\theta\,ds$$

$$= 0 + \int_{(2)} \frac{K}{2a} 2a\,d\theta + 0 + \int_{(4)} \frac{K}{a}(-a\,d\theta) = K\left(\frac{3\pi}{2}\right) + K\left(-\frac{3\pi}{2}\right) = \mathbf{0} \quad \textit{Ans.}$$

There is zero circulation **about all closed paths which do not enclose the origin**.

8.13 A well-known exact solution to the Navier-Stokes equation (4.38) is the unsteady circulating motion [15]

$$\upsilon_\theta = \frac{K}{2\pi r}\left[1 - \exp\left(-\frac{r^2}{4\nu t}\right)\right] \qquad \upsilon_r = \upsilon_z = 0$$

where K is a constant and ν is the kinematic viscosity. Does this flow have a polar-coordinate stream function and/or velocity potential? Explain. Evaluate the circulation Γ for this motion, plot it versus r for a given finite time, and interpret compared to ordinary line vortex motion.

Solution: Since v_θ does not depend upon θ and since $v_r = 0$, this distribution exactly satisfies the continuity equation (4.12b). Therefore **a stream function exists**:

$$\psi = -\int v_\theta\,dr = -\frac{K}{2\pi}\ln(r) + \frac{K}{2\pi}\int \frac{1}{r}\exp\left(-\frac{r^2}{4\nu t}\right)dr$$

(I can't work out the last integral.) *Ans.*

However, the flow is **not irrotational** and therefore not a potential flow:

$$\omega_z = \frac{1}{r}\frac{\partial}{\partial r}(r v_\theta) \neq \mathbf{0} \quad \textbf{therefore } \boldsymbol{\phi} \textbf{ does not exist} \quad \textit{Ans.}$$

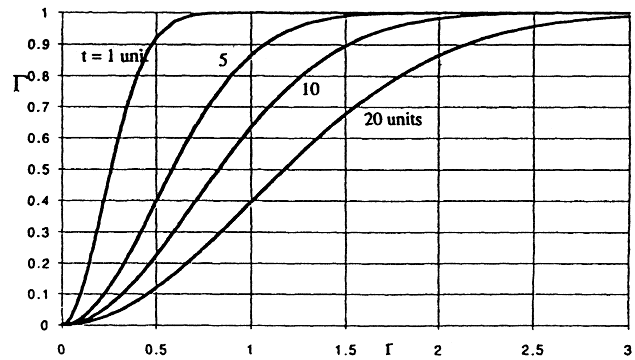


Fig. P8.13

Since we have purely circulating motion, the circulation is easy to compute around a circular path enclosing the origin:

$$\Gamma = \int_0^{2\pi} v_\theta r\, d\theta = K[1 - e^{-r^2/4\nu t}] \quad \textit{Ans.}$$

The distribution of Γ(r) is shown in the figure (in arbitrary units). At t = 0, Γ is uniform and represents a potential vortex. As time increases, the vortex decays from the inside out due to viscosity and the circulation in the inner core vanishes.

8.14 A tornado may be modeled as the circulating flow shown in Fig. P8.14, with $\upsilon_r = \upsilon_z = 0$ and $\upsilon_\theta(r)$ such that

$$\upsilon_\theta = \begin{cases} \omega r & r \le R \\ \dfrac{\omega R^2}{r} & r > R \end{cases}$$

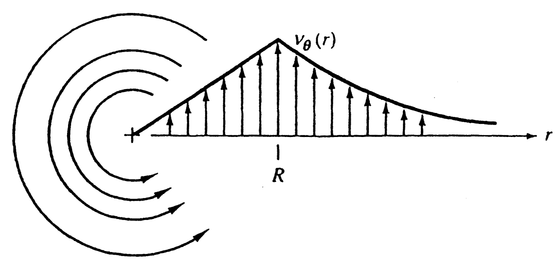


Fig. P8.14

Determine whether this flow pattern is irrotational in either the inner or outer region. Using the *r*-momentum equation (D.5) of App. D, determine the pressure distribution *p*(*r*) in the tornado, assuming $p = p_\infty$ as $r \to \infty$. Find the location and magnitude of the lowest pressure.

Solution: The inner region is solid-body **rotation**, the outer region is **irrotational**:

$$\text{Inner region:}\quad \Omega_z = \frac{1}{r}\frac{d}{dr}(r v_\theta) = \frac{1}{r}\frac{d}{dr}(r\omega r) = \mathbf{2\omega = constant \neq 0} \quad \textit{Ans. (inner)}$$

$$\text{Outer region:}\quad \Omega_z = \frac{1}{r}\frac{d}{dr}(\omega R^2/r) = \mathbf{0} \quad \text{(irrotational)} \quad \textit{Ans. (outer)}$$

The pressure is found by integrating the r-momentum equation (D-5) in the Appendix:

$$\frac{dp}{dr} = \rho v_\theta^2/r, \quad \text{or:} \quad p_{\text{outer}} = \int \frac{\rho}{r}\left(\frac{\omega R^2}{r}\right)^2 dr = -\rho\omega^2 R^4/2r^2 + \text{constant}$$

$$\text{when} \quad r = \infty,\ p = p_\infty, \quad \text{hence} \quad \boldsymbol{p_{\text{outer}} = p_\infty - \rho\omega^2 R^4/(2r^2)} \quad \textit{Ans.}\ \text{(outer)}$$

$$\text{At the match point, } r = R, \quad p_{\text{outer}} = p_{\text{inner}} = p_\infty - \rho\omega^2 R^2/2$$

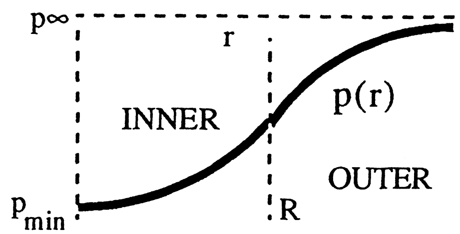
In the inner region, we integrate the radial pressure gradient and match at $r = R$:

$$p_{\text{inner}} = \int \frac{\rho}{r}(\omega r)^2 dr = \rho\omega^2 r^2/2 + \text{constant}, \quad \text{match to} \ \ p(R) = p_\infty - \rho\omega^2 R^2/2$$

$$\text{finally,} \ \ \boldsymbol{p_{\text{inner}} = p_\infty - \rho\omega^2 R^2 + \rho\omega^2 r^2/2} \quad \textit{Ans.}\ \text{(inner)}$$

The minimum pressure occurs at the origin, $r = 0$:

$$p_{\min} = \boldsymbol{p_\infty - \rho\omega^2 R^2} \quad \textit{Ans.}\ \text{(min)}$$



8.15 A *category-3* hurricane on the Saffir-Simpson scale (www.encyclopedia.com) has a maximum velocity of 130 mi/h. Let the match-point radius be $R = 18$ km (see Fig. P8.14). Assuming sea-level standard conditions at large r, (a) find the minimum pressure; (b) find the pressure at the match-point; and (c) show that both minimum and match-point pressures are independent of R.

Solution: Convert 130 mi/h = 58.1 m/s = ωR. Let ρ = 1.22 kg/m^3. (a) From Prob. 8.14,

$$p_{\min} = p_\infty - \rho(\omega R)^2 = 101350\,\text{Pa} - (1.22\ \text{kg/m}^3)(58.1\ \text{m/s})^2 = \mathbf{97200\ Pa} \quad \textit{Ans.}\ \text{(a)}$$

(b) Again from Prob. 8.14, the match pressure only drops half as low as the minimum pressure:

$$p_{match} = p_\infty - \frac{\rho}{2}(\omega R)^2 = 101350 - \frac{1.22}{2}(58.1)^2 = \mathbf{99300\ Pa} \quad \textit{Ans.}\ \text{(b)}$$

(c) We see from above that both $p_{\min}$ and p_{match} have R in their formulas, but only in conjunction with ω. That is, these pressures depend only upon $V_{\max}$, wherever it occurs.

8.16 Consider inviscid stagnation flow $\psi = Kxy$, superimposed with a source at the origin of strength m. Plot the resulting streamlines in the upper-half plane, using the length scale $(m/K)^{1/2}$. Give a physical interpretation of the flow pattern.

Solution: The sum of a stagnation flow plus a source at the origin is:

$$\psi = Kxy + m\theta \quad \text{Define dimensionless } x^* = x\sqrt{\frac{K}{m}} \text{ and } y^* = y\sqrt{\frac{K}{m}}$$

$$\text{Then we obtain } \frac{\psi}{m} = x^*y^* + \theta, \quad \text{where } \theta = \tan^{-1}\left(\frac{y^*}{x^*}\right)$$

The MATLAB plot is given below: It represents **stagnation flow toward a bump**.

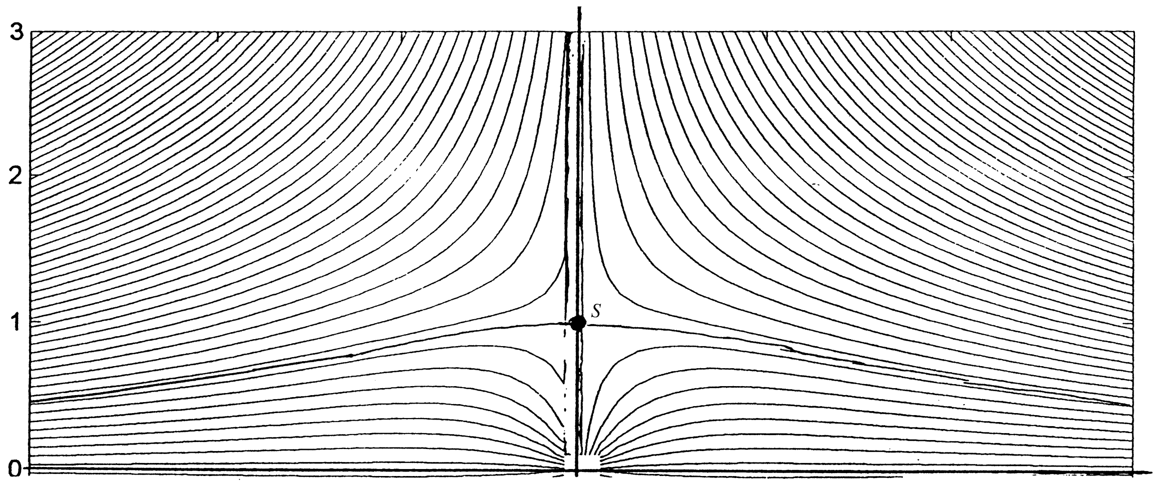


Fig. P8.16

8.17 Find the position (x, y) on the upper surface of the half-body in Fig. 8.5a for which the local velocity equals the uniform stream velocity. What should the pressure be at this point?

Solution: The surface velocity and surface contour are given by Eq. (8.18):

$$V^2 = U_\infty^2\left(1 + \frac{a^2}{r^2} + \frac{2a}{r}\cos\theta\right) \quad \text{along the surface } \frac{r}{a} = \frac{\pi - \theta}{\sin\theta}$$

If $V = U_\infty$, then $a^2/r^2 = -2a\cos\theta/r$, or $\cos\theta = -a/(2r)$. Combine with the surface profile above, and we obtain an equation for θ alone: $\mathbf{\tan\theta = -2(\pi - \theta)}$. The final solution is:

$$\theta = 113.2°; \quad r/a = 1.268; \quad \mathbf{x/a = -0.500}; \quad \mathbf{y/a = 1.166} \quad \textit{Ans.}$$

Since the velocity equals U_∞ at this point, the surface pressure $\boldsymbol{p = p_\infty}$. *Ans.*

8.18 Using the graphical method of Fig. 8.4, plot the streamlines and potential lines of the flow due to a line source of strength m at (a, 0) plus a line source $3m$ at (–a, 0). What is the flow pattern viewed from afar?

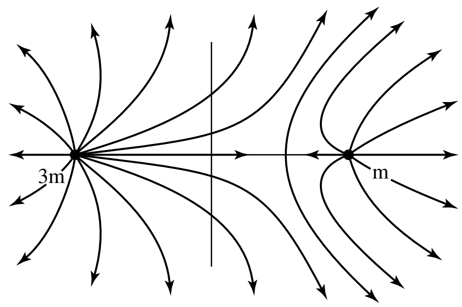
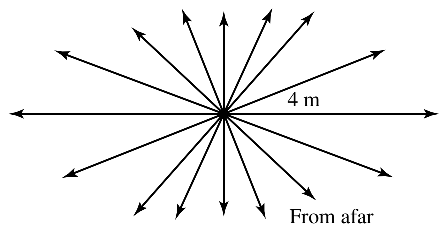


Fig. P8.18

Solution: The pattern viewed close-up is shown above. The pattern viewed from afar is at right and represents a single source of strength **4*m***. *Ans.*



8.19 Plot the streamlines and potential lines of the flow due to a line source of strength $3m$ at (a, 0) plus a line sink of strength $-m$ at (–a, 0). What is the pattern viewed from afar?

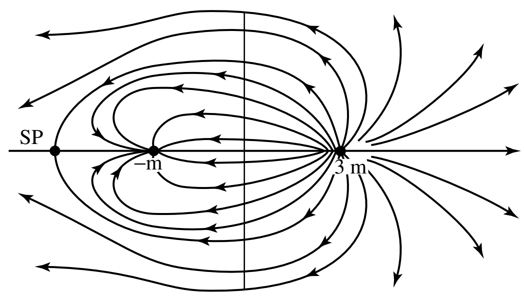
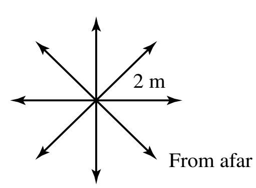


Fig. P8.19

Solution: The pattern viewed close-up is shown at upper right—there is a stagnation point to the left of the sink, at (x, y) = (–2a, 0). The pattern viewed from afar is at right and represents a single source of strength **+2*m***. *Ans.*



8.20 Plot the streamlines of the flow due to a line vortex of strength $+K$ at (0, +a) plus a line vortex of strength $-K$ at (0, –a). What is the pattern viewed from afar?

Solution: The pattern viewed close-up is shown at right (see Fig. 8.17*b* of the text). The pattern viewed from afar represents **little or nothing**, since the two vortices cancel strengths and cause no flow at ∞. *Ans.*

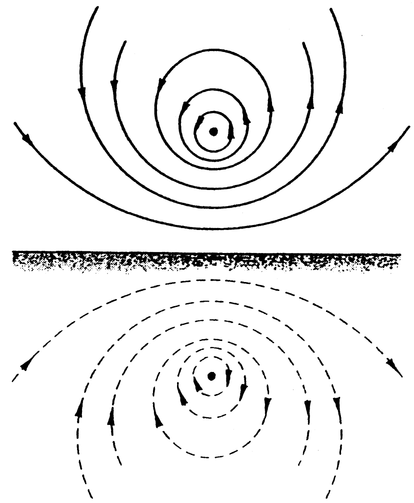


Fig. P8.20

8.21 Plot the streamlines of the flow due to a line vortex $+K$ at (+a, 0) and a vortex $(-2K)$ at (–a, 0). What is the pattern viewed from afar?

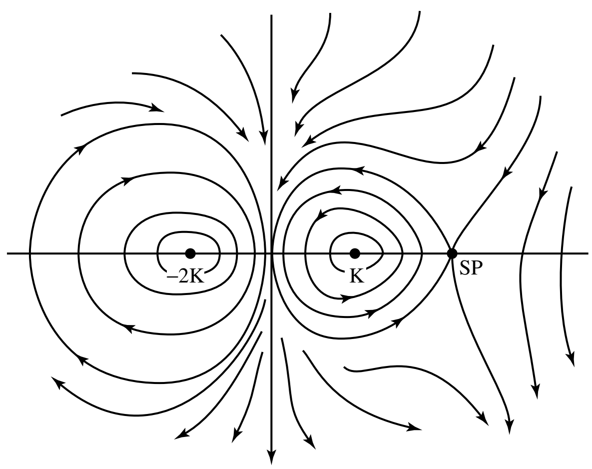


Fig. P8.21

Solution: From close-up, the flow looks like the "cat's eyes" at right. There is a stagnation point at (x, y) = (2a, 0). From afar (not shown), the pattern looks like a *clockwise* vortex of strength $-\boldsymbol{K}$.

8.22 Plot the streamlines of a uniform stream $\mathbf{V} = \mathbf{i}U$ plus a clockwise line vortex $-K$ at the origin. Are there any stagnation points?

Solution: This pattern is the same as Fig. 8.6 in the text, except it is upside down. There is a stagnation point at (x, y) = (0, $-K/U$). *Ans.*

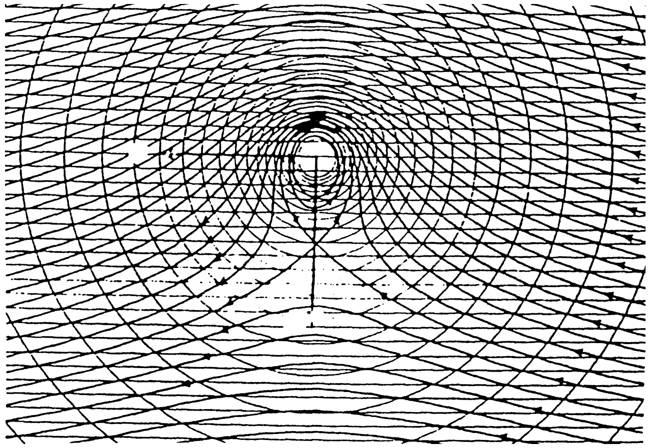


Fig. P8.22

8.23 Find the resultant velocity vector induced at point A in Fig. P8.23 due to the combination of uniform stream, vortex, and line source.

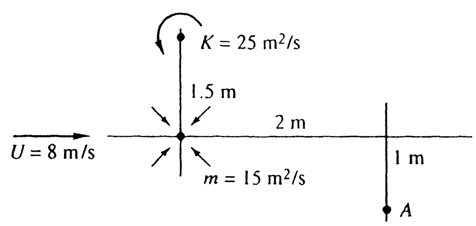
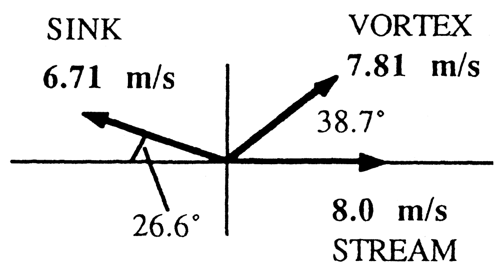


Fig. P8.23

Solution: The velocities caused by each term—stream, vortex, and sink—are shown at right. They have to be added together vectorially to give the final result:



$$\mathbf{V} = \mathbf{11.3}\ \frac{\mathbf{m}}{\mathbf{s}} \quad \text{at } \theta = \mathbf{44.2°}\ \angle \quad \textit{Ans.}$$

8.24 Line sources of equal strength $m = Ua$, where U is a reference velocity, are placed at $(x, y) = (0, a)$ and $(0, -a)$. Sketch the stream and potential lines in the upper half plane. Is $y = 0$ a "wall"? If so, sketch the pressure coefficient

$$C_p = \frac{p - p_0}{\frac{1}{2}\rho U^2}$$

along the wall, where p_0 is the pressure at (0, 0). Find the minimum pressure point and indicate

where flow separation might occur in the boundary layer.

Solution: This problem is an "image" flow and is sketched in Fig. 8.17a of the text. Clearly $y = 0$ is a "wall" where

$$u = 2u_s = \frac{2Ua}{\sqrt{x^2+a^2}}\cdot\frac{x}{\sqrt{x^2+a^2}}$$
$$= 2Ua/(x^2+a^2)$$

From Bernoulli, $p + \rho u^2/2 = p_o$,

$$C_p = \frac{p-p_o}{(1/2)\rho U^2} = -\frac{u^2}{U^2}$$
$$= -\left[\frac{\mathbf{2x/a}}{\mathbf{1+(x/a)^2}}\right]^2 \quad \textit{Ans.}$$

The minimum pressure coefficient is $C_{p,min} = -1.0$ at x = a, as shown in the figure. Beyond this point, pressure increases (*adverse* gradient) and separation is possible.

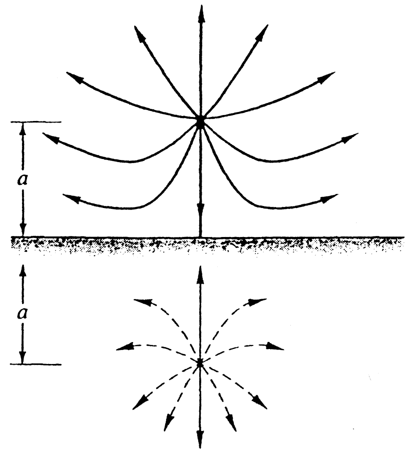
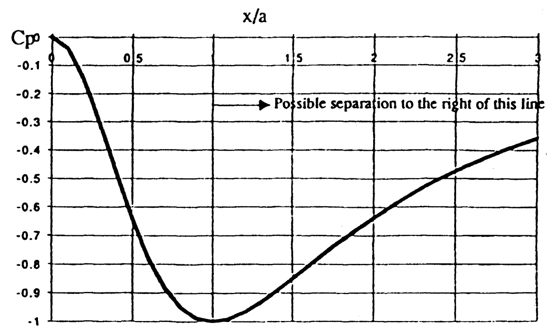


Fig. P8.24



8.25 Let the vortex/sink flow of Eq. (4.134) simulate a tornado as in Fig. P8.25. Suppose that the circulation about the tornado is $\Gamma = 8500\ m^2/s$ and that the pressure at $r = 40$ m is 2200 Pa less than the far-field pressure. Assuming inviscid flow at sea-level density, estimate (a) the appropriate sink strength $-m$, (b) the pressure at $r = 15$ m, and (c) the angle β at which the streamlines cross the circle at $r = 40$ m (see Fig. P8.25).

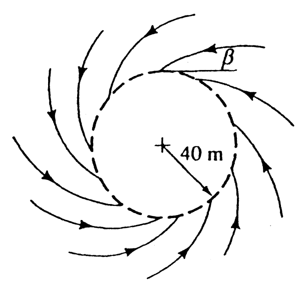


Fig. P8.25

Solution: The given circulation yields the circumferential velocity at $r = 40$ m:

$$v_\theta = \frac{\Gamma}{2\pi r} = \frac{8500}{2\pi(40)} \approx 33.8\ \frac{m}{s}$$

Assuming sea-level density $\rho = 1.225$ kg/m^3, we use Bernoulli to find the radial velocity:

$$p_\infty + \frac{\rho}{2}(0)^2 = (p_\infty - \Delta p) + \frac{\rho}{2}\left(v_\theta^2 + v_r^2\right) = p_\infty - 2200 + \frac{1.225}{2}\left[(33.8)^2 + v_r^2\right]$$

$$\text{Solve for}\quad v_r \approx 49.5\ \frac{m}{s} = \frac{m}{r} = \frac{m}{40},\quad \therefore\ \boldsymbol{m \approx 1980\ \frac{m^2}{s}}\quad \textit{Ans.}\ (a)$$

With circumferential and radial (inward) velocity known, the streamline angle β is

$$\beta = \tan^{-1}\left(\frac{v_r}{v_\theta}\right) = \tan^{-1}\left(\frac{49.5}{33.8}\right) \approx \mathbf{55.6°}\quad \textit{Ans.}\ (c)$$

(b) At r = 15 m, compute $v_r = m/r = 1980/15 \approx 132$ m/s (unrealistically high) and $v_\theta = \Gamma/2\pi r = 8500/[2\pi(15)] \approx 90$ m/s (high again, there is probably a viscous core here). Then we use Bernoulli again to compute the pressure at r = 15 m:

$$p + \frac{1.225}{2}[(132)^2 + (90)^2] = p_\infty,\quad \text{or}\quad p \approx p_\infty - \mathbf{15700\ Pa}\quad \textit{Ans.}\ (b)$$

If we assume sea-level pressure of 101 kPa at ∞, then $p_{absolute} = 101 - 16 \approx$ **85 kPa**.

8.26 Find the resultant velocity induced at point A in Fig. P8.26 by the uniform stream, line source, line sink, and line vortex.

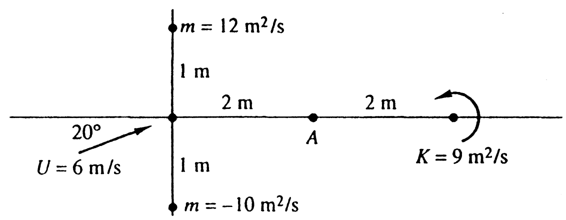
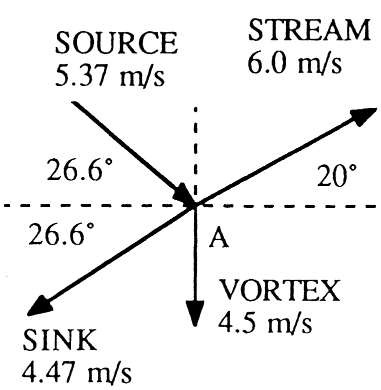


Fig. P8.26

Solution: The source and sink are each $\sqrt{5}$ = 2.24 m from point A, so the sink velocity is 10/2.24 = 4.47 m/s and the source velocity is 12/2.24 = 5.37 m/s, as shown. The vortex velocity is 9/2 = 4.5 m/s.

The net horizontal component is 6.44 m/s. The net vertical component is –6.85 m/s (down). Then the resultant induced velocity at A is



$$V = \sqrt{(6.85)^2 + (6.44)^2} \approx \mathbf{9.40\ \frac{m}{s}}$$

$$\text{at}\ \theta = \tan^{-1}\left(\frac{-6.85}{6.44}\right) \approx \mathbf{-46.8°}\quad \textit{Ans.}$$

8.27 Water at 20°C flows past a half-body as shown in Fig. P8.27. Measured pressures at points A and B are 160 kPa and 90 kPa, respectively, with uncertainties of 3 kPa each. Estimate the stream velocity and its uncertainty.

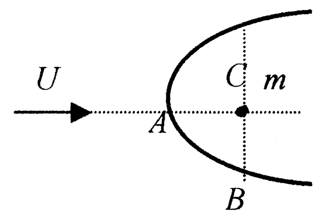


Fig. P8.27

Solution: Since Eq. (8.18) is for the upper surface, use it by noting that $V_C = V_B$ in the figure:

$$\frac{r_C}{a} = \frac{\pi - \pi/2}{\sin(\pi/2)} = \frac{\pi}{2}, \quad V_C^2 = V_B^2 = U_\infty^2\left[1 + \left(\frac{2}{\pi}\right)^2 + \frac{2}{(\pi/2)}\cos(\pi/2)\right] = 1.405U_\infty^2$$

$$\text{Bernoulli:} \quad p_A + \frac{\rho}{2}V_A^2 = 160000 + 0 = p_B + \frac{\rho}{2}V_B^2 = 90000 + \frac{998}{2}\left(1.405U_\infty^2\right)$$

Solve for $\mathbf{U_\infty \approx 10.0\ m/s}$ *Ans.*

The uncertainty in ($p_A - p_B$) is as high as 6000 Pa, hence the uncertainty in U_∞ is **±0.4 m/s**. *Ans.*

8.28 Sources of equal strength *m* are placed at the four symmetric positions (a, a), (–a, a), (a, –a), and (–a, –a). Sketch the streamline and potential-line patterns. Do any plane "walls" appear?

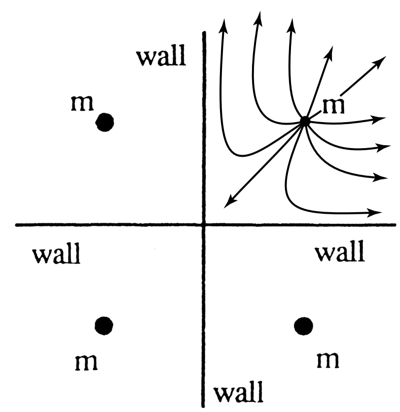


Fig. P8.28

Solution: This is a double-image flow and creates two walls, as shown. Each quadrant has the same pattern: a source in a "corner." *Ans.*

8.29 A uniform water stream, $U_\infty = 20$ m/s and $\rho = 998$ kg/m^3, combines with a source at the origin to form a half-body. At (x, y) = (0, 1.2 m), the pressure is 12.5 kPa less than p_∞.

(a) Is this point outside the body? Estimate (b) the appropriate source strength m and (c) the pressure at the nose of the body.

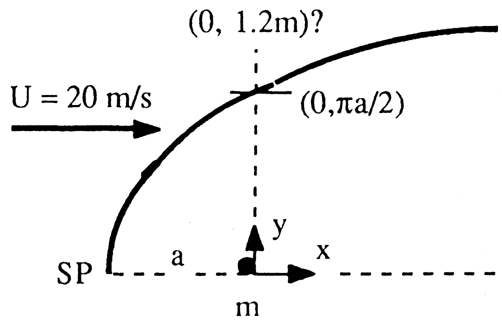


Fig. P8.29

Solution: We know, from Fig. 8.5 and Eq. 8.18, the point on the half-body surface just above "m" is at y = πa/2, as shown, where a = m/U. The Bernoulli equation allows us to compute the necessary source strength m from the pressure at (x, y) = (0, 1.2 m):

$$p_\infty + \frac{\rho}{2}U_\infty^2 = p_\infty + \frac{998}{2}(20)^2 = p_\infty - 12500 + \frac{998}{2}\left[(20)^2 + \left(\frac{m}{1.2}\right)^2\right]$$

$$\text{Solve for}\quad \boldsymbol{m \approx 6.0\ \frac{m^2}{s}}\quad \textit{Ans. (b)}\quad \text{while}\ \ a = \frac{m}{U} = \frac{6.0}{20} = 0.3\ \text{m}$$

The body surface is thus at y = πa/2 = **0.47 m** above m. Thus the point in question, y = 1.2 m above m, is **outside the body**. *Ans.* (a)

At the nose SP of the body, (x, y) = (–a, 0), the velocity is zero, hence we predict

$$p_\infty + \frac{\rho}{2}U_\infty^2 = p_\infty + \frac{998}{2}(20)^2 = p_{nose} + \frac{\rho}{2}(0)^2,\quad \text{or}\quad \mathbf{p_{nose} \approx p_\infty + 200\ kPa}\quad \textit{Ans. (c)}$$

8.30 Suppose that the total discharge from the manifold in Fig. P8.11 is 450 m^3/s and that there is a uniform ocean current of 60 cm/s to the right. Sketch the flow pattern from above, showing the dimensions and the region where the cooling-water discharge is confined.

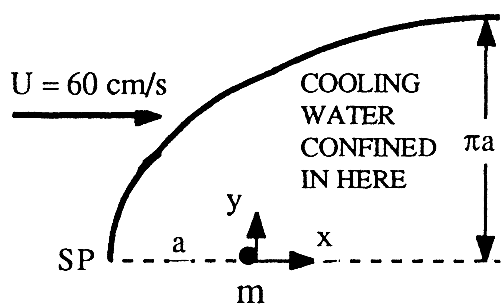


Fig. P8.30

Solution: From Prob. 8.11, with Q known and b = 8 m, we compute the source strength:

$$m = \frac{Q}{2\pi b} = \frac{450\ m^3/s}{2\pi(8\ m)} = 8.95\ \frac{m^2}{s},\quad \text{hence}\ \ \mathbf{a} = \frac{m}{U} = \frac{8.95}{0.6} \approx \mathbf{15\ m}\quad \textit{Ans. (a)}$$

The half-width of the confined region = πa = π(15) ≈ **47 m** *Ans.* (b)

The discharge water is confined to a region 94 m wide and 15 m in front of the manifold.

8.31 A Rankine half-body is formed as shown in Fig. P8.31. For the conditions shown, compute (a) the source strength m in m^2/s; (b) the distance a; (c) the distance h; and (d) the total velocity at point A.

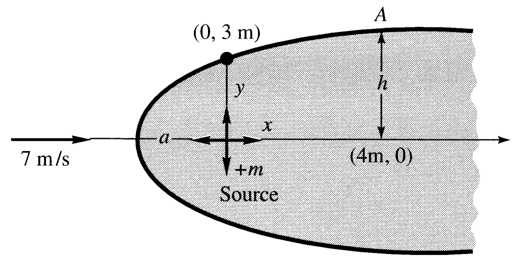


Fig. P8.31

Solution: The vertical distance above the origin is a known multiple of m and a:

$$y_{x=0} = 3\text{ m} = \frac{\pi m}{2U} = \frac{\pi m}{2(7)} = \frac{\pi a}{2},$$

$$\text{or} \quad \boldsymbol{m \approx 13.4\ \frac{m^2}{s}} \quad \text{and} \quad \boldsymbol{a \approx 1.91\text{ m}} \quad \textit{Ans.}\text{ (a, b)}$$

The distance h is found from the equation for the body streamline:

$$\text{At} \quad x = 4\text{ m}, \quad r_{body} = \frac{m(\pi-\theta)}{U\sin\theta} = \frac{13.4(\pi-\theta)}{7\sin\theta} = \frac{4.0}{\cos\theta}, \quad \text{solve for } \theta \approx 47.8^\circ$$

$$\text{Then} \quad r_A = 4.0/\cos(47.8^\circ) = 5.95\text{ m} \quad \text{and} \quad \mathbf{h} = r\sin\theta \approx \mathbf{4.41\ m} \quad \textit{Ans.}\text{ (c)}$$

The resultant velocity at point A is then computed from Eq. (8.18):

$$V_A = U\left(1 + \frac{a^2}{r^2} + \frac{2a}{r}\cos\theta_A\right)^{1/2} = 7\left[1 + \left(\frac{1.91}{5.95}\right)^2 + 2\left(\frac{1.91}{5.95}\right)\cos 47.8^\circ\right]^{1/2} \approx \mathbf{8.7\ \frac{m}{s}} \quad \textit{Ans.}\text{ (d)}$$

8.32 Sketch the streamlines, especially the body shape, due to equal line sources m at (–a, 0) and (+a, 0) plus a uniform stream $U_\infty = ma$.

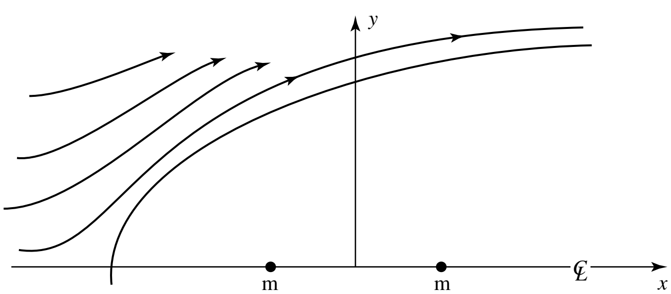


Fig. P8.32

Solution: As shown, a half-body shape is formed quite similar to the Rankine half-body. The stagnation point, for this special case $U_\infty = ma$, is at $x = (-1 - \sqrt{2})a = -2.41a$. The half-body shape would vary with the dimensionless source-strength parameter ($U_\infty a/m$).

8.33 Sketch the streamlines, especially the body shape, due to equal line sources m at (0, +a) and (0, –a) plus a uniform stream U_∞ = ma.

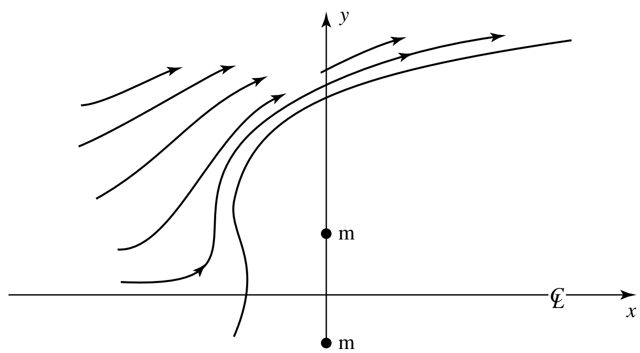


Fig. P8.33

Solution: As shown, a half-body shape is formed which has a dimple at the nose. The stagnation point, for this special case U_∞ = ma, is at x = –a. The half-body shape varies with the parameter (U_∞a/m).

8.34 Consider three equally spaced line sources m placed at (x, y) = (+a, 0), (0, 0), and (–a, 0). Sketch the resulting streamlines and note any stagnation points. What would the pattern look like from afar?

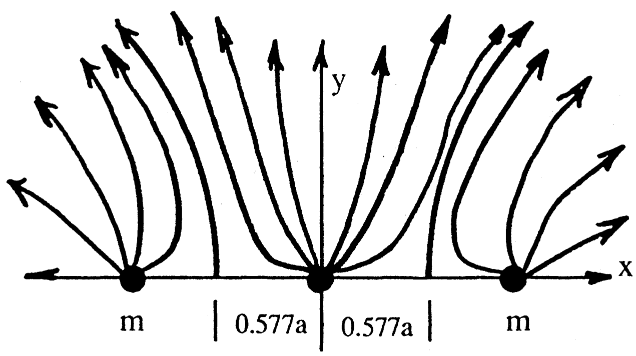


Fig. P8.34

Solution: The pattern (symmetrical about the x-axis) is shown above. There are two stagnation points, at x = $\pm a/\sqrt{3}$ = **±0.577a**. Viewed from afar, the pattern would look like a single source of strength **3 m**.

8.35 Consider three equal sources in a triangular configuration: one at (a/2, 0), one at (–a/2, 0), and one at (0, a). Plot the streamlines for this flow. Are there any stagnation points? *Hint*: Try the MATLAB Contour command [Ref. 34].

Solution: We have $\psi = m\tan^{-1}(y/(x - 0.5a)) + m\tan^{-1}(y/(x + 0.5a)) + m\tan^{-1}((y - a)/x)$. The streamlines are shown in the figure below. There is *one* stagnation point: on the *y*-axis at (0, 0.5a). Viewed from afar, it looks like a **single source of strength (3*m*)**.

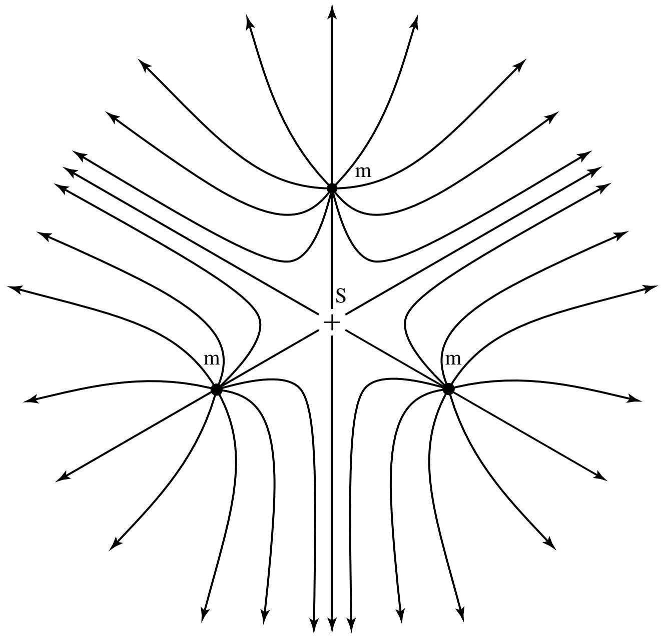


Fig. P8.35

8.36 When a line source-sink pair with m = 2 m^2/s is combined with a uniform stream, it forms a Rankine oval whose minimum dimension is 40 cm, as shown. If a = 15 cm, what are the stream velocity and the maximum velocity? What is the length?

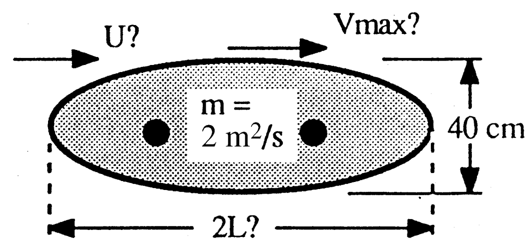


Fig. P8.36

Solution: We know h/a = 20/15, so from Eq. (8.30) we may determine the stream velocity:

$$\frac{h}{a} = \frac{20}{15} = \cot\left[\frac{h/a}{2m/(U_\infty a)}\right] = \cot\left[\frac{20/15}{2(2)/(0.15U_\infty)}\right], \quad \text{solve for } \mathbf{U_\infty \approx 12.9\ \frac{m}{s}} \quad \textit{Ans.}$$

$$\text{Then}\quad \frac{m}{U_\infty a} = \frac{2}{12.9(0.15)} = 1.036, \quad \frac{L}{a} = [1 + 2(1.036)]^{1/2} = 1.75, \quad \mathbf{2L \approx 53\ cm} \quad \textit{Ans.}$$

$$\text{Finally,}\quad \frac{V_{max}}{U_\infty} = 1 + \frac{2m/(U_\infty a)}{1 + (h/a)^2} = 1 + \frac{2(1.036)}{1 + (20/15)^2} = 1.75, \quad \mathbf{V_{max} \approx 22.5\ \frac{m}{s}} \quad \textit{Ans.}$$

8.37 A Rankine oval 2 m long and 1 m high is immersed in a stream $U_\infty = 10$ m/s, as in Fig. P8.37. Estimate (a) the velocity at point A and (b) the location of point B where a particle approaching the stagnation point achieves its maximum deceleration.

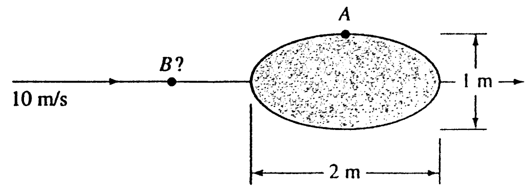


Fig. P8.37

Solution: (a) With L/h = 2.0, we may evaluate Eq. (8.30) to find the source-sink strength:

$$\frac{h}{a} = \cot\left[\frac{h/a}{2m/(U_\infty a)}\right] \quad \text{and} \quad \frac{L}{a} = \left(1 + \frac{2m}{U_\infty a}\right)^{1/2}$$

$$\text{converges to} \quad \frac{L}{h} = 2.0 \quad \text{if} \quad \frac{m}{U_\infty a} = \mathbf{0.3178}$$

$$\text{Meanwhile,} \quad \frac{h}{a} = 0.6395 \ \text{and} \ \frac{L}{a} = 1.2789 \quad \text{thus} \ a = \frac{1 \text{ meter}}{1.2789} \approx \mathbf{0.782\ m}$$

Also compute $V_{max}/U_\infty = 1.451$, hence $V_{max} = 1.451(10) \approx \mathbf{14.5\ m/s}$. *Ans.* (a)

(b) Along the x-axis, at any x ≤ –L, the velocity toward the body nose has the form

$$u = U_\infty + \frac{m}{a+x} + \frac{m}{a-x}, \quad \text{where } m \approx 0.3178 U_\infty a$$

$$\text{Then} \quad \frac{du}{dt} = u\frac{\partial u}{\partial x} = \left[U_\infty + \frac{m}{a+x} + \frac{m}{a-x}\right](-m)\left[\frac{1}{(a+x)^2} - \frac{1}{(a-x)^2}\right]$$

For this value of m, the maximum deceleration occurs at $\mathbf{x = -1.41a}$ *Ans.*

This is quite near the nose (which is at x = –1.28a). The numerical value of the maximum deceleration is $\mathbf{(du/dt)_{max} \approx -0.655U_\infty^2/a}$.

8.38 A uniform stream U in the *x* direction combines with a source *m* at (+a, 0) and a sink –*m* at (–a, 0). Plot the resulting streamlines and note any stagnation points.

Solution: There are two cases. (a) For $m > U_\infty a/2$, there are two stagnation points on the **y-axis**; (b) for $m < U_\infty a/2$, there are two stagnation points on the **x-axis**:

$$\text{(a)}\ m > \frac{U_\infty a}{2}: \quad \mathbf{y_{stag}} = \pm\sqrt{\frac{\mathbf{2m}}{\mathbf{U_\infty a}} - \mathbf{1}}; \quad \text{(b)}\ m < \frac{U_\infty a}{2}: \quad \mathbf{x_{stag}} = \pm\sqrt{\mathbf{1} - \frac{\mathbf{2m}}{\mathbf{U_\infty a}}} \quad \textit{Ans.}$$

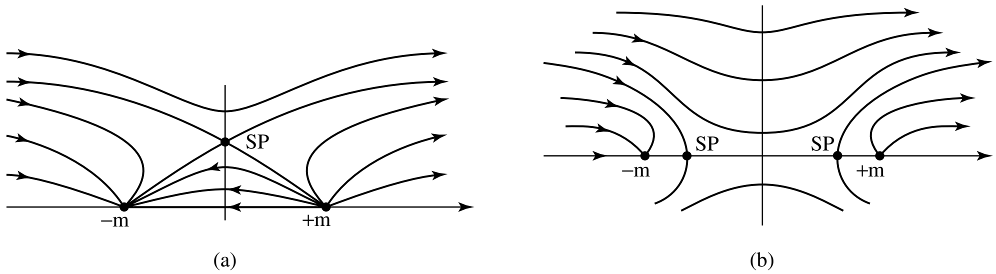


Fig. P8.38

8.39 Find the value of $m/(U_\infty a)$ for which the velocity in the inside center of a Rankine oval exactly equals $3U_\infty$.

Solution: From the geometry of Fig. 8.9, the velocity in the center of the oval is:

$$V_{Origin} = U_\infty + \frac{m}{a}\bigg|_{source} + \frac{m}{a}\bigg|_{sink} = 3U_\infty \quad \text{if} \quad \boldsymbol{\frac{m}{U_\infty a} = 1.0} \quad \textit{Ans.}$$

8.40 Consider a uniform stream U_∞ plus line sources (+m) at (+a, 0) and (−a, 0) and a single line sink (−2m) at (0, 0). Does a closed body shape appear? If so, plot its shape for $m/(U_\infty a)$ equal to (a) 1.0; and (b) 5.0.

Solution: Although the flow of the sources balances the sink, there **is no decent body shape**. The sink is too strong. The two cases requested are shown below.

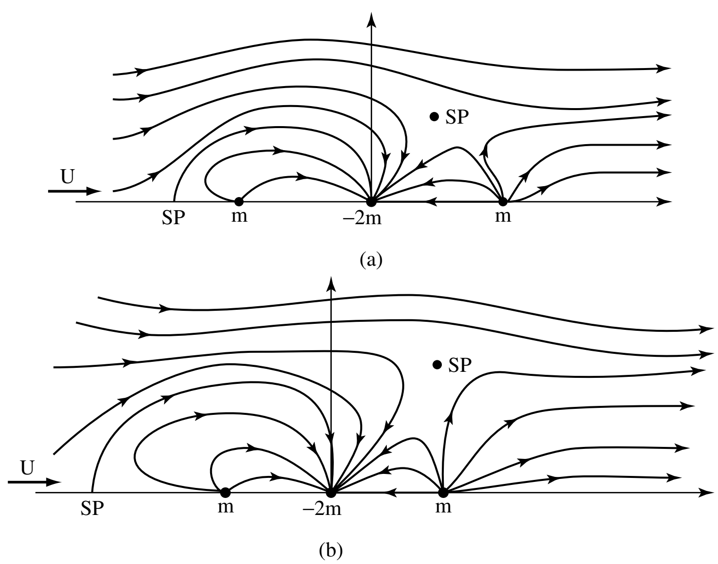


Fig. P8.40

8.41 A Kelvin oval is formed by a line-vortex pair with K = 9 m^2/s, a = 1 m, and U = 10 m/s. What are the height, width, and shoulder velocity of this oval?

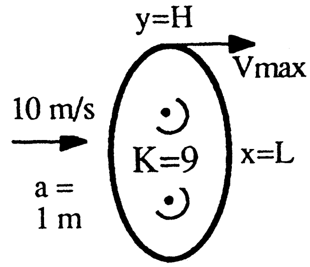


Fig. P8.41

Solution: With reference to Fig. 8.12 and Eq. (8.41), the oval is described by

$$\psi = 0,\ x = 0,\ y = H:\quad UH = \frac{K}{2}\ln\left[\frac{(H+a)^2}{(H-a)^2}\right],\quad \text{with } \frac{K}{Ua} = \frac{9}{10(1)} = \mathbf{0.9}$$

Solve by iteration for $H/a \approx 1.48$, or **2H** = oval height ≈ **2.96 m** *Ans.*

$$\text{Similarly,}\quad \frac{L}{a} = \left(\frac{2K}{Ua} - 1\right)^{1/2} = [2(0.9)-1]^{1/2} = 0.894,\quad \mathbf{2L} = \text{width} \approx \mathbf{1.79\ m} \quad \textit{Ans.}$$

$$\text{Finally,}\quad V_{max} = U + \frac{K}{H-a} - \frac{K}{H+a} = 10 + \frac{9}{0.48} - \frac{9}{2.48} \approx \mathbf{25.1\ \frac{m}{s}} \quad \textit{Ans.}$$

8.42 For what value of $K/(U_\infty a)$ does the velocity at the shoulder of a Kelvin oval equal $4U_\infty$? What is height h/a of this oval?

Solution: Following up on the formulas from Prob. 8.41, we are to iterate between the oval-height and maximum-velocity formulas from Eq. (8.41):

$$V_{max} = U_\infty + \frac{K}{h-a} - \frac{K}{h+a} = \mathbf{4U_\infty} \quad \text{plus} \quad \frac{h}{a} = \frac{K}{U_\infty a}\ln\left[\frac{h/a+1}{h/a-1}\right],\quad \text{solve for h and K.}$$

After effort (PC calculation is best), we find $\mathbf{K/(U_\infty a) \approx 0.396,\ h/a \approx 1.124}$. *Ans.*

8.43 Consider water at 20°C flowing past a 1-m-diameter cylinder. What doublet strength in m^2/s is required to simulate this flow? If the stream pressure is 200 kPa, use inviscid theory to estimate the surface pressure at (a) 180°; (b) 135°; and (c) 90°.

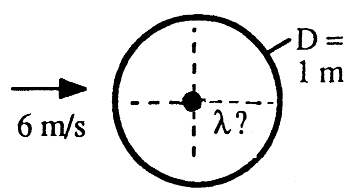


Fig. P8.43

Solution: For water at 20°C, take $\rho = 998\ kg/m^3$. The required doublet strength is

$$\lambda = U_\infty a^2 = (6.0\ m/s)(0.5\ m)^2 \approx \mathbf{1.5\ \frac{m^3}{s}} \quad \textit{Ans.}$$

The surface pressures are computed from Bernoulli's equation, with $V_{surface} = 2U_\infty \sin\theta$:

$$p_s + \frac{\rho}{2}(2U_\infty \sin\theta)^2 = p_\infty + \frac{\rho}{2}U_\infty^2, \quad \text{or:} \quad p_s = 200000 + \frac{998}{2}(6)^2(1 - 4\sin^2\theta)$$

(a) at 180°, $p_s \approx$ **218000 Pa**; (b) at 135°, **182000 Pa**; (c) at 90°, **146000 Pa** *Ans.* (a, b, c)

8.44 Suppose that circulation is added to the cylinder flow of Prob. 8.43 sufficient to place the stagnation points at $\theta = 35°$ and 145°. What is the required vortex strength K in m^2/s? Compute the resulting pressure and surface velocity at (a) the stagnation points, and (b) the upper and lower shoulders. What will be the lift per meter of cylinder width?

Solution: Recall that Prob. 8.43 was for water at 20°C flowing at 6 m/s past a 1-m-diameter cylinder, with p_∞ = 200 kPa. From Eq. (8.35),

$$\sin\theta_{stag} = \sin(35°) = \frac{K}{2U_\infty a} = \frac{K}{2(6\text{ m/s})(0.5\text{ m})}, \quad \text{or:} \quad \mathbf{K = 3.44\ m/s} \quad \textit{Ans.}$$

(a) At the stagnation points, velocity is zero and pressure equals stagnation pressure:

$$p_{stag} = p_\infty + \frac{\rho}{2}U_\infty^2 = 200{,}000\ Pa + \frac{998\text{ kg/m}^3}{2}(6\text{ m/s})^2 = \mathbf{218{,}000\ Pa} \quad \textit{Ans.}\text{ (a)}$$

(b) At any point on the surface, from Eq. (8.37),

$$p_{stag} = 218000 = p_{surf} + \frac{\rho}{2}\left(-2U_\infty\sin\theta + \frac{K}{a}\right)^2 = p_{surf} + \frac{998}{2}\left[-2(6)\sin\theta + \frac{3.44}{0.5}\right]^2$$

At the upper shoulder, $\theta = 90°$,

$$p = 218000 - \frac{998}{2}(-5.12)^2 \approx \mathbf{204{,}900\ Pa} \quad \textit{Ans.}\text{ (b—upper)}$$

At the lower shoulder, $\theta = 270°$,

$$p = 218000 - \frac{998}{2}(-18.88)^2 \approx \mathbf{40{,}100\ Pa} \quad \textit{Ans.}\text{ (b—lower)}$$

8.45 If circulation K is added to the cylinder flow in Prob. 8.43, (a) for what value of K will the flow begin to cavitate at the surface? (b) Where on the surface will cavitation begin? (c) For this condition, where will the stagnation points lie?

Solution: Recall that Prob. 8.43 was for water at 20°C flowing at 6 m/s past a 1-m-diameter cylinder, with p_∞ = 200 kPa. From Table A.5, p_{vap} = 2337 Pa. (b) Cavitation will occur at the lowest pressure point, which is **at the bottom shoulder ($\boldsymbol{\theta}$ = 270°)** in Fig. 8.10. *Ans.* (b)

(a) Use Bernoulli's equation to estimate the velocity at θ = 270° if the pressure there is p_{vap}:

$$p_o = 218000\text{ Pa} = p_{vap} + \frac{\rho}{2}V_{surf}^2 = 2337\text{ Pa} + \frac{998\text{ kg/m}^3}{2}\left[(-2)(6\text{ m/s})\sin(270°) + \frac{K}{0.5\text{ m}}\right]^2$$

$$\text{Solve for:}\quad \mathbf{K_{cavitation} \approx 4.39\,m^2/s} \quad \textit{Ans.}\text{ (a)}$$

(c) The locations of the two stagnation points are given by Eq. (8.35):

$$\sin\theta_{stag} = \frac{K}{2U_\infty a} = \frac{4.39\text{ m}^2\text{/s}}{2(6\text{ m/s})(0.5\text{ m})} = 0.732, \quad \theta_{stag} = \mathbf{47°}\ \textit{and}\ \mathbf{133°} \quad \textit{Ans.}\text{ (c)}$$

8.46 A cylinder is formed by bolting two semicylindrical channels together on the inside, as shown in Fig. P8.46. There are 10 bolts per meter of width on each side, and the inside pressure is 50 kPa (gage). Using potential theory for the outside pressure, compute the tension force in each bolt if the fluid outside is sea-level air.

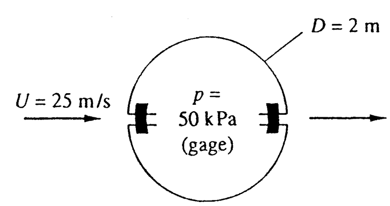


Fig. P8.46

Solution: For sea-level air take ρ = 1.225 kg/m³. Use Bernoulli to find surface pressure:

$$p_\infty + \frac{\rho}{2}U_\infty^2 = 0 + \frac{1.225}{2}(25)^2 = p_s + \frac{1.225}{2}(2U_\infty\sin\theta)^2, \quad \text{or:}\quad p_s = 383 - 1531\sin^2\theta$$

$$\text{compute}\quad F_{down} = 2\int_0^{\pi/2} p\sin\theta\, b a\, d\theta = 2\int_0^{\pi/2}(383 - 1531\sin^2\theta)\sin\theta(1\text{ m})(1\text{ m})\,d\theta = -1276\ \frac{\text{N}}{\text{m}}$$

This is small potatoes compared to the force due to *inside* pressure:

$$F_{up} = 2p_{inside}ab = 2(50000)(1)(1) = 100000\ \frac{\text{N}}{\text{m}}$$

$$\text{Total force per meter} = 100000 - (-1276) = 101276 \div 20\text{ bolts} \approx \mathbf{5060\ \frac{N}{bolt}} \quad \textit{Ans.}$$

8.47 A circular cylinder is fitted with two pressure sensors, to measure pressure at "a" (180°) and "b" (105°), as shown. The intent is to use this cylinder as a stream velocimeter. Using inviscid theory, derive a formula for calculating U_∞ from p_a, p_b, ρ, and radius *a*.

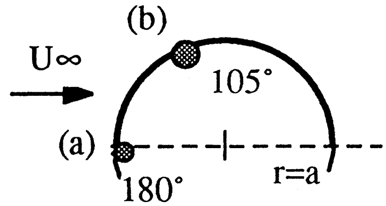


Fig. P8.47

Solution: We relate the pressures to surface velocities from Bernoulli's equation:

$$p_\infty + \frac{\rho}{2}U_\infty^2 = p_a + \frac{\rho}{2}(0)^2 = p_b + \frac{\rho}{2}(2U_\infty \sin 105°)^2, \quad \text{or:} \quad \mathbf{U_\infty = \frac{1}{\sin 105°}\sqrt{\frac{p_a - p_b}{2\rho}}} \quad \textit{Ans.}$$

This is not a bad idea for a velocimeter, except that (1) it should be calibrated; and (2) it must be carefully aligned so that sensor "a" exactly faces the oncoming stream.

8.48 Wind at U_∞ and p_∞ flows past a Quonset hut which is a half-cylinder of radius a and length L (Fig. P8.48). The internal pressure is p_i. Using inviscid theory, derive an expression for the upward force on the hut due to the difference between p_i and p_s.

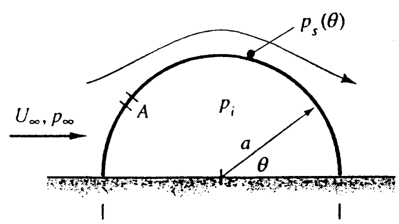


Fig. P8.48

Solution: The analysis is similar to Prob. 8.46 on the previous page. If p_o is the stagnation pressure at the nose ($\theta = 180°$), the surface pressure distribution is

$$p_s = p_o - \frac{\rho}{2}U_s^2 = p_o - \frac{\rho}{2}(2U_\infty \sin\theta)^2 = p_o - 2\rho U_\infty^2 \sin^2\theta$$

Then the net upward force on the half-cylinder is found by integration:

$$F_{up} = \int_0^\pi (p_i - p_s)\sin\theta\, ab\, d\theta = \int_0^\pi (p_i - p_o + 2\rho U_\infty^2 \sin^2\theta)\sin\theta\, ab\, d\theta,$$

$$\text{or:} \quad \mathbf{F_{up} = (p_i - p_o)2ab + \frac{8}{3}\rho U_\infty^2\, ab} \quad \textit{Ans.} \left(\text{where } p_o = p_\infty + \frac{\rho}{2}U_\infty^2\right)$$

8.49 In strong winds, the force in Prob. 8.48 above can be quite large. Suppose that a hole is introduced in the hut roof at point A (see Fig. P8.48) to make p_i equal to the surface pressure p_A. At what angle θ should hole A be placed to make the net force zero?

Solution: Set F = 0 in Prob. 8.48 and find the proper pressure from Bernoulli:

$$F_{up} = 0 \text{ if } \quad p_i = p_o - \frac{4}{3}\rho U_\infty^2, \quad \text{but also } \; p_i = p_A = p_o - \frac{\rho}{2}(2U_\infty \sin\theta_A)^2$$

$$\text{Solve for} \quad \sin\theta_A = \sqrt{2/3} = 0.817 \quad \text{or} \quad \theta_A \approx \mathbf{125°} \quad \textit{Ans.}$$

(or **55°** = poor position on rear of body)

8.50 It is desired to simulate flow past a ridge or "bump" by using a streamline *above* the flow over a cylinder, as shown in Fig. P8.50. The bump is to be $a/2$ high, as shown. What is the proper elevation h of this streamline? What is U_{max} on the bump compared to U_∞?

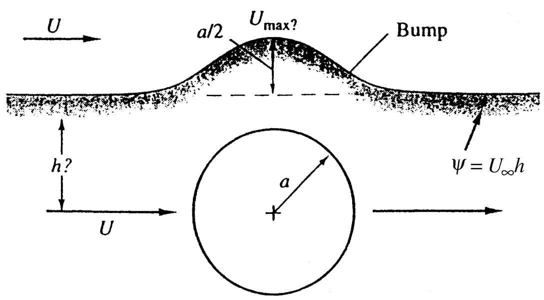


Fig. P8.50

Solution: Apply the equation of the streamline (Eq. 8.32) to θ=180° and also 90°:

$$\psi = U_\infty \sin\theta\left(r - \frac{a^2}{r}\right) \quad \text{at } \theta = 180° \text{ (the freestream) gives } \psi = U_\infty h$$

$$\text{Then, at } \theta = 90°, \quad r = h + \frac{a}{2}, \quad \psi = U_\infty h = U_\infty \sin 90°\left(h + \frac{a}{2} - \frac{a^2}{h + a/2}\right)$$

$$\text{Solve for} \quad \mathbf{h = \frac{3}{2}a} \quad \textit{Ans.} \quad \text{(corresponds to r = 2a)}$$

The velocity at the hump (r = 2a, θ = 90°) then follows from Eq. (8.33):

$$U_{max} = U_\infty \sin 90°\left[1 + \frac{a^2}{(2a)^2}\right] \quad \text{or} \quad \mathbf{U_{max} = \frac{5}{4}U_\infty} \quad \textit{Ans.}$$

8.51 Modify Prob. 8.50 above as follows: Let the bump be such that $U_{max} = 1.5U_\infty$. Find (a) the upstream elevation h; and (b) the height Z of the bump.

Solution: We use the analysis but modify it for unknown bump height *Z*:

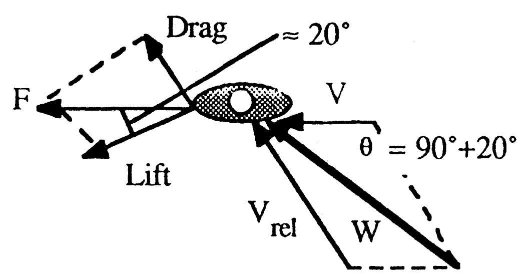
$$\text{At } \theta = 90°,\ r = h + Z:\quad U_{max} = 1.5U_\infty = U_\infty \sin 90°\left[1 + \left(\frac{a}{h+Z}\right)^2\right],\quad \text{solve } h + Z = \mathbf{a\sqrt{2}}$$

$$\text{Then}\quad \psi = U_\infty h = U_\infty \sin 90°\left(h + Z - \frac{a^2}{h+Z}\right),\quad \text{solve } \mathbf{h = Z = \frac{a}{\sqrt{2}}}\quad \textit{Ans.}$$

8.52 The Flettner-rotor sailboat in Fig. E8.2 has a water drag coefficient of 0.006 based on a wetted area of 45 ft^2. If the rotor spins at 220 rev/min, find the maximum boat speed that can be achieved in 15 mi/h winds. Find the optimum wind angle.

Fig. E8.2

Solution: Recall that the rotor has a diameter of 2.5 ft and is 10 ft high. Standard air density is 0.00238 slug/ft^3. As in Ex. 8.2, estimate $C_L \approx 3.3$ and $C_D \approx 1.2$. If the boat speed is V and the wind is W, the relative velocity V_{rel} is shown in the figure at right. Thrust = drag:



$$F = \left(C_L^2 + C_D^2\right)^{1/2}\frac{\rho}{2}V_{rel}^2 DL$$

$$= \text{Boat drag} = C_{d,boat}\frac{\rho_{water}}{2}V^2 A_{wetted},$$

$$\text{or:}\quad [(3.3)^2 + (1.2)^2]^{1/2}\left(\frac{0.00238}{2}\right)V_{rel}^2(2.5)(10) = (0.006)\left(\frac{1.99}{2}\right)V^2(45),$$

$$\text{or:}\quad \mathbf{V_{rel} = 1.604V}$$

Convert W = 15 mi/h = 22 ft/s. As shown in the figure, the angle between the wind lift and wind drag is $\tan^{-1}(C_D/C_L) = \tan^{-1}(1.2/3.3) \approx 20.0°$. Then, by geometry, the angle θ between the relative wind and the boat speed (see figure above) is $\theta = 180 - 70 = 110°$. The law of cosines, applied to the wind-vector triangle above, then determines the boat speed:

$$W^2 = V^2 + V_{rel}^2 - 2VV_{rel}\cos\theta,$$

$$\text{or:}\quad (22)^2 = V^2 + (1.604V)^2 - 2V(1.604V)\cos(110°)$$

$$\text{Solve for}\quad \mathbf{V_{boat} \approx 10.2\ \frac{ft}{s}}\quad \textit{Ans.}$$

For this "optimum" condition (which directs the resultant wind force along the keel or path of the boat), the angle β between the wind and the boat direction (see figure on the previous page) is

$$\frac{\sin\beta}{1.604(10.2)} = \frac{\sin(110°)}{22}, \quad \text{or:} \quad \beta \approx \mathbf{44°} \quad \textit{Ans.}$$

8.53 Modify Prob. P8.52 as follows. For the same sailboat data, find the wind velocity, in mi/h, which will drive the boat at an optimum speed of **8 kn** parallel to its keel.

Solution: Convert 8 knots = 13.5 ft/s. Again estimate $C_L \approx 3.3$ and $C_D \approx 1.2$. The geometry is the same as in Prob. P8.52, hence V_{rel} still equals $1.604V = 21.66$ ft/s. The law of cosines still holds for the velocity diagram in the figure:

$$W^2 = V^2 + V_{rel}^2 - 2VV_{rel}\cos\theta = (13.5)^2 + (21.66)^2 - 2(13.5)(21.66)\cos(110°) = 851 \text{ ft}^2/\text{s}^2$$

$$\text{Solve for} \quad W_{wind} = 29.2 \text{ ft/s} = \mathbf{19.9\ mi/h} \quad \textit{Ans.}$$

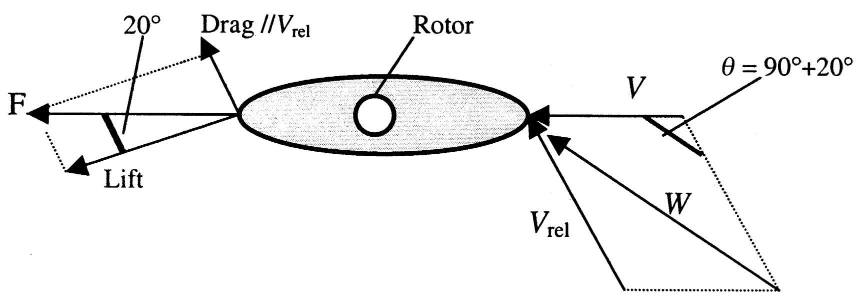


Fig. P8.52

8.54 The original Flettner rotor ship was approximately 100 ft long, displaced 800 tons, and had a wetted area of 3500 ft^2. As sketched in Fig. P8.54, it had two rotors 50 ft high and 9 ft in diameter rotating at 750 r/min, which is far outside the range of Fig. 8.11. The measured lift and drag coefficients for each rotor were about 10 and 4, respectively. If the ship is moored and subjected to a crosswind of 25 ft/s, as in Fig. P8.54, what will the wind force parallel and normal to the ship centerline be? Estimate the power required to drive the rotors.

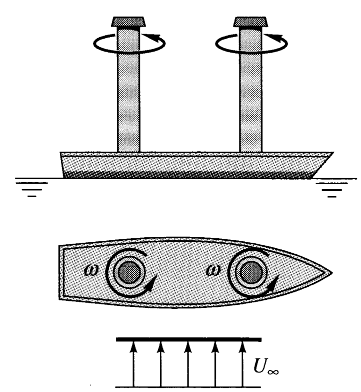
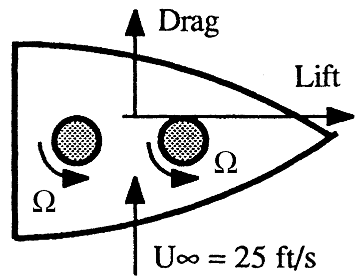


Fig. P8.54

Solution: For sea-level air take ρ = 0.00238 slug/ft^3 and μ = 3.71E–7 slug/ft·s. Then compute the forces:



$$\text{Lift} = C_L \frac{\rho}{2} U_\infty^2 DL$$

$$= 10\left(\frac{0.00238}{2}\right)(25)^2(9\text{ ft})(50\text{ ft}) \times (2\text{ rotors}) \approx \mathbf{6700\ lbf}\text{ (parallel)} \quad Ans.$$

$$\text{Drag} = (4/10)\text{Lift} \approx \mathbf{2700\ lbf}\text{ (normal)} \quad Ans.$$

We don't have any *formulas* in the book for the (viscous) torque of a rotating cylinder (you could find results in refs. 1 and 2 of Chap. 7). As a good approximation, assume the cylinder simulates a flat plate of length $2\pi R = 2\pi(4.5) = 28.3$ ft. Then the shear stress is:

$$\tau_w = C_f \frac{\rho}{2} U^2 \approx \frac{0.027}{Re_L^{1/7}} \frac{\rho}{2} (\Omega R)^2, \quad \Omega R = 750\left(\frac{2\pi}{60}\right)(4.5) = 353\ \frac{\text{ft}}{\text{s}}, \quad \text{and}$$

$$Re_L = \frac{0.00238(353)(28.3)}{3.71\text{E–}7} \approx 6.42\text{E}7, \quad \text{whence } \tau_w \approx \mathbf{0.308}\ \frac{\text{lbf}}{\text{ft}^2}$$

$$\text{Then} \quad \text{Torque} = \tau_w(\pi D L)R = 0.308\pi(9)(50)(4.5) \approx 1958\ \text{ft·lbf}$$

$$\text{Total power} = T\Omega = (1958)\left(750\frac{2\pi}{60}\right) \times (2\text{ rotors}) \div 550\ \frac{\text{hp}}{\text{ft·lbf/s}} \approx \mathbf{560\ hp} \quad Ans.$$

8.55 Assume that the Flettner rotor ship of Fig. P8.54 has a water-resistance coefficient of 0.005. How fast will the ship sail in seawater at 20°C in a 20 ft/s wind if the keel aligns itself with the resultant force on the rotors? [This problem involves relative velocities.]

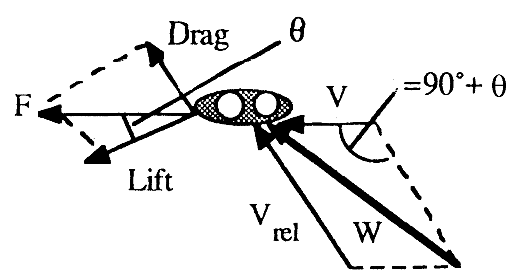


Fig. P8.55

Solution: For air, take ρ = 0.00238 slug/ft^3. For seawater, take ρ = 1.99 slug/ft^3 and μ = 2.09E–5 slug/ft·s. Recall D = 9 ft, L = 50 ft, 2 rotors at 750 rev/min, $C_L \approx 10.0$, $C_D \approx 4.0$. In the sketch above, the drag and lift combine along the ship's keel. Then

$$\theta = \tan^{-1}\left(\frac{4}{10}\right) \approx 21.8°, \quad \text{so angle between } \mathbf{V} \text{ and } \mathbf{V_{rel}} = 90 + \theta \approx \mathbf{111.8°}$$

$$\text{Thrust}\quad F = [(10)^2 + (4)^2]^{1/2}\left(\frac{0.00238}{2}\right)V_{rel}^2(9)(50)(2\text{ rotors})$$

$$= \text{Drag} = C_d(\rho/2)V^2A_{wetted} = (0.005)(1.99/2)(3500)V^2, \quad \text{solve } \mathbf{V_{rel} \approx 1.23V}$$

$$\text{Law of cosines:}\quad W^2 = V^2 + V_{rel}^2 - 2VV_{rel}\cos(\theta + 90°),$$

$$\text{or:}\quad (20)^2 = V^2 + (1.23V)^2 - 2V(1.23V)\cos(111.8°), \quad \text{solve for } \mathbf{V_{ship} \approx 10.8\ \frac{ft}{s}} \quad \textit{Ans.}$$

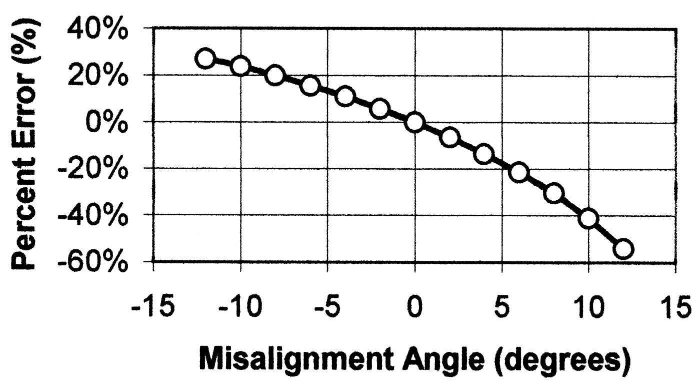
8.56 A proposed freestream velocimeter would use a cylinder with pressure taps at $\theta =$ 180° and at 150°. The pressure difference would be a measure of stream velocity U_∞. However, the cylinder must be aligned so that one tap exactly faces the freestream. Let the misalignment angle be δ, that is, the two taps are at (180° + δ) and (150° + δ). Make a plot of the percent error in velocity measurement in the range –20° < δ < +20° and comment on the idea.

Solution: Recall from Eq. (8.34) that the surface velocity on the cylinder equals $2U_\infty\sin\theta$. Apply Bernoulli's equation at both points, 180° and 150°, to solve for stream velocity:

$$p_1 + \frac{\rho}{2}[2U_\infty\sin(180° + \delta)]^2 = p_2 + \frac{\rho}{2}[2U_\infty\sin(150° + \delta)]^2,$$

$$\text{or:}\quad U_\infty = \frac{\sqrt{\Delta p/2\rho}}{\sqrt{\sin^2(150° + \delta) - \sin^2(180° + \delta)}}$$

The error is zero when $\delta = 0°$. Thus we can plot the percent error versus δ. When $\delta = 0°$, the denominator above equals 0.5. When $\delta = 5°$, the denominator equals 0.413, giving an error on the low side of (0.413/0.5) – 1 = –17%! The plot below shows that this is a very poor idea for a velocimeter, since even a small misalignment causes a large error.



Problem 8.56

8.57 In principle, it is possible to use rotating cylinders as aircraft wings. Consider a cylinder 30 cm in diameter, rotating at 2400 rev/min. It is to lift a 55-kN airplane flying at 100 m/s. What should the cylinder length be? How much power is required to maintain this speed? Neglect end effects on the rotating wing.

Solution: Assume sea-level air, $\rho = 1.23\ \text{kg/m}^3$. Use Fig. 8.11 for lift and drag:

$$\frac{a\omega}{U_\infty} = \frac{(0.15)[2400(2\pi/60)]}{100} \approx 0.38. \quad \text{Fig. 8.11: Read } C_L \approx 1.8,\ C_D \approx 1.1$$

$$\text{Then} \quad \text{Lift} = 55000\ \text{N} = C_L \frac{\rho}{2} U_\infty^2 DL = (1.8)\left(\frac{1.23}{2}\right)(100)^2(0.3)L,$$

$$\text{solve} \quad \mathbf{L \approx 17\ m} \quad \textit{Ans.}$$

$$\text{Drag} = C_D \frac{\rho}{2} U_\infty^2 DL = (1.1)\left(\frac{1.23}{2}\right)(100)^2(0.3)(17) \approx 33600\ \text{N}$$

$$\text{Power required} = FU = (33600)(100) \approx \mathbf{3.4\ MW!} \quad \textit{Ans.}$$

The power requirements are ridiculously high. This airplane has way too much drag.

8.58 Plot the streamlines due to a line sink ($-m$) at the origin, plus line sources ($+m$) at (a, 0) and (4a, 0). *Hint*: A cylinder of radius 2a appears.

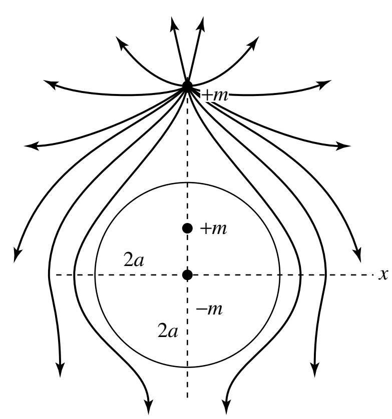


Fig. P8.58

Solution: The overall stream function is

$$\psi = m\tan^{-1}\left(\frac{y-4a}{x}\right) + m\tan^{-1}\left(\frac{y-a}{x}\right) - m\tan^{-1}(y/x)$$

The cylinder shape, of radius $2a$, is the streamline $\psi = -\pi/2$. *Ans.*

8.59 By analogy with Prob. 8.58 above, plot the streamlines due to counterclockwise line vortices +K at (0, 0) and (4a, 0) plus a clockwise line vortex (–K) at (a, 0). *Hint*: Again a cylinder of radius $2a$ appears.

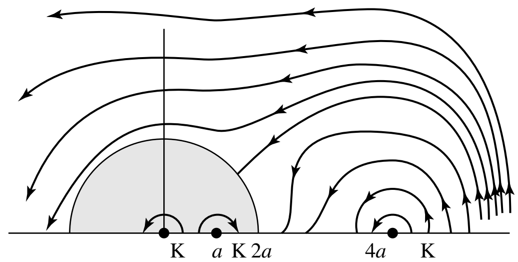


Fig. P8.59

8.60 One of the corner-flow patterns of Fig. 8.15 is given by the cartesian stream function $\psi = A(3yx^2 - y^3)$. Which one? Can this correspondence be proven from Eq. (8.49)?

Solution: This ψ is Fig. 8.15a, **flow in a 60° corner**. [Its velocity potential was given earlier Eq. (8.49) of the text.] The trigonometric form (Eq. 8.49 for n = **3**) is

$$\psi = Ar^3\sin(3\theta), \quad \text{but} \quad \sin(3\theta) \equiv 3\sin\theta\cos^2\theta - \sin^3\theta.$$

Introducing $y = r\sin\theta$ and $x = r\cos\theta$, we obtain $\boldsymbol{\psi = A(3yx^2 - y^3)}$ *Ans.*

8.61 Plot the streamlines of Eq. (8.49) in the upper right quadrant for $n = 4$. How does the velocity increase with x outward along the x axis from the origin? For what corner angle and value of n would this increase be linear in x? For what corner angle and n would the increase be as x^5?

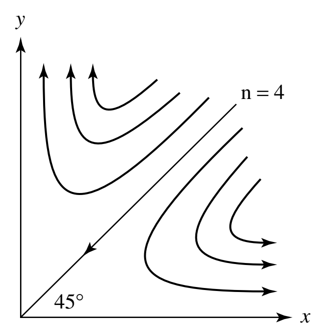


Fig. P8.61

Solution: For $n = 4$, we have **flow in a 45° corner**, as shown. Compute

$$n = 4: \quad \psi = Ar^4\sin(4\theta), \quad v_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = 4Ar^3\cos(4\theta)$$

$$\text{Along the } x\text{-axis,} \quad \theta = 0,\ r = x,\ v_r = u = (\text{const})\boldsymbol{x^3} \quad \textit{Ans. (a)}$$

In general, for any n, the flow along the x-axis is $u = (\text{const})x^{n-1}$. Thus u is linear in x for $n = \mathbf{2}$ (a 90° corner). *Ans*. (b). And $u = Cx^5$ if $n = \mathbf{6}$ (a 30° corner). *Ans*. (c)

8.62 Combine stagnation flow, Fig. 8.14b, with a source at the origin:

$$f(z) = Az^2 + m\ln(z)$$

Plot the streamlines for $m = AL^2$, where L is a length scale. Interpret.

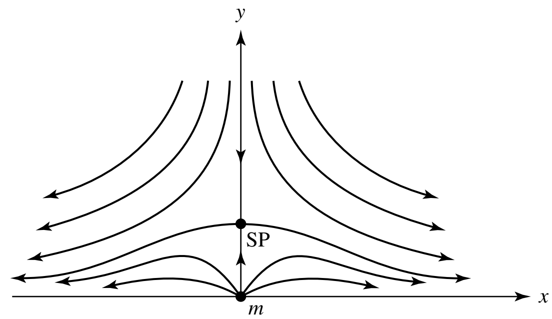


Fig. P8.62

Solution: The imaginary part of this complex potential is the stream function:

$$\psi = 2Axy + m\tan^{-1}\left(\frac{y}{x}\right), \quad \text{with } m = AL^2$$

The streamlines are shown on the previous page. The source pushes the oncoming stagnation flow away from the vicinity of the origin. There is a stagnation point above the source, at (x, y) = (**0, L/√2**). Thus we have "stagnation flow near a bump." *Ans.*

8.63 The superposition in Prob. 8.62 above leads to stagnation flow near a curved *bump*, in contrast to the flat wall of Fig. 8.15*b*. Determine the maximum height *H* of the bump as a function of the constants *A* and *m*. The bump crest is a stagnation point:

$$v_{\text{bump crest}} = -2AH + \frac{m}{H} = 0 \quad \text{whence } \boldsymbol{H_{\text{bump}} = \sqrt{\frac{m}{2A}}} \quad \textit{Ans.}$$

8.64 Consider the polar-coordinate velocity potential $\phi = Br^{1.2}\cos(1.2\theta)$, where B is a constant. (a) Determine whether $\nabla^2\phi = 0$. If so, (b) find the associated stream function $\psi(r, \theta)$ and (c) plot the full streamline which includes the x-axis ($\theta = 0$) and interpret.

Solution: (a) It is laborious, but the velocity potential satisfies Laplace's equation in polar coordinates:

$$\nabla^2\phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{\partial^2\phi}{\partial\theta^2}\right) \equiv 0 \quad \text{if } \boldsymbol{\phi = Br^{1.2}\cos(1.2\theta)} \quad \textit{Ans. } \text{(a)}$$

(b) This example is one of the family of "corner flow" solutions in Eq. (8.49). Thus:

$$\boldsymbol{\psi = Br^{1.2}\sin(1.2\theta)} \quad \textit{Ans. } \text{(b)}$$

(c) This function represents **flow around a 150° corner**, as shown below. *Ans*. (c)

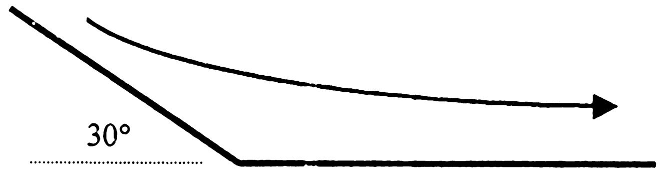


Fig. P8.64

8.65 Potential flow past a wedge of half-angle θ leads to an important application of laminar-boundary-layer theory called the *Falkner-Skan flows* [Ref. 15 of Chap. 8, pp. 242–247]. Let x denote distance along the wedge wall, as in Fig. P8.65, and let $\theta = 10°$. Use Eq. (8.49) to find the variation of surface velocity $U(x)$ along the wall. Is the pressure gradient adverse or favorable?

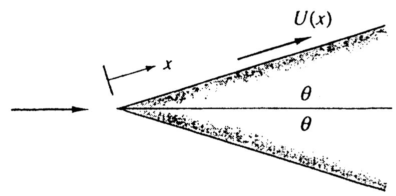


Fig. P8.65

Solution: As discussed above, all wedge flows are "corner flows" and have a velocity along the wall of the form $u = (\text{const})x^{n-1}$, where n = π/(turning angle). In this case, the turning angle is $\beta = (\pi - \theta)$, where $\theta = 10° = \pi/18$. Hence the proper value of n here is:

$$n = \frac{\pi}{\beta} = \frac{\pi}{\pi - \pi/18} = \frac{18}{17}, \quad \text{hence } U = Cx^{n-1} = \mathbf{Cx^{1/17}} \quad \textbf{(favorable gradient)} \quad \textit{Ans.}$$

8.66 The inviscid velocity along the wedge in Prob. 8.65 has the form U(x) = Cx^m, where m = n − 1 and n is the exponent in Eq. (8.49). Show that, for any C and n, computation of the laminar boundary-layer by Thwaites' method, Eqs. (7.53) and (7.54), leads to a unique value of the Thwaites parameter λ. Thus wedge flows are called *similar* [Ref. 15 of Chap. 8, p. 244].

Solution: The momentum thickness is computed by Eq. (7.54), assuming $\theta_o = 0$:

$$\theta^2 = \frac{0.45\nu}{U^6}\int_0^x U^5\,dx = \frac{0.45\nu}{C^6x^{6m}}\int_0^x C^5x^{5m}\,dx = \frac{0.45\nu\, x^{1-m}}{C(5m+1)} \quad \text{Then use Eq. (7.53):}$$

$$\lambda = \frac{\theta^2}{\nu}\frac{dU}{dx} = \left(\frac{0.45x^{1-m}}{C(5m+1)}\right)(mCx^{m-1}) = \frac{\mathbf{0.45m}}{\mathbf{5m+1}} \quad \text{(independent of C and m)} \quad \textit{Ans.}$$

8.67 Investigate the complex potential function $f(z) = U_\infty(z + a^2/z)$, where a is a constant, and interpret the flow pattern.

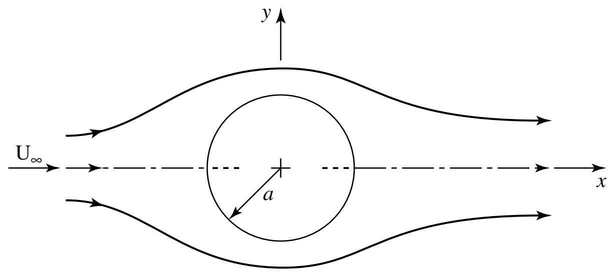


Fig. P8.67

Solution: This represents flow past a **circular cylinder** of radius a, with stream function and velocity potential identical to the expressions in Eqs. (8.31) and (8.32) with K = 0. [There is no circulation.]

8.68 Investigate the complex potential function $f(z) = U_\infty z + m \ln[(z + a)/(z - a)]$, where m and a are constants, and interpret the flow pattern.

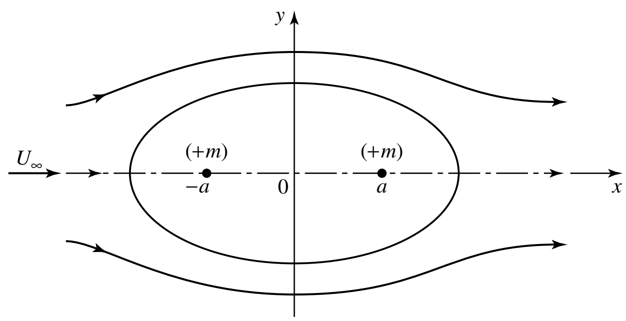


Fig. P8.68

Solution: This represents flow past a **Rankine oval**, with stream function identical to that given by Eq. (8.29).

8.69 Investigate the complex potential function $f(z) = A\cosh(\pi z/a)$, where a is a constant, and plot the streamlines inside the region shown in Fig. P8.69. What hyphenated French word might describe this flow pattern?

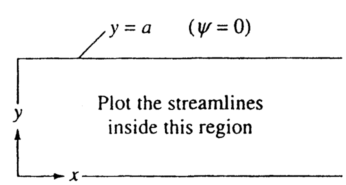
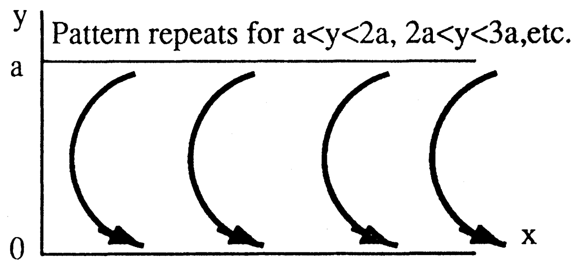


Fig. P8.69

Solution: This potential splits into

$$\psi = A\sinh(\pi x/a)\sin(\pi y/a)$$
$$\phi = A\cosh(\pi x/a)\cos(\pi y/a)$$

and represents flow in a "cul-de-sac" or blind alley.



8.70 Show that the complex potential $f(z) = U_\infty[z + (a/4)\coth(\pi z/a)]$ represents flow past an oval shape placed midway between two parallel walls $y = \pm a/2$. What is a practical application?

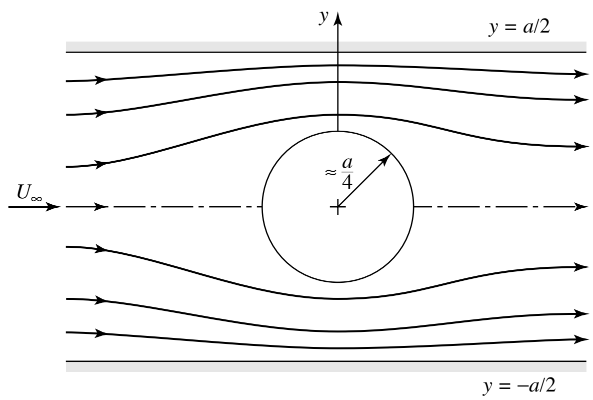


Fig. P8.70

Solution: The stream function of this flow is

$$\psi = U_\infty\left[y - \frac{(a/4)\sin(2\pi y/a)}{\cosh(2\pi x/a) - \cos(2\pi y/a)}\right]$$

The streamlines are shown in the figure. The body shape, trapped between y = ±a/2, is nearly a cylinder, with width a/2 and height 0.51a. A nice application is the estimate of wall "blockage" effects when a body (say, in a wind tunnel) is trapped between walls.

8.71 Figure P8.71 shows the streamlines and potential lines of flow over a thin-plate weir as computed by the complex potential method. Compare qualitatively with Fig. 10.16*a*. State the proper boundary conditions at all boundaries. The velocity potential has equally spaced values. Why do the flow-net "squares" become smaller in the overflow jet?

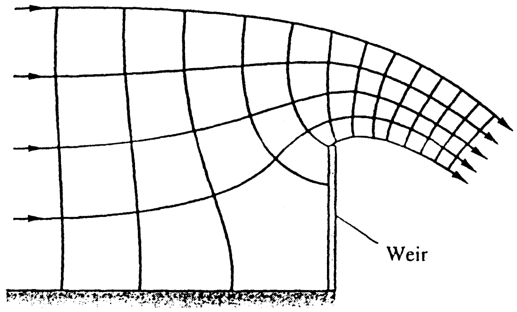
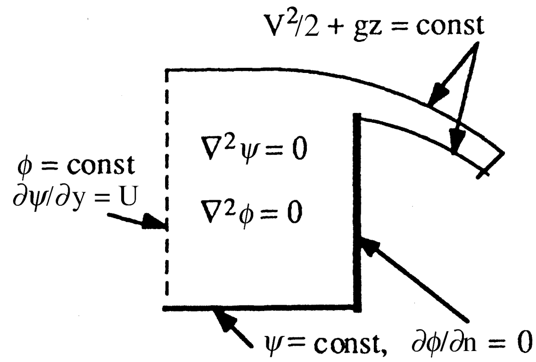


Fig. P8.71

Solution: Solve Laplace's equation for either ψ or ϕ (or both), find the velocities $u = \partial\phi/\partial x$, $v = \partial\phi/\partial y$, force the (constant) pressure to match Bernoulli's equation on the free surfaces (whose shape is *a priori* unknown). The squares become smaller in the overfall jet because the velocity is increasing.



8.72 Use the method of images to construct the flow pattern for a source $+m$ near <u>two</u> walls, as in Fig. P8.72. Sketch the velocity distribution along the lower wall ($y = 0$). Is there any danger of flow separation along this wall?

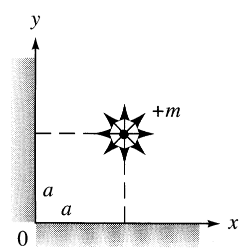
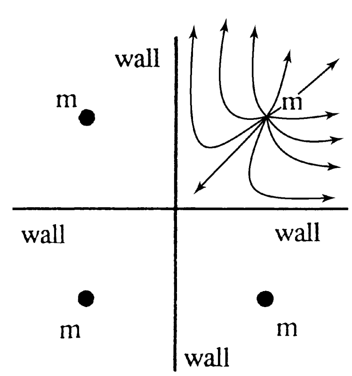


Fig. P8.72

Solution: This pattern is the same as that of Prob. 8.28. It is created by placing **four** identical sources at $(x, y) = (\pm a, \pm a)$, as shown. Along the wall ($x \geq 0$, $y = 0$), the velocity first increases from 0 to a maximum at $x = a$. Then the velocity *decreases* for $x > a$, which is an *adverse* pressure gradient—**separation may occur**. *Ans.*



8.73 Set up an image system to compute the flow of a source at *unequal* distances from *two* walls, as shown in Fig. P8.73. Find the point of maximum velocity on the y-axis.

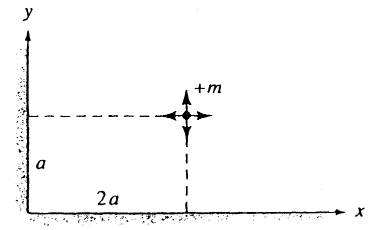
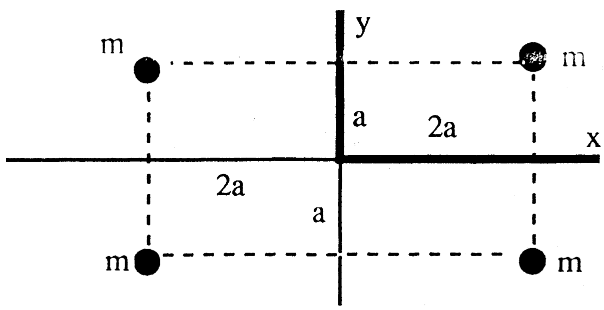


Fig. P8.73

Solution: Similar to Prob. 8.72 on the previous page, we place identical sources (+m) at the symmetric (but non-square) positions (x, y) = (±2a, ±a) as shown below. The induced velocity along the wall (x > 0, y = 0) has the form

$$U = \frac{2m(x+2a)}{(x+2a)^2 + a^2} + \frac{2m(x-2a)}{(x-2a)^2 + a^2}$$



This velocity has a maximum (to the *right*) at **x ≈ 2.93a**, **U ≈ 1.387 m/a**. *Ans.*

8.74 A positive line vortex *K* is trapped in a corner, as in Fig. P8.74. Compute the total induced velocity at point B, (x, y) = (2a, a), and compare with the induced velocity when no walls are present.

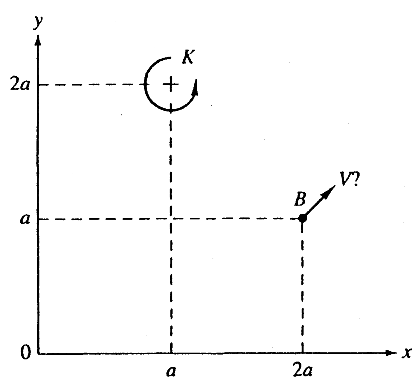
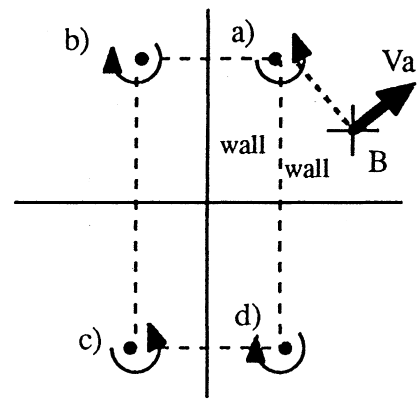


Fig. P8.74

Solution: The two walls are created by placing vortices, as shown at right, at (x, y) = (±a, ±2a). With only one vortex (#a), the induced velocity $\mathbf{V}_a$ would be

$$\mathbf{V}_a = \frac{K}{2a}\mathbf{i} + \frac{K}{2a}\mathbf{j}, \quad \text{or} \quad \frac{K}{a\sqrt{2}} \text{ at } 45^\circ \angle$$



as shown at right. With the walls, however, we have to add this vectorially to the velocities induced by vortices b, c, and d.

$$\text{With walls:} \quad \mathbf{V} = \sum \mathbf{V}_{a,b,c,d} = \frac{K}{a}\left(\frac{1}{2} - \frac{1}{10} - \frac{1}{6} + \frac{3}{10}\right)\mathbf{i} + \frac{K}{a}\left(\frac{1}{2} - \frac{3}{10} + \frac{1}{6} - \frac{1}{10}\right)\mathbf{j},$$

$$\text{or:} \quad \mathbf{V}_B = 0.533\frac{K}{a}\mathbf{i} + 0.267\frac{K}{a}\mathbf{j} = \frac{8K}{15a}\mathbf{i} + \frac{4K}{15a}\mathbf{j} \quad \textit{Ans.}$$

The presence of the walls thus causes a significant change in the magnitude and direction of the induced velocity at point B.

8.75 Using the four-source image pattern needed to construct the flow near a corner shown in Fig. P8.72, find the value of the source strength *m* which will induce a wall velocity of 4.0 m/s at the point $(x, y) = (a, 0)$ just below the source shown, if $a = 50$ cm.

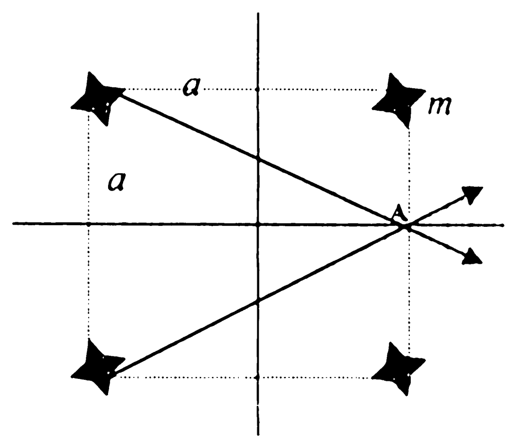


Fig. P8.75

Solution: The flow pattern is formed by four equal sources *m* in the 4 quadrants, as in the figure at right. The sources above and below the point A(a, 0) cancel each other at A, so the velocity at A is caused only by the two left sources. The velocity at A is the sum of the two horizontal components from these 2 sources:

$$V_A = 2\frac{m}{\sqrt{a^2+(2a)^2}}\frac{2a}{\sqrt{a^2+(2a)^2}} = \frac{4ma}{5a^2} = \frac{4m}{5(0.5m)} = 4\ \frac{\text{m}}{\text{s}} \quad \text{if } \boldsymbol{m = 2.5\ \frac{m^2}{s}} \qquad \textit{Ans.}$$

8.76 Use the method of images to approximate the flow past a cylinder at distance 4*a* from the wall, as in Fig. P8.76. To illustrate the effect of the wall, compute the velocities at points A, B, C, and D, comparing with a cylinder flow in an infinite expanse of fluid (without walls).

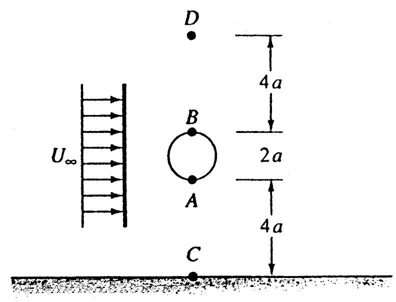


Fig. P8.76

Solution: Let doublet #1 be above the wall, as shown, and let image doublet #2 be below the wall, at (x, y) = (0, –5*a*). Then, at any point on the y-axis, the total velocity is

$$V_{x=0} = -v_\theta\big|_{90^\circ} = U_\infty[1+(a/r_1)^2+(a/r_2)^2]$$

Since the images are 10*a* apart, the cylinders are only slightly out-of-round and the velocities at A, B, C, D may be tabulated as follows:

Point:	**A**	**B**	**C**	**D**
r_1:	a	a	$5a$	$5a$
r_2:	$9a$	$11a$	$5a$	$15a$
V_{walls}:	$2.012U_\infty$	$2.008U_\infty$	$1.080U_\infty$	$1.044U_\infty$
$V_{no\ walls}$:	$2.0U_\infty$	$2.0U_\infty$	$1.04U_\infty$	$1.04U_\infty$

The presence of the walls causes only a slight change in the velocity pattern.

8.77 Discuss how the flow pattern of Prob. 8.58 might be interpreted to be an *image*-system construction for circular walls. Why are there two images instead of one?

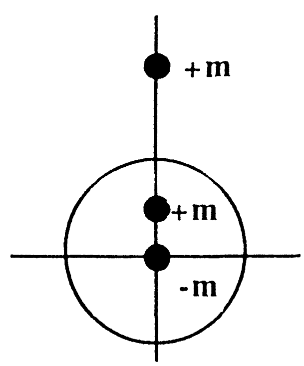


Fig. P8.77

Solution: The missing "image sink" in this problem is at y = +∞ so is not shown. If the source is placed at y = *a* and the image source at y = *b*, the radius of the cylinder will be $R = \sqrt{(\mathbf{ab})}$. For further details about this type of imaging, see Chap. 8, Ref. 3, p. 230.

8.78 Indicate the system of images needed to construct the flow of a uniform stream past a Rankine half-body centered between parallel walls, as in Fig. P8.78. For the particular dimensions shown, estimate the position ℓ of the nose of the resulting half-body.

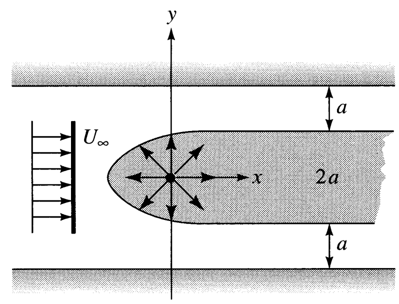
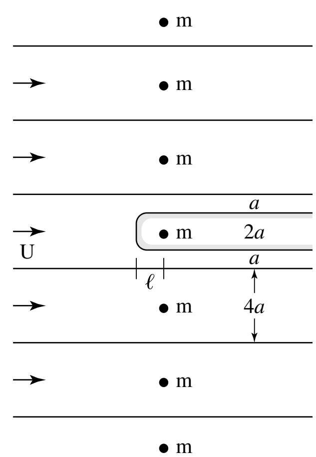


Fig. P8.78

Solution: A body between *two* walls is created by an infinite array of sources, as shown at right. The source strength *m* fits the half-body size "2a":

$$Q = U(2a) = 2\pi m, \quad \text{or:} \quad \mathbf{m = Ua/\pi}$$



The distance ℓ to the nose denotes the stagnation point, where

$$U = \frac{m}{L}\left[1 + \frac{2}{1+(4a/L)^2} + \frac{2}{1+(8a/L)^2} + \cdots\right]$$

where m = Ua/π, as shown. We solve this series summation for $\ell \approx$ **0.325a**. *Ans.*

8.79 Indicate the system of images needed to simulate the flow of a line source placed unsymmetrically between two parallel walls, as in Fig. P8.79. Compute the velocity on the lower wall at x = +*a*. How many images are needed to establish this velocity to within ±1%?

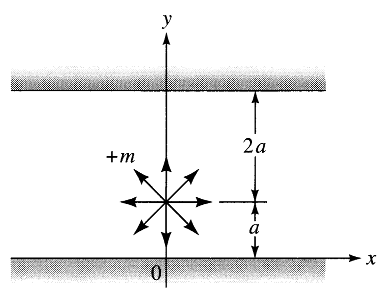
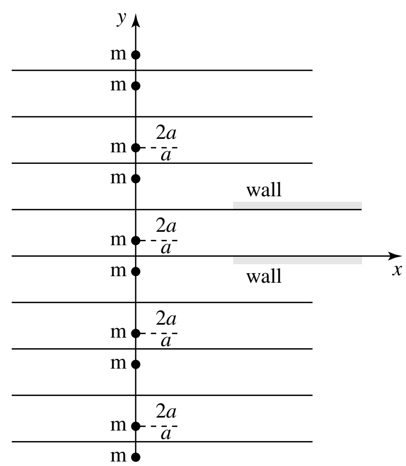


Fig. P8.79

Solution: To form a wall at y = 0 and also at y = 3a, with the source located at (0, a), one needs an infinite number of **pairs** of sources, as shown. The velocity at (a, 0) is an



infinite sum:

$$u(a,\,0)=\frac{2m}{a}\left[\frac{1}{1^2+1}+\frac{1}{5^2+1}+\frac{1}{7^2+1}+\frac{1}{11^2+1}+\frac{1}{13^2+1}+\cdots\right]$$

Accuracy within 1% is reached after <u>18 terms</u>: **u(a, 0) ≈ 1.18 m/a**. *Ans.*

8.80 The beautiful expression for lift of a two-dimensional airfoil, Eq. (8.69), arose from applying the *Joukowski transformation*, $\zeta = z + a^2/z$, where $z = x + iy$ and $\zeta = \eta + i\beta$. The constant a is a length scale. The theory transforms a certain circle in the z plane into an airfoil in the ζ plane. Taking $a = 1$ unit for convenience, show that (a) a circle with center at the origin and radius >1 will become an *ellipse* in the ζ plane, and (b) a circle with center at $x = -\varepsilon \ll 1, y = 0,$ and radius $(1+\varepsilon)$ will become an *airfoil* shape in the ζ plane. *Hint*: Excel is excellent for solving this problem.

Solution: Introduce z = x + iy into the transformation and find real and imaginary parts:

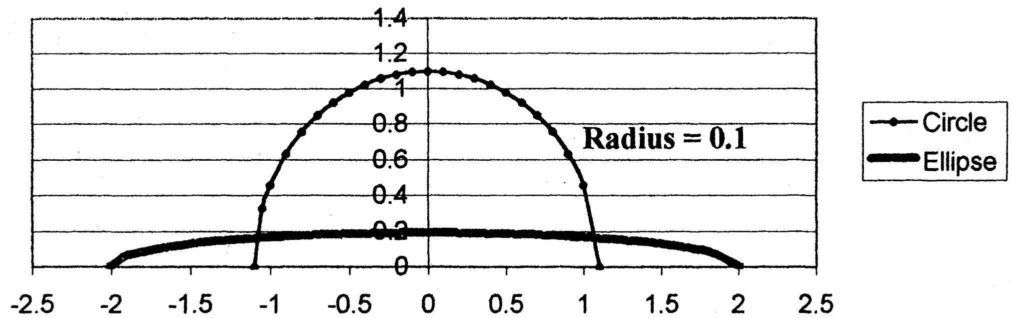
$$\zeta = (x+iy)+\frac{1}{x+iy}\left(\frac{x-iy}{x-iy}\right)=x\left(1+\frac{1}{x^2+y^2}\right)+iy\left(1-\frac{1}{x^2+y^2}\right)=\eta+i\beta$$

Thus η and β are simple functions of x and y, as shown. Thus, if the circle in the z plane has radius C > 1, the coordinates in the ζ plane will be

$$\eta = x\left(1+\frac{1}{C^2}\right) \qquad \beta = y\left(1-\frac{1}{C^2}\right)$$

The circle in the z plane will transform into an *ellipse* in the ζ plane of major axis $(1 + 1/C^2)$ and minor axis $(1 - 1/C^2)$. This is shown on the next page for C = 1.1. If the

circle center is at $(-\varepsilon,0)$ and radius $C = 1+\varepsilon$, an *airfoil* will form, because a sharp (trailing) edge will form on the right and a fat (elliptical) leading edge will form on the left. This is also shown below for $\varepsilon = 0.1$.



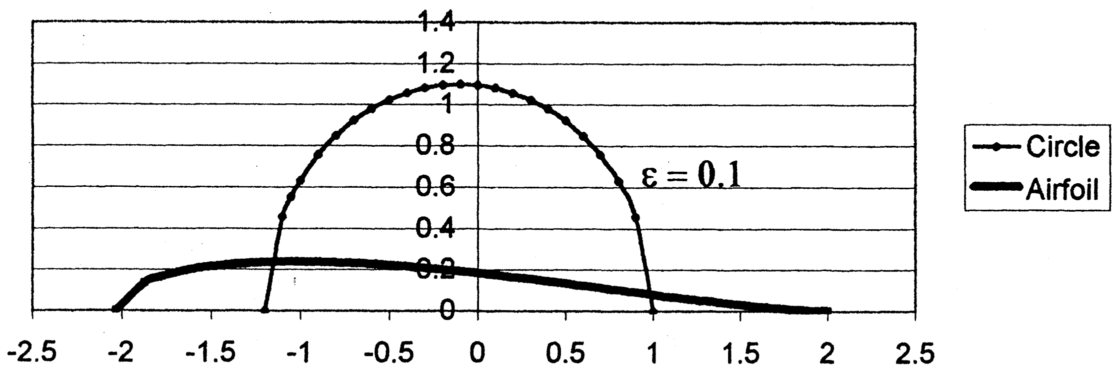


Fig. P8.80

8.81 A very wide NACA 4412 airfoil, with a chord of 75 cm, is tested in a sea-level wind tunnel at 45 m/s and found to have a lift of 65 lbf per foot of span. Estimate the angle of attack for this condition.

Solution: For sea-level air take ρ =1.22 kg/m^3. From Fig. 8.20 and Table 8.3, for the 4412 airfoil, $\alpha_{ZL} \approx -4°$ and $dC_L/d\alpha \approx 6.0$. The lift coefficient is computed from the measured lift and velocity. Convert 65 lbf/ft to 949 N/m and evaluate:

$$C_L = \frac{2(Lift)}{\rho V^2 bC} = \frac{2(949\text{ N/m})}{(1.22\text{ kg/m}^3)(45\text{ m/s})^2(1\text{ m})(0.75\text{ m})} = 1.024 \approx 6.0\,\mathbf{sin}(\alpha + 4.0°)$$

$$\text{Solve for}\quad \alpha = 9.8° \quad\text{or better,}\quad \alpha \approx \mathbf{10° \pm 0.5°} \quad \textit{Ans.}$$

8.82 The ultralight plane *Gossamer Condor* in 1977 was the first to complete the Kremer Prize figure-eight course solely under human power. Its wingspan was 29 m, with C_{av} = 2.3 m and a total mass of 95 kg. Its drag coefficient was approximately 0.05. The pilot was able to deliver 1/4 horsepower to propel the plane. Assuming two-dimensional flow at sea level, estimate (a) the cruise speed attained, (b) the lift coefficient; and (c) the horsepower required to achieve a speed of 15 knots.

Solution: For sea-level air, take $\rho = 1.225$ kg/m^3. With C_D known, we may compute V:

$$Power = F_{drag}V = \frac{1}{4}\text{hp}(745.7) = 186\text{ W} = \left[C_D\frac{\rho}{2}V^2bC\right]V$$

$$= 0.05\left(\frac{1.225}{2}\right)V^2(29)(2.3)V = 186;$$

$$\text{Solve}\quad V^3 = 91.3 \quad or \quad \mathbf{V = 4.5\ \frac{m}{s}} \quad Ans.\text{ (a)}$$

Then, with V known, we may compute the lift coefficient from the known weight:

$$C_L = \frac{weight}{(\rho/2)V^2bC} = \frac{95(9.81)\text{ N}}{(1.225/2)(4.5)^2(29)(2.3)} = \mathbf{1.13} \quad Ans.\text{ (b)}$$

Finally, compute the power if V = 15 knots = 7.72 m/s:

$$P = FV = \left(C_D\frac{\rho}{2}V^2bC\right)V = 0.05\left(\frac{1.225}{2}\right)(7.72)^2(29)(2.3)(7.72)$$

$$= 940\text{ W} = \mathbf{1.26\ hp} \quad Ans.\text{ (c)}$$

8.83 Two-dimensional lift-drag data for the NACA 2412 airfoil with 2 percent camber (from Ref. 12) may be curve-fitted accurately as follows:

$$C_L \approx 0.178 + 0.109\alpha - 0.00109\alpha^2$$

$$C_D \approx 0.0089 + 1.97\text{E}{-4}\alpha + 8.45\text{E}{-5}\alpha^2 - 1.35\text{E}{-5}\alpha^3 + 9.92\text{E}{-7}\alpha^4$$

with α in degrees in the range $-4° < \alpha < +10°$. Compare (a) the lift-curve slope and (b) the angle of zero lift with theory, Eq. (8.69). (c) Prepare a polar lift-drag plot and compare with Fig. 7.25.

Solution: The lift formula is not quite linear in α, but a reasonable lift-curve slope and the zero-lift angle of attack may be computed as follows:

$$\left.\frac{dC_L}{d\alpha}\right|_{\alpha=0} = 0.109 \text{ per degree} \times \frac{180}{\pi} \approx \mathbf{6.25} \text{ per radian} \quad \textit{Ans.} \text{ (a)}$$

$$\text{Angle of zero lift:} \quad 0.178 + 0.109\alpha - 0.00109\alpha^2 = 0 \quad \text{at } \alpha \approx \mathbf{-1.61°} \quad \textit{Ans.} \text{ (b)}$$

$$\text{From Eq. (8.70) theory,} \quad \alpha_{\text{zero lift}} = -\tan^{-1}[2(0.02)] \approx -2.3° \quad \text{(fair accuracy)}$$

The polar lift-drag curve is simply prepared by plotting lift versus drag for all α in the given range $-4° < \alpha < 10°$. The curve is shown below, with negative lift omitted. It resembles the NACA 0009 foil in Fig. 7.25, except it has slightly less drag.

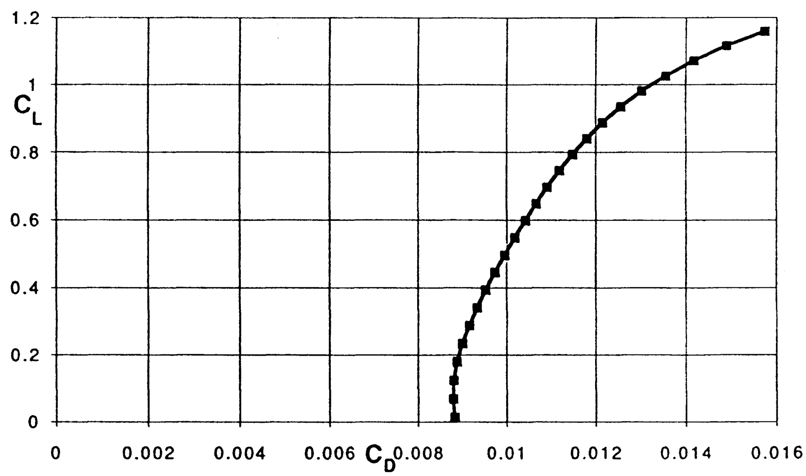


Fig. P8.83

8.84 Reference 12 contains inviscid theory calculations for the surface velocity distributions V(x) over an airfoil, where x is the chordwise coordinate. A typical result for small angle of attack is shown below. Use these data, plus Bernoulli's equation, to estimate (a) the lift coefficient; and (b) the angle of attack if the airfoil is symmetric.

x/c	***V/U***$_\infty$ **(upper)**	***V/U***$_\infty$ **(lower)**
0.0	0.0	0.0
0.025	0.97	0.82
0.05	1.23	0.98
0.1	1.28	1.05
0.2	1.29	1.13
0.3	1.29	1.16
0.4	1.24	1.16
0.6	1.14	1.08
0.8	0.99	0.95
1.0	0.82	0.82

Solution: From Bernoulli's equation, the surface pressures may be computed, whence the lift coefficient then follows from an integral of the pressure difference:

$$\text{Surface:}\quad p = p_\infty + \frac{\rho}{2}\left(U_\infty^2 - V^2\right),\quad L = \int_0^C (p_{lower} - p_{upper})\,b\,dx = \int_0^C \frac{\rho}{2}\left(V_{upper}^2 - V_{lower}^2\right)b\,dx$$

$$\text{Non-dimensionalize:}\quad C_L = \frac{L}{(\rho/2)U_\infty^2 bC} = \int_0^1 \left[(V/U)_{upp}^2 - (V/U)_{low}^2\right] d\left(\frac{x}{C}\right)$$

Thus the lift coefficient is an integral of the difference in $(V/U)^2$ on the airfoil. Such a plot is shown below. The area between the curves is approximately

$$\int_0^1 \Delta(V/U)^2 \approx C_L \approx \mathbf{0.21} \quad \textit{Ans.}\text{ (a)};\quad \alpha = \sin^{-1}\left(\frac{0.21}{2\pi}\right) \approx \mathbf{1.9°}\quad \textit{Ans.}\text{ (b)}$$

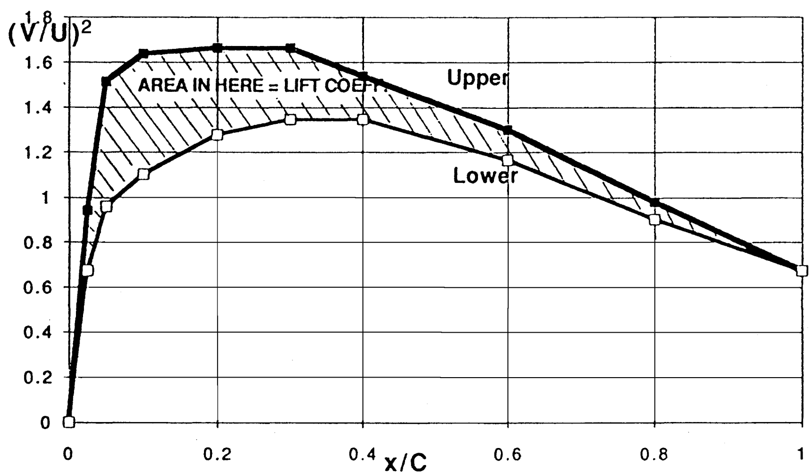


Fig. P8.84

8.85 A wing of 2 percent camber, 5-in chord, and 30-in span is tested at a certain angle of attack in a wind tunnel with sea-level standard air at 200 ft/s and is found to have lift of 30 lbf and drag of 1.5 lbf. Estimate from wing theory (a) the angle of attack, (b) the minimum drag of the wing and the angle of attack at which it occurs, and (c) the maximum lift-to-drag ratio.

Solution: For sea-level air take $\rho = 0.00238$ slug/ft^3. Establish the lift coefficient first:

$$C_L = \frac{2L}{\rho V^2 bC} = \frac{2(30\text{ lbf})}{0.00238(200)^2(30/12)(5/12)} = 0.605 = \frac{2\pi\sin[\alpha + \tan^{-1}(0.04)]}{1 + 2/(6.0)}$$

$$\text{Solve for}\quad \alpha \approx \mathbf{5.1°}\quad \textit{Ans.}\text{ (a)}$$

Find the induced drag and thence the minimum drag (when lift = zero):

$$C_D = \frac{C_L}{20} = 0.0303 = C_{D\infty} + \frac{C_L^2}{\pi AR} = C_{D\infty} + \frac{(0.605)^2}{\pi(6.0)}, \quad \text{solve for } C_{D\infty} = 0.0108$$

Then $\quad D_{min} = C_{D\infty}\frac{\rho}{2}V^2bC = 0.0108\left(\frac{0.00238}{2}\right)(200)^2\left(\frac{30}{12}\right)\left(\frac{5}{12}\right) \approx \mathbf{0.54\ lbf}$ *Ans.* (b)

Finally, the maximum L/D ratio occurs when $C_D = 2C_{D\infty}$, or:

$$C_L\left(\text{at max } \frac{L}{D}\right) = \sqrt{\pi ARC_{D\infty}} = \sqrt{\pi(6)(0.0108)} = 0.451,$$

$$\text{whence} \quad (L/D)_{max} = \frac{0.451}{2(0.0108)} \approx \mathbf{21} \quad \textit{Ans. (c)}$$

8.86 An airplane has a mass of 20,000 kg and flies at 175 m/s at 5000-m standard altitude. Its rectangular wing has a 3-m chord and a symmetric airfoil at 2.5° angle of attack. Estimate (a) the wing span; (b) the aspect ratio; and (c) the induced drag.

Solution: For air at 5000-m altitude, take $\rho = 0.736\ kg/m^3$. We know W, find b:

$$L = W = 20000(9.81) = C_L\frac{\rho}{2}V^2bC = \frac{2\pi\sin 2.5^\circ}{1+2(3/b)}\left(\frac{0.736}{2}\right)(175)^2 b(3.0)$$

$$\text{Rearrange to} \quad b^2 - 21.2b - 127 = 0, \quad \text{or} \quad b \approx \mathbf{26.1\ m} \quad \textit{Ans. (a)}$$

$$\text{Aspect ratio} \quad AR = b/C = 26.1/3.0 \approx \mathbf{8.7} \quad \textit{Ans. (b)}$$

With aspect ratio known, we can solve for lift and induced-drag coefficients:

$$C_L = \frac{2\pi\sin 2.5^\circ}{1+2/8.7} = 0.223, \quad C_{Di} = \frac{C_L^2}{\pi AR} = \frac{(0.223)^2}{\pi(8.7)} = 0.00182$$

$$\text{Induced drag} = (0.00182)(0.736/2)(175)^2(26.1)(3) \approx \mathbf{1600\ N} \quad \textit{Ans. (c)}$$

8.87 A freshwater boat of mass 400 kg is supported by a rectangular hydrofoil of aspect ratio 8, 2% camber, and 12% thickness. If the boat travels at 7 m/s and α = 2.5°, estimate (a) the chord length; (b) the power required if $C_{D\infty}$ = 0.01, and (c) the top speed if the boat is refitted with an engine which delivers 20 hp to the water.

Solution: For fresh water take $\rho = 998$ kg/m^3. (a) Use Eq. (8.82) to estimate the chord length:

$$Lift = C_L\frac{\rho}{2}V^2bC = \frac{2\pi\sin\left[\dfrac{2.5}{57.3}+2(0.02)\right]}{1+2/8}\left(\frac{998\text{ kg/m}^3}{2}\right)(7\text{ m/s})^2(8C)C,$$

$$= (400\text{ kg})(9.81\text{ m/s}^2)$$

$$\text{Solve for}\quad C^2 = 0.0478\text{ m}^2 \quad or \quad \mathbf{C \approx 0.219\ m} \quad \textit{Ans. (a)}$$

(b) At 7 m/s, the power required depends upon the drag, with $C_L = 0.420$ from part (a):

$$C_D = C_{D\infty} + \frac{C_L^2}{\pi(AR)} = 0.01 + \frac{(0.420)^2}{\pi(8)} = 0.0170$$

$$F = C_D\frac{\rho}{2}V^2bC = (0.0170)\left(\frac{998\text{ kg/m}^3}{2}\right)(7\text{ m/s})^2(8\times 0.219\text{ m})(0.219\text{ m}) = 159\text{ N}$$

$$Power = FV = (159\text{ N})(7\text{ m/s}) = 1120\text{ W} = \mathbf{1.5\ hp} \quad \textit{Ans. (b)}$$

(c) Set up the power equation again with velocity unknown:

$$Power = FV = \left[\left(C_{D\infty} + \frac{C_L^2}{\pi(AR)}\right)\left(\frac{\rho}{2}\right)V^2bC\right]V, \quad C_L = \frac{2(Weight)}{\rho V^2bC} = \frac{2\pi\sin(\alpha-\alpha_{ZL})}{1+2/(AR)}$$

Enter the data: $C_{D\infty} = 0.01$, $C = 0.219$ m, $b = 1.75$ m, $AR = 8$, Weight = 3924 N, Power = 20 hp = 14914 W, $\alpha_{ZL} = -2.29°$, $\rho = 998$ kg/m^3. Iterate (or use EES) to find the results:

$$C_L = 0.0526,\ C_D = 0.0101,\ \alpha = -1.69°,\ \mathbf{V_{max} = 19.8\ m/s \approx 44\ mi/h} \quad \textit{Ans. (c)}$$

8.88 The Boeing 727 airplane has a gross weight of 125000 lbf, a wing area of 1200 ft^2, and an aspect ratio of 6. It has two turbofan engines and cruises at 532 mi/h at 30000 feet standard altitude. Assume that the airfoil is the NACA 2412 section from Prob. 8.83. If we neglect all drag except the wing, what thrust is required from each engine for these conditions?

Solution: At 30000 ft = 9144 m, $\rho \approx 0.000888$ slug/ft^3. Convert 532 mi/h = 780 ft/s. We have sufficient information to calculate the lift coefficient:

$$C_L = \frac{2W}{\rho V^2 A} = \frac{2(125000)}{8.88E\text{–}4(780)^2(1200)} = 0.385 = \frac{C_{L\infty}}{1+2/6}, \quad \text{hence}\ \ C_{L\infty} = 0.514$$

$$\text{From Prob. 8.83, } \alpha \approx 3.18°, \quad \therefore\ C_{D\infty} = 0.0100, \text{ or } C_D = 0.010 + \frac{(0.385)^2}{\pi(6)} \approx \mathbf{0.0179}$$

With drag coefficient estimated, the drag force (or thrust) follows easily:

$$F = C_D \frac{\rho}{2} V^2 A = 0.0179\left(\frac{8.88\text{E–}4}{2}\right)(780)^2(1200) = 5800 \text{ lbf}$$

Hence required engine thrust = F/2 ≈ **2900 lbf** each *Ans.*

8.89 The Beechcraft T-34C airplane has a gross weight of 5500 lbf, a wing area of 60 ft^2, and cruises at 322 mi/h at 10000 feet standard altitude. It is driven by a propeller which delivers 300 hp to the air. Assume that the airfoil is the NACA 2412 section from Prob. 8.83 and neglect all drag except the wing. What is the appropriate aspect ratio for this wing?

Solution: At 10000 ft = 3048 m, $\rho \approx 0.00176$ slug/ft^3. Convert 322 mi/h = 472 ft/s. From the weight and power we can compute the lift and drag coefficients:

$$C_L = \frac{2W}{\rho V^2 A} = \frac{2(5500)}{0.00176(472)^2(60)} = 0.468 = \frac{C_{L\infty}}{1 + 2/AR}$$

$$\text{Power} = 300 \text{ hp} \times 550 = 165000 \ \frac{\text{ft·lbf}}{\text{s}}$$

$$= C_D\left(\frac{0.00176}{2}\right)(472)^2(60)(472), \quad \text{or} \quad C_D \approx 0.0297$$

With C_L & C_D known and their values at AR = ∞ from Prob. 8.83, we can compute AR:

$$0.468 = \frac{C_{L\infty}}{1 + 2/AR}; \quad C_D = 0.0297 = C_{D\infty} + \frac{(0.468)^2}{\pi AR}, \quad \text{with “}\infty\text{” from Prob. 8.83}$$

By iteration, the solution converges to $\alpha \approx 5.25°$ and **AR ≈ 3.73**. *Ans.*

8.90 NASA is developing a swing-wing airplane called the Bird of Prey [37]. As shown in Fig. P8.90, the wings pivot like a pocketknife blade: forward (a), straight (b), or backward (c). Discuss a possible advantage for each of these wing positions. If you can’t think of any, read the article [37] and report to the class.

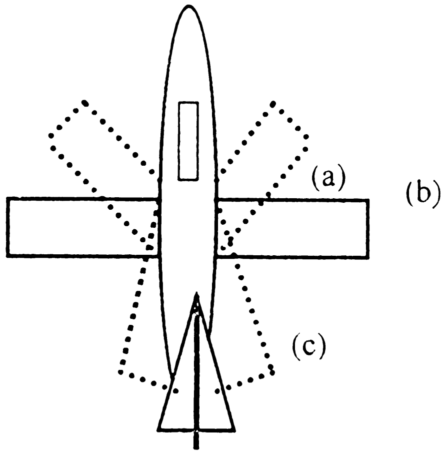


Fig. P8.90

Solution: Each configuration has a different advantage: (a) highly maneuverable but unstable, needs computer control; (b) maximum lift at low speeds, best for landing and take-off; (c) maximum speed possible with wings swept back.

8.91 If $\phi(r, \theta)$ in axisymmetric flow is defined by Eq. (8.85) and the coordinates are given in Fig. 8.24, determine what partial differential equation is satisfied by ϕ.

Solution: The velocities are related to ϕ by Eq. (8.87), and direct substitution gives

$$\frac{\partial}{\partial r}(r^2 v_r \sin\theta) + \frac{\partial}{\partial \theta}(r v_\theta \sin\theta) = 0, \quad \text{with } v_r = \frac{\partial \phi}{\partial r} \quad \text{and} \quad v_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta}$$

Thus the PDE for ϕ is: $$\mathbf{\sin\theta \frac{\partial}{\partial r}\left(r^2 \frac{\partial \phi}{\partial r}\right) + \frac{\partial}{\partial \theta}\left(\frac{\partial \phi}{\partial \theta}\sin\theta\right) = 0} \quad \textit{Ans.}$$

This linear but complicated PDE is *not* Laplace's Equation in axisymmetric coordinates.

8.92 A point source with volume flow Q = 30 m^3/s is immersed in a uniform stream of speed 4 m/s. A Rankine half-body of revolution results. Compute (a) the distance from the source to the stagnation point; and (b) the two points (r, θ) on the body surface where the local velocity equals 4.5 m/s. [See Fig. 8.26]

Solution: The properties of the Rankine half-body follow from Eqs. (8.89) and (8.94):

$$m = \frac{Q}{4\pi} = \frac{30}{4\pi} = \mathbf{2.39}\ \frac{m^3}{s}, \quad \text{hence } \mathbf{a} = \sqrt{\frac{m}{U}} = \sqrt{\frac{2.39}{4}} \approx \mathbf{0.77\ m} \quad \textit{Ans.}\text{ (a)}$$

[It's a *big* half-body, to be sure.] Some iterative computation is needed to find the body shape and the velocities along the body surface:

$$\text{Surface, } \psi = -\mathrm{Ua}^2: \quad \mathrm{r} = \mathrm{a}\frac{\sqrt{2(1+\cos\theta)}}{\sin\theta} = \mathrm{a}\csc\left(\frac{\theta}{2}\right), \quad \text{with velocity components}$$

$$\mathrm{v_r} = \mathrm{U}\cos\theta + \mathrm{m/r^2} \quad \text{and} \quad \mathrm{v_\theta} = -\mathrm{U}\sin\theta$$

A brief tabulation of surface velocities reveals *two* solutions, both for $\theta < 90°$:

θ:	40°	50°	**50.6°**	60°	70°	80°	**88.1°**	90°	100°
V, m/s:	4.37	4.49	**4.50**	4.58	4.62	4.59	**4.50**	4.47	4.27

As in Fig. 8.26, V rises to a peak of 4.62 m/s (1.155U) at 70.5°, passing through 4.5 m/s at (r, θ) ≈ (**1.808 m, 50.6°**) and (**1.111 m, 88.1°**). *Ans.*

8.93 The Rankine body of revolution of Fig. 8.26 could simulate the shape of a pitot-static tube (Fig. 6.30). According to inviscid theory, how far downstream from the nose should the static-pressure holes be placed so that the local surface velocity is within ±0.5% of U? Compare your answer with the recommendation x ≈ 8D in Fig. 6.30.

Solution: We search iteratively along the surface until we find **V = 1.005U**:

$$\text{Along} \quad \mathrm{r/a} = \csc(\theta/2), \quad \mathrm{v_r} = \mathrm{U}\cos\theta + \frac{\mathrm{m}}{\mathrm{r}^2}, \quad \mathrm{v_\theta} = -\mathrm{U}\sin\theta, \quad \mathrm{V} = \sqrt{\mathrm{v_r^2 + v_\theta^2}}, \quad \mathrm{m} = \mathrm{Ua}^2$$

The solution is found at $\theta \approx$ **8.15°**, x ≈ 13.93*a*, **Δx/D** = 14.93*a*/4*a* ≈ **3.73**. *Ans.*
[Further downstream, at Δx = 8D, we find that V ≈ 1.001U, or within 0.1%.]

8.94 Determine whether the Stokes streamlines from Eq. (8.86) are everywhere orthogonal to the Stokes potential lines from Eq. (8.87), as is the case for cartesian and plan polar coordinates.

Solution: Compare the ratio of velocity components for lines of constant ψ:

$$\left.\frac{\mathrm{dr}}{\mathrm{r\,d}\theta}\right|_{\text{streamline}} = \frac{\mathrm{v_r}}{\mathrm{v_\theta}} = \frac{-(1/\mathrm{r}^2\sin\theta)(\partial\psi/\partial\theta)}{(1/\mathrm{r}\sin\theta)(\partial\psi/\partial\mathrm{r})} = \frac{\partial\phi/\partial\mathrm{r}}{(1/\mathrm{r})(\partial\phi/\partial\theta)},$$

$$\text{or:} \quad \frac{-(1/\mathrm{r})(\partial\psi/\partial\theta)}{(\partial\psi/\partial\mathrm{r})} = -1\Big/\left[\frac{-(1/\mathrm{r})(\partial\phi/\partial\theta)}{\partial\phi/\partial\mathrm{r}}\right], \quad \text{thus } [\text{slope}]_{\psi\text{ line}} = \frac{-1}{[\text{slope}]_{\phi\text{ line}}}$$

They **are** orthogonal. *Ans.*

8.95 Show that the axisymmetric potential flow formed by a point source $+m$ at (–a, 0), a point sink-($-m$) at (+a, 0), and a stream U in the x direction becomes a Rankine body of revolution as in Fig. P8.95. Find analytic expressions for the length 2L and diameter 2R of the body.

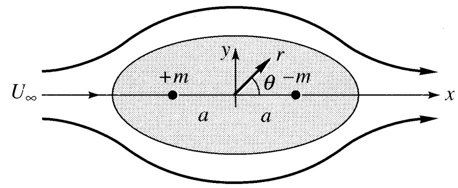


Fig. P8.95

Solution: The stream function for this three-part superposition is given below, and the body shape (the *Rankine ovoid*) is given by $\psi = 0$.

$$\psi = -\frac{U}{2}r^2\sin^2\theta + m(\cos\theta_2 - \cos\theta_1),$$ where "2" and "1" are from the source/sink.

Stagnation points at $r = L$, $\theta = 0, \pi$, or $U + \dfrac{m}{(L+a)^2} - \dfrac{m}{(L-a)^2} = 0$

Solve for $\mathbf{[(L/a)^2 - 1]^2 = \dfrac{4m}{Ua^2}(L/a)}$ *Ans.*

Similarly, the maximum radius R of the ovoid occurs at $\psi = 0$, $\theta = \pm 90°$:

$$0 = -\frac{U}{2}R^2 + 2m\cos\theta_2, \quad \text{where } \cos\theta_2 = \frac{a}{\sqrt{a^2+R^2}},$$

or: $\mathbf{(R/a)^2\sqrt{1+(R/a)^2} = \dfrac{4m}{Ua^2}}$ *Ans.*

Some numerical values of length and diameter are as follows:

m/Ua^2:	0.01	0.1	1.0	10.0	100.0
L/a:	1.100	1.313	1.947	3.607	7.458
R/a:	0.198	0.587	1.492	3.372	7.458
L/R:	5.553	2.236	1.305	1.070	1.015

As m/Ua^2 increases, the ovoid approaches a large spherical shape, $L/R \approx 1.0$.

8.96 Suppose that a sphere with a single stagnation hole is to be used as a velocimeter. The pressure at this hole is used to compute the stream velocity U, but there are errors if the stream is not perfectly aligned with the oncoming stream. Using inviscid incompressible theory, plot the % error in stream velocity as a function of misalignment angle ϕ. At what angle is the error 10%?

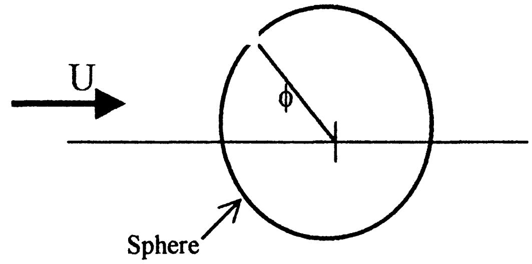


Fig. P8.96

Solution: It is assumed that the pressure gage reads the difference between the pressure p_s at the hole and the ambient pressure p_a. When perfectly aligned, we have actual stagnation pressure p_o and may compute

$$p_s = p_o = p_a + \frac{\rho}{2}U^2, \quad or: \quad U = \sqrt{\frac{2(p_s - p_a)}{\rho}}$$

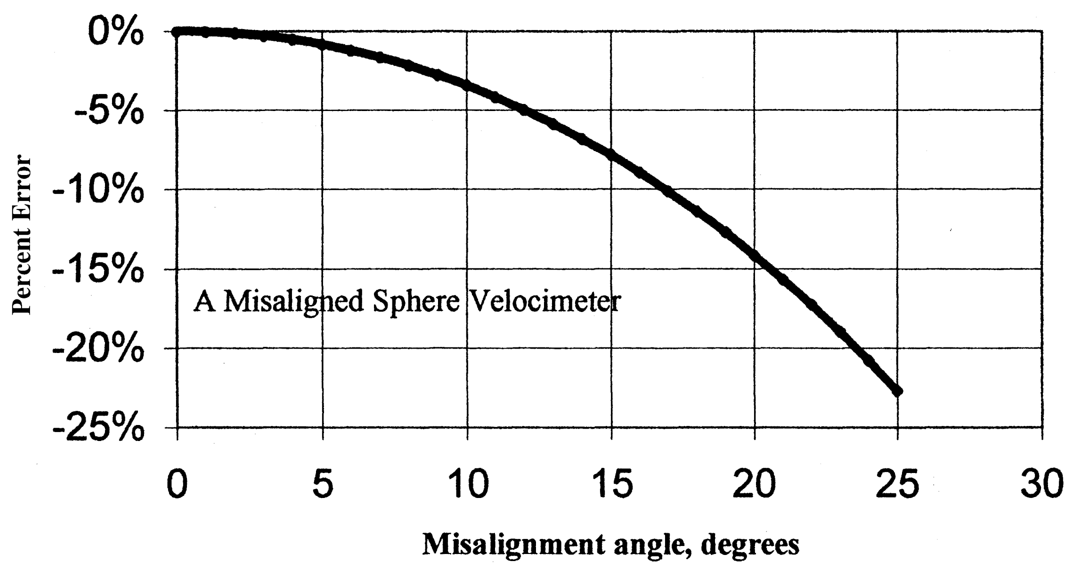
If we are misaligned by an angle ϕ, there is a non-zero velocity V at the hole, given by Eq. (8.100): $V = (3/2)U\sin\phi$. Bernoulli's equation gives

$$p_a + \frac{\rho}{2}U^2 = p_s + \frac{\rho}{2}V^2, \quad or: \quad U^2 - V^2 = \frac{2(p_s - p_a)}{\rho} = U^2\left(1 - \frac{9}{4}\sin^2\phi\right)$$

Thus the instrument reads **low** by the amount $\sqrt{\mathbf{1 - (9/4)\sin^2\phi}}$ *Ans.*

The error is 10%, that is, $U_{measured} = 0.9U_{actual}$, when ϕ **= 16.9°**. *Ans.*

A plot of the percent error in velocity is given below as a function of ϕ.



8.97 The Rankine body or revolution in Fig. P8.97 is 60 cm long and 30 cm in diameter. When it is immersed in the low-pressure water tunnel as shown, cavitation may appear at point *A*. Compute the stream velocity *U*, neglecting surface wave formation, for which cavitation occurs.

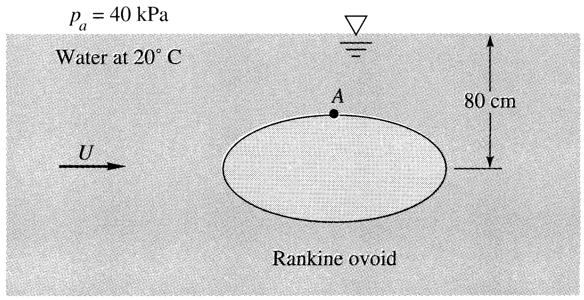


Fig. P8.97

Solution: For water at 20°C, take ρ = 998 kg/m^3 and p_v = 2337 Pa. For an ovoid of ratio L/R = 60/30 = **2.0**, we may interpolate in the Table of Prob. 8.95 to find

$$\frac{m}{Ua^2} = 0.1430, \quad \frac{L}{a} = 1.3735, \quad \text{hence } a = \frac{30}{1.3735} = 21.84 \text{ cm}, \quad R = 0.687a = 15.0 \text{ cm},$$

$$m = 0.143Ua^2 = 0.00682U, \quad U_{max} = U + \frac{2ma}{r_2^3} = U + \frac{2(0.00682U)(0.2184)}{[(.2184)^2 + (.15)^2]^{3/2}} \approx \mathbf{1.16U}$$

Then Bernoulli's equation allows us to compute U when point A reaches vapor pressure:

$$p_\infty + \frac{\rho}{2}U^2 + \rho g z_\infty \approx p_A + \frac{\rho}{2}V_A^2 + \rho g z_A$$

$$\text{where} \quad V_A = 1.16U \quad \text{and} \quad p_\infty = p_{atm} + \rho g(z_{surf} - z_\infty)$$

$$40000 + 9790(0.8) + \frac{998}{2}U^2 + 0 = 2337 + \frac{998}{2}(1.16U)^2 + 9790(0.15)$$

$$\text{Solve for cavitation speed} \quad \mathbf{U \approx 16 \ \frac{m}{s}} \quad \textit{Ans.}$$

8.98 We have studied the point source (sink) and the line source (sink) of infinite depth into the paper. Does it make any sense to define a finite-length line sink (source) as in Fig. P8.98? If so, how would you establish the mathematical properties

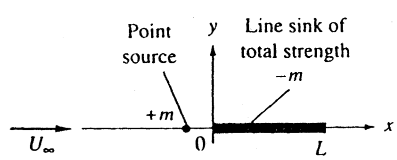
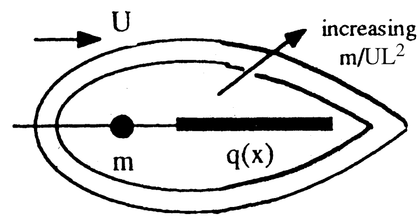


Fig. P8.98

of such a finite line sink? When combined with a uniform stream and a point source of equivalent strength as in Fig. P8.98, should a closed-body shape be formed? Make a guess and sketch some of these possible shapes for various values of the dimensionless parameter $m/(U_x L^2)$.

Solution: Yes, the "sheet" sink makes good sense and will create a body with a sharper trailing edge. If q(x) is the local sink strength, then $m = \int q(x)\,dx$, and the body shape is a teardrop which becomes fatter with increasing m/UL^2.



8.99 Consider air flowing past a hemisphere resting on a flat surface, as in Fig. P8.99. If the internal pressure is p_i, find an expression for the pressure force on the hemisphere. By analogy with Prob. 8.49 at what point *A* on the hemisphere should a hole be cut so that the pressure force will be zero according to inviscid theory?

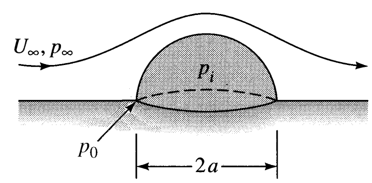


Fig. P8.99

Solution: Recall from Eq. (8.100) that the velocity along the sphere surface is

$$V_s = \frac{3}{2}U_\infty \sin\theta \quad \text{and} \quad F_{up} = \int_0^{\pi/2} (p_i - p_s)\, 2\pi a \sin\theta\, a\, d\theta \cos\theta, \quad \text{where}$$

$$p_s(\text{Bernoulli}) = p_o - \frac{\rho}{2}\left(\frac{3}{2}U_\infty \sin\theta\right)^2, \quad \text{work out } F = \pi a^2(p_i - p_o) + \frac{9\pi}{16}\rho U_\infty^2 a^2$$

This force is <u>zero</u> if we put a hole at point **A** ($\theta = \theta_A$) such that

$$p_A = p_o - \frac{9}{16}\rho U_\infty^2 = p_o - \frac{\rho}{2}V_A^2, \quad \text{or} \quad V_A = \sqrt{\frac{9}{8}}U_\infty = 1.061 U_\infty \stackrel{?}{=} \frac{3}{2}U_\infty \sin\theta_A$$

Solve for $\boldsymbol{\theta_A \approx 45°}$ or **135°** *Ans.*

8.100 A 1-m-diameter sphere is being towed at speed V in fresh water at 20°C as shown in Fig. P8.100. Assuming inviscid theory with an undistorted free surface, estimate the speed V in m/s at which cavitation will first appear on the sphere surface. Where will cavitation appear? For this condition, what will be the pressure at point A on the sphere which is 45° up from the direction of travel?

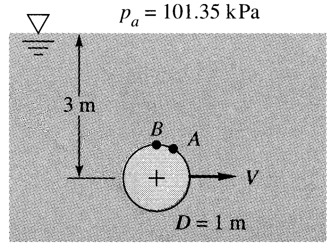


Fig. P8.100

Solution: For water at 20°C, take ρ = 998 kg/m^3 and p_v = 2337 Pa. Cavitation will occur at the lowest-pressure point, which is point **B** on the *top* of the cylinder, at $\theta = 90°$:

$$p_\infty + \frac{\rho}{2}U_\infty^2 + \rho g z_\infty = p_B + \frac{\rho}{2}V_B^2 + \rho g z_B, \quad \text{where } p_\infty = p_{atm} + \rho g(z_{surface} - z_\infty)$$

$$\text{Thus} \quad [101350 + 9790(3)] + \frac{998}{2}U_\infty^2 + 0 = 2337 + \frac{998}{2}(1.5U_\infty)^2 + 9790(0.5)$$

$$\text{Solve for} \quad \mathbf{U_\infty \approx 14.1\ \frac{m}{s}} \quad \textit{Ans.} \quad \text{to cause cavitation at point B } (\theta = 90°)$$

With the stream velocity known, we may now solve for the pressure at point A ($\theta = 45°$):

$$p_A + \frac{998}{2}\left[\frac{3}{2}(14.1)\sin 45°\right]^2 + 9790(0.5\sin 45°) = 101350 + 9790(3) + \frac{998}{2}(14.1)^2$$

$$\text{Solve for} \quad \mathbf{p_A \approx 115000\ Pa} \quad \textit{Ans.}$$

8.101 Consider a steel sphere (SG = 7.85) of diameter 2 cm, dropped from rest in water at 20°C. Assume a constant drag coefficient C_D = 0.47. Accounting for the sphere's hydrodynamic mass, estimate (a) its terminal velocity; and (b) the time to reach 99% of terminal velocity. Compare these to the results when hydrodynamic mass is neglected, $V_{terminal} \approx 1.95$ m/s and $t_{99\%} \approx 0.605$ s, and discuss.

Solution: For water take $\rho = 998$ kg/m^3. Add hydrodynamic mass to the differential equation:

$$(m + m_h)\frac{dV}{dt} = W_{net} - C_D\frac{\rho}{2}V^2 A, \quad A = \frac{\pi}{4}D^2 \quad \text{and} \quad W_{net} = (\rho_{steel} - \rho_{water})g\frac{\pi}{6}D^3$$

$$\text{Separate the variables and integrate:} \quad V = \sqrt{\frac{2W_{net}}{C_D\rho A}}\tanh\left(t\sqrt{\frac{W_{net}C_D\rho A}{2(m+m_h)^2}}\right)$$

(a) The terminal velocity is the coefficient of the *tanh* function in the previous equation:

$$V_f = \sqrt{\frac{2W_{net}}{C_D\rho A}} = \sqrt{\frac{2(7834-998)(9.81)(\pi/6)(0.02)^3}{(0.47)(998)(\pi/4)(0.02)^2}}$$

$$= \mathbf{1.95\ \frac{m}{s}} \quad \text{(same as when } m_h = 0) \quad \textit{Ans.}\text{ (a)}$$

(b) Noting that tanh(2.647) = 0.99, we find the time to approach 99% of V_f to be

$$t\sqrt{\frac{W_{net}C_D\rho A}{2(m+m_h)^2}} = 2.647$$

$$= t\sqrt{\frac{(7834-998)(9.81)(\pi/6)(0.02)^3(0.47)(998)(\pi/4)(0.02)^2}{2[(7834+998/2)(\pi/6)(0.02)^3]^2}} = 4.122t$$

Solve for $t = 2.647/4.122 = \mathbf{0.642\ s}$ (6% more than when $m_h = 0$) *Ans.* (b)

8.102 A golf ball weighs 0.102 lbf and has a diameter of 1.7 in. A professional golfer strikes the ball at an initial velocity of 250 ft/s, an upward angle of 20°, and a backspin (front of the ball rotating upward). Assume that the lift coefficient on the ball (based on frontal area) follows Fig. P7.108. If the ground is level and drag is neglected, make a simple analysis to predict the impact point (a) without spin and (b) with backspin of 7500 r/min.

Solution: For sea-level air, take ρ = 0.00238 slug/ft^3. (a) If we neglect drag and spin, we just use classical particle physics to predict the distance travelled:

$$\mathrm{V} = \mathrm{V_{oz}} - \mathrm{gt} = 0 \quad \text{when} \quad \mathrm{t} = \frac{\mathrm{V_o}\sin\theta_\mathrm{o}}{\mathrm{g}}, \quad \mathrm{x} = \mathrm{V_{ox}}(2\mathrm{t}) = \frac{\mathrm{V_o^2}}{\mathrm{g}}2\sin\theta_\mathrm{o}\cos\theta_\mathrm{o}$$

$$\text{Substitute} \quad \Delta\mathrm{x}_{\text{impact}} = \frac{(250)^2}{32.2}2\sin 20°\cos 20° \approx \mathbf{1250\ ft} \quad \textit{Ans.}\text{ (a)}$$

For part (b) we have to estimate the lift of the spinning ball, using Fig. P7.108:

$$\omega = 7500\left(\frac{2\pi}{60}\right) = 785\ \frac{\mathrm{rad}}{\mathrm{s}}, \quad \frac{\omega\mathrm{R}}{\mathrm{U}} = \frac{785(1.7/24)}{250} \approx 0.22\text{: Read } \mathrm{C_L} \approx 0.02$$

$$\text{Estimate average lift } \mathrm{L} \approx \mathrm{C_L}(\rho/2)\mathrm{V}^2\pi\mathrm{R}^2$$

$$= (0.02)\left(\frac{0.00238}{2}\right)(250)^2\pi\left(\frac{1.7}{24}\right)^2 \approx 0.023\ \mathrm{lbf}$$

Write the equations of motion in the z and x directions and assume average values:

$$m\ddot{x} = -L\sin\theta, \text{ with } \theta_{avg} \approx 10°, \quad \ddot{x}_{avg} \approx -\frac{0.023}{0.102/32.2}\sin 10° \approx -1.3\ \frac{ft}{s^2}$$

$$\text{Then } x \approx V_{ox}t - \frac{1}{2}\ddot{x}_{avg}t^2, \quad \text{where } t = 2\times(\text{time to reach the peak})$$

$$m\ddot{z} = L\cos\theta - W, \quad \text{or} \quad \ddot{z}_{avg} \approx \frac{0.023}{0.102/32.2}\cos 10° - 32.2 \approx -25\ \frac{ft}{s^2}$$

Using these admittedly crude estimates, the travel distance to impact is estimated:

$$t_{peak} = \frac{V_{oz}}{a_z} = \frac{250\sin 20°}{25} \approx 3.4\text{ s}, \quad t_{impact} = 2t_{peak} \approx 6.8\text{ s},$$

$$\Delta x_{impact} = V_{ox}t - \frac{1}{2}a_x t^2 = 250\cos 20°(6.8) - \frac{1.3}{2}(6.8)^2 \approx \mathbf{1570\ ft} \quad \textit{Ans.}\ \text{(b)}$$

These are 400-yard to 500-yard drives, on the fly! It would be nice, at least when teeing off, to have zero viscous drag on the golfball.

8.103 Modify Prob. 8.102 as follows. Golf balls are dimpled, not smooth, and have higher lift and lower drag ($C_L \approx 0.2$ and $C_D \approx 0.3$ for typical backspin). Using these values, make a computer analysis of the ball trajectory for the initial conditions of Prob. 8.102. If time permits, investigate the effect of initial angle for the range $10° < \theta_0 < 50°$.

Solution: Again take $\rho = 0.00238$ slug/ft^3 for sea-level air. The equations of motion are

$$m\ddot{x} = -D\cos\theta - L\sin\theta, \quad \text{where } D = C_D\frac{\rho}{2}V^2\pi R^2 = 0.3\left(\frac{0.00238}{2}\right)(\dot{x}^2+\dot{z}^2)\pi\left(\frac{1.7}{24}\right)^2$$

$$m\ddot{z} = -W - D\sin\theta + L\cos\theta, \quad \text{where } L = \mathbf{0.2}\left(\frac{0.00238}{2}\right)(\dot{x}^2+\dot{z}^2)\pi(1.7/24)^2,$$

$$W = 0.102\text{ lbf} \quad x_o = 0,\ \dot{x}_o = 250\cos 20°\ \frac{ft}{s}; \quad z_o = 0,\ \dot{z}_o = 250\sin 20°\ \frac{ft}{s}$$

These may be solved numerically, by Runge-Kutta or whatever, until impact. The complete trajectory is shown in the graph on the next page for the specific problem stated here. The impact point is at t = 7.7 s, **x ≈ 723 ft = 241 yards**. *Ans.* (A mediocre drive)

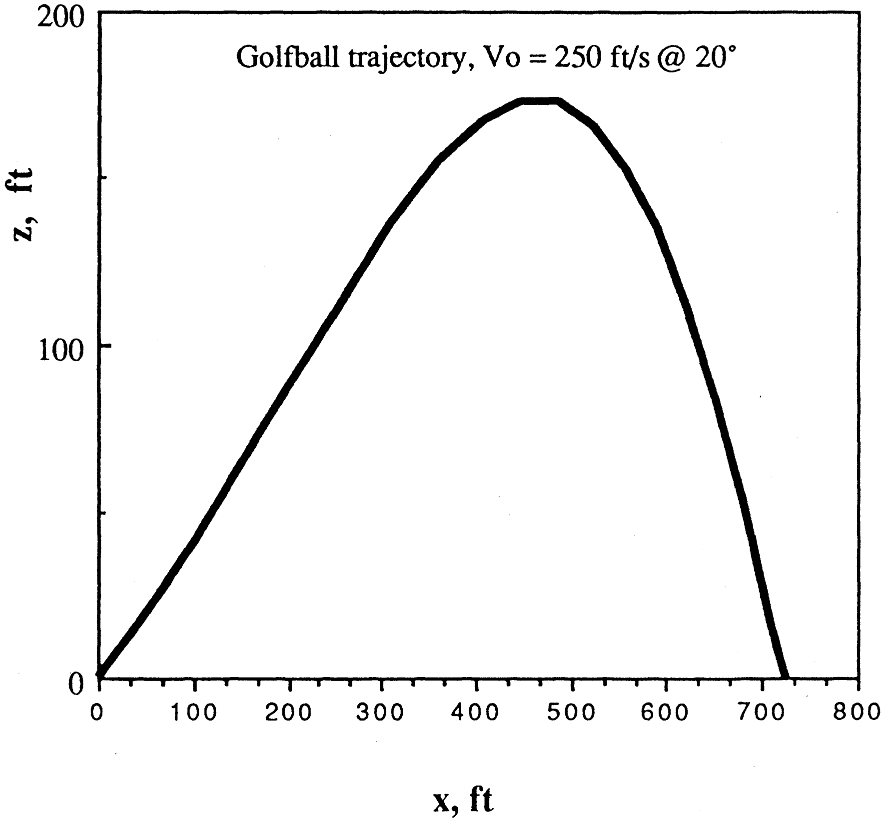


Fig. P8.103

Other impacts: for θ_o = (10°, **20°**, 30°, 40°, 50°), Δx = (705, **723 ft**, 684, 598, 469 ft).

8.104 Consider a cylinder of radius a moving at speed U_∞ through a still fluid, as in Fig. P8.104. Plot the streamlines relative to the cylinder by modifying Eq. (8.32) to give the relative flow with $K = 0$. Integrate to find the total relative kinetic energy, and verify the hydrodynamic mass of a cylinder from Eq. (8.104).

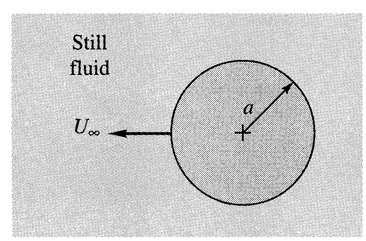


Fig. P8.104

Solution: For this two-dimensional polar-coordinate system, a differential **mass** is:

$$dm = \rho b r\, dr\, d\theta, \quad \text{where } b = \text{width into the paper}$$

Subtract off the stream U to get $v_r\big|_{rel} = -U\cos\theta(a^2/r^2),\ v_\theta\big|_{rel} = -U\sin\theta(a^2/r^2)$

That is, the velocities "relative" to the cylinder are, in fact, the velocities induced by the doublet. Now introduce the element kinetic energy into Eq. (8.102) and integrate:

$$KE = \int_{fluid} \frac{1}{2}dm\, V_{rel}^2 = \int_0^{2\pi}\int_a^{\infty} \frac{1}{2}(\rho br\, dr\, d\theta)[\{U\cos\theta a^2/r^2\}^2 + \{U\sin\theta a^2/r^2\}^2] = \frac{\boldsymbol{\pi}}{\mathbf{2}}\boldsymbol{\rho}\mathbf{Ua^2b}$$

$$\text{Then, by definition,}\quad m_{hydro} = \frac{KE}{U^2/2} = \boldsymbol{\pi\rho}\mathbf{a^2b} = \text{cylinder displaced mass} \quad \textit{Ans.}$$

8.105 In Table 7.2 the drag coefficient of a 4:1 elliptical cylinder in laminar-boundary-layer flow is 0.35. According to Patton [17], the hydrodynamic mass of this cylinder is $\pi\rho hb/4$, where b is width into the paper and h is the maximum thickness. Use these results to derive a formula from the time history $U(t)$ of the cylinder if it is accelerated from rest in a still fluid by the sudden application of a constant force F.

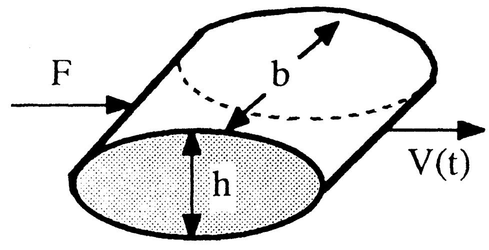


Fig. P8.105

Solution: The equation of motion is

$$\sum F_x = (m + m_{hydro})\frac{dU}{dt} = F - Drag = F - C_D\frac{\rho}{2}U^2bh, \quad \text{separate and integrate:}$$

$$\int_0^V \frac{dU}{F - \zeta U^2} = \int_0^t \frac{dt}{m + m_h}, \quad \text{or:} \quad U = \sqrt{\frac{\mathbf{F}}{\boldsymbol{\zeta}}}\,\mathbf{tanh}\left[\frac{\mathbf{t}\sqrt{\mathbf{F}\boldsymbol{\zeta}}}{\mathbf{m + m_h}}\right], \quad \zeta = \frac{\rho}{2}C_D bh \quad \textit{Ans.}$$

For numerical work, we would use $m_h = \pi\rho hb/4$, $C_D \approx 0.35$ from Table 7.2.

8.106 Laplace's equation in polar coordinates, Eq. (8.11), is complicated by the variable radius r. Consider the finite-difference mesh in Fig. P8.106, with nodes (i, j) at equally spaced $\Delta\theta$ and Δr. Derive a finite-difference model for Eq. (8.11) similar to our cartesian expression in Eq. (8.109).

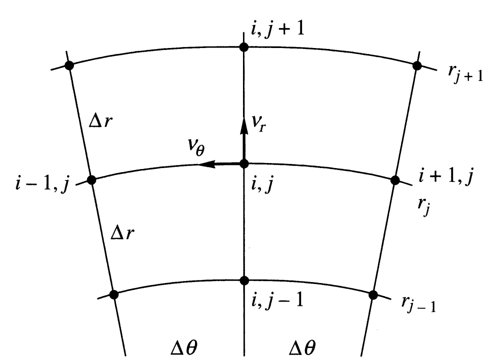


Fig. P8.106

Solution: We are asked to model

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} = 0$$

There are two possibilities, depending upon whether you split up the first term. I suggest

$$\frac{1}{r_{ij}\,\Delta r}\left[\left(r_{ij}+\frac{\Delta r}{2}\right)\left(\frac{\psi_{i,j+1}-\psi_{i,j}}{\Delta r}\right)-\left(r_{ij}-\frac{\Delta r}{2}\right)\left(\frac{\psi_{i,j}-\psi_{i,j-1}}{\Delta r}\right)\right]+\frac{\psi_{i+1,j}-2\psi_{i,j}+\psi_{i-1,j}}{r_{ij}^2(\Delta\theta)^2}\approx 0$$

Clean up: $\mathbf{(2+2\zeta)\psi_{i,j} \approx \psi_{i+1,j}+\psi_{i-1,j}+\zeta(1+\eta)\psi_{i,\,j+1}+\zeta(1-\eta)\psi_{i,j-1}}$ *Ans.*

where $\zeta = (r_{ij}\,\Delta\theta/\Delta r)^2$ and $\eta = \Delta r/(2r_{ij})$

8.107 Set up the numerical problem of Fig. 8.30 for an expansion angle of 30°. A new grid system and non-square mesh may be needed. Give the proper nodal equation and boundary conditions. If possible, program this 30° expansion and solve on a digital computer.

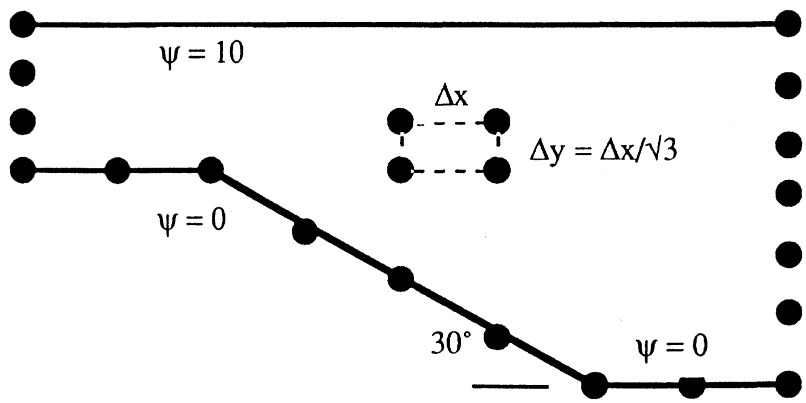


Fig. P8.107

Solution: Assuming the *same* 2:1 expansion, from U(in) = 10 m/s to U(out) = 5 m/s, we need a non-square mesh to make nodes fall along the slanted line at a 30° slope. The mesh size, as shown, should be $\Delta y = \Delta x/\sqrt{3}$. The scale-ratio β from Equation (8.108) equals $(\Delta x/\Delta y)^2 = \mathbf{3.0}$, and the model for each node is

$$2(1+\beta)\psi_{i,j} = \psi_{i-1,j}+\psi_{i+1,j}+\beta(\psi_{i,j-1}+\psi_{i,j+1}),\quad \text{where } \beta = 3.0 \quad \textit{Ans.}$$

The stream function would retain the same boundary values as in Fig. 8.30: $\psi = \mathbf{0}$ along the lower wall, $\psi = \mathbf{10}$ along the upper wall, and linear variation, between 0 and 10, along the inlet and exit planes. [See Fig. 8.31 also for boundary values.]

If we keep the same **vertical** nodal spacing in Fig. 8.30, Δy = **20** cm, then we need a horizontal spacing Δx = **34.64** cm, and the total length of the duct—from i = 1 to i = 16—will be 519.6 cm, whereas in Fig. 8.30 this total length is only 300 cm.

The numerical results will not be given here.

8.108 Consider two-dimensional potential flow into a step contraction as in Fig. P8.108. The inlet velocity $U_1 = 7$ m/s, and the outlet velocity U_2 is uniform. The nodes (i, j) are labelled in the figure. Set up the complete finite-difference algebraic relation for all nodes. Solve, if possible, on a digital computer and plot the streamlines.

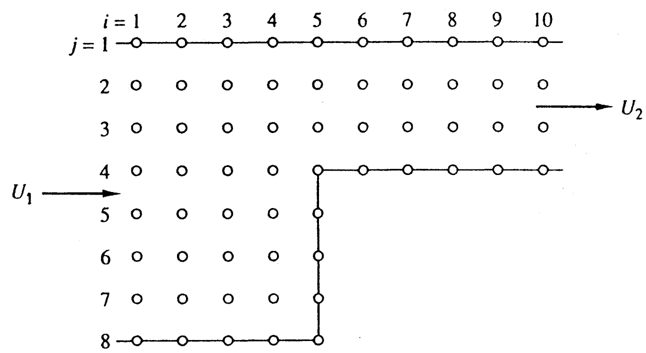


Fig. P8.108

Solution: By continuity, $U_2 = U_1(7/3) = 16.33$ m/s. For a square mesh, the standard Laplace model, Eq. (8.109), holds. For simplicity, assume unit mesh widths $\Delta x = \Delta y = 1$.

$$\text{Solve} \quad \psi_{i,j} = \frac{1}{4}(\psi_{i,j+1} + \psi_{i,j-1} + \psi_{i+1,j} + \psi_{i-1,j})$$

with $\psi = 0$ on the lower wall and $\psi = 49$ on the upper wall.

The writer's numerical solution is tabulated below.

i =	1	2	3	4	5	6	7	8	9	10
j = 1, ψ =	49.00	49.00	49.00	49.00	49.00	49.00	49.00	49.00	49.00	49.00
j = 2	42.00	40.47	38.73	36.66	34.54	33.46	32.98	32.79	32.79	26.67
j = 3	35.00	32.17	28.79	24.38	19.06	17.30	16.69	16.46	16.38	16.33
j = 4	28.00	24.40	19.89	13.00	0.00	0.00	0.00	0.00	0.00	0.00
j = 5	21.00	17.53	13.37	7.73	0.00					
j = 6	14.00	11.36	8.32	4.55	0.00					
j = 7	7.00	5.59	4.02	2.14	0.00					
j = 8	0.00	0.00	0.00	0.00	0.00					

8.109 Consider inviscid potential flow through a two-dimensional 90° bend with a contraction, as in Fig. P8.109. Assume uniform flow at the entrance and exit. Make a finite-difference computer model analysis for small grid size (at least 150 nodes), determine the dimensionless pressure distribution along the walls, and sketch the streamlines. [You may use either square or rectangular grids.]

Solution: This problem is "digital computer enrichment" and will not be presented here.

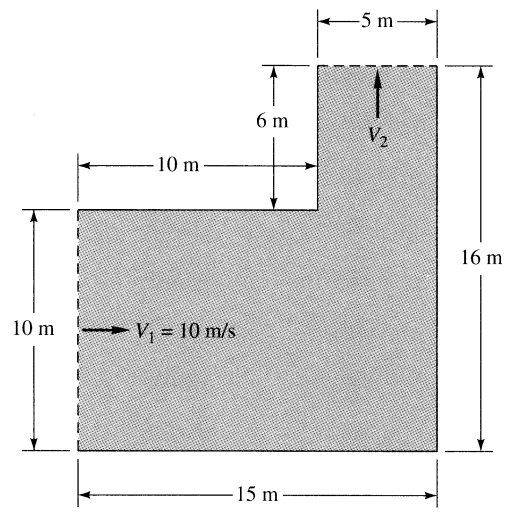


Fig. P8.109

8.110 For fully developed laminar incompressible flow through a straight noncircular duct, as in Sec. 6.8, the Navier-Stokes Equation (4.38) reduce to

$$\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=\frac{1}{\mu}\frac{dp}{dx}=\text{const}<0$$

where (y, z) is the plane of the duct cross section and x is along the duct axis. Gravity is neglected. Using a nonsquare rectangular grid (Δx, Δy), develop a finite-difference model for this equation, and indicate how it may be applied to solve for flow in a rectangular duct of side lengths a and b.

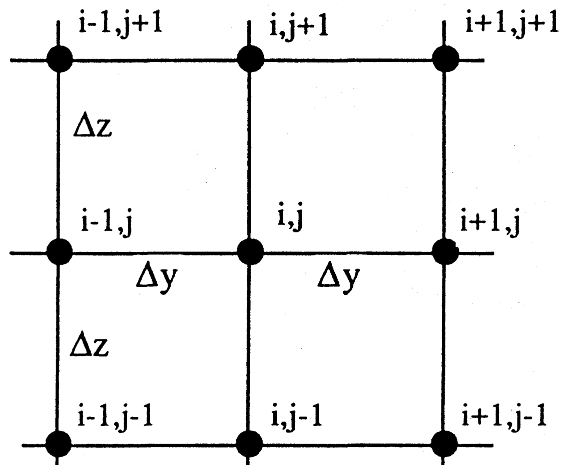


Fig. P8.110

Solution: An appropriate square grid is shown above. The finite-difference model is

$$\frac{u_{i+1,j}-2u_{i,j}+u_{i-1,j}}{(\Delta y)^2}+\frac{u_{i,j+1}-2u_{i,j}+u_{i,j-1}}{(\Delta z)^2}\approx\frac{1}{\mu}\frac{dp}{dx},\quad \text{or,}\quad \text{if}\ \ \Delta y=\Delta z,$$

$$\mathbf{u_{i,j}=\frac{1}{4}\left[u_{i,j+1}+u_{i,j-1}+u_{i+1,j}+u_{i-1,j}-\frac{(\Delta y)^2}{\mu}\frac{dp}{dx}\right]}\quad \textit{Ans.}$$

This is "Poisson's equation," it looks like the Laplace model plus the constant "source" term involving the mesh size (Δy) and the pressure gradient and viscosity.

8.111 Solve Prob. 8.110 numerically for a rectangular duct of side length b by $2b$, using at least 100 nodal points. Evaluate the volume flow rate and the friction factor, and compare with the results in Table 6.4:

$$Q\approx 0.1143\frac{b^4}{\mu}\left(-\frac{dp}{dx}\right)\qquad f\,\mathrm{Re}_{D_h}\approx 62.19$$

where $D_h = 4A/P = 4b/3$ for this case. Comment on the possible truncation errors of your model.

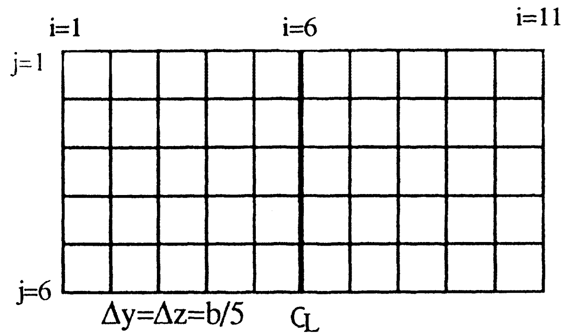


Fig. P8.111

Solution: A typical square mesh is shown in the figure above. It is appropriate to nondimensionalize the velocity and thus get the following dimensionless model:

$$\mathbf{V}=\frac{u}{(b^2/\mu)(-dp/dx)};\ \text{Then}\ \ V_{ij}=\frac{1}{4}\left[V_{i,j+1}+V_{i,j-1}+V_{i+1,j}+V_{i-1,j}+\left(\frac{\Delta y}{b}\right)^2\right],\ \text{with}\ \Delta y=\Delta z$$

The boundary conditions are: No-slip along all the outer surfaces: V = 0 along i = 1, i =11, j = 1, and j = 6. The internal values V_{ij} are then computed by iteration and sweeping over the interior field. Some computed results for this mesh, $\Delta y/b = 0.2$, are as follows:

i =		1	2	3	4	5	6 (centerline)
j = 1,	V =	0.000	0.000	0.000	0.000	0.000	0.000
j = 2,	V =	0.000	0.038	0.058	0.067	0.072	0.073
j = 3,	V =	0.000	0.054	0.084	0.100	0.107	0.109
j = 4,	V =	0.000	0.054	0.084	0.100	0.107	0.109
j = 5,	V =	0.000	0.038	0.058	0.067	0.072	0.073
j = 6,	V =	0.000	0.000	0.000	0.000	0.000	0.000

The solution is doubly symmetric because of the rectangular shape. [This mesh is too **coarse**, it only has 27 interior points.] After these dimensionless velocities are computed, the volume flow rate is computed by integration:

$$Q = \int_0^b \int_0^{2b} u\, dy\, dz = \frac{b^4}{\mu}\left(-\frac{dp}{dx}\right)\int_0^1 \int_0^2 V \frac{dy}{b}\frac{dz}{b} = \text{constant}\,\frac{b^4}{\mu}\left(-\frac{dp}{dx}\right)$$

The double integral was evaluated numerically by summing over all the mesh squares. Two mesh sizes were investigated by the writer, with good results as follows:

$$\Delta y/b = 0.2\text{:}\quad Q/(b^4/\mu)(-dp/dx) = \text{“constant”} \approx 0.1063\,(7\%\ \text{off})\quad (27\ \text{grid nodes})$$

$$= \mathbf{0.1} \qquad\qquad \text{“constant”} \approx \mathbf{0.1123}\,(2\%\ \text{off})\quad (171\ \text{nodes})\quad \textit{Ans.}$$

The accuracy is good, and the numerical model is very simple to program.

8.112 In his CFD textbook, Patankar [Ref. 5] replaces the left-hand side of Eq. (8.119*b*) and (8.119*c*), respectively, with the following two expressions:

$$\textit{Replace}\quad u\frac{\partial u}{\partial x} + \upsilon\frac{\partial u}{\partial y}\quad \textit{by}\quad \frac{\partial}{\partial x}(u^2) + \frac{\partial}{\partial y}(\upsilon u);$$

$$\textit{Replace}\quad u\frac{\partial \upsilon}{\partial x} + \upsilon\frac{\partial \upsilon}{\partial y}\quad \textit{by}\quad \frac{\partial}{\partial x}(u\upsilon) + \frac{\partial}{\partial y}(\upsilon^2)$$

Are these equivalent expressions, or are they merely simplified approximations? Either way, why might these forms be better for finite-difference purposes?

Solution: These expressions are indeed *equivalent* because of the 2-D incompressible continuity equation. In the first example,

$$\frac{\partial}{\partial x}(u^2) + \frac{\partial}{\partial y}(u\upsilon) \equiv 2u\frac{\partial u}{\partial x} + u\frac{\partial \upsilon}{\partial y} + \upsilon\frac{\partial u}{\partial y} = \left[u\frac{\partial u}{\partial x} + \upsilon\frac{\partial \upsilon}{\partial y}\right] + u\left(\cancel{\frac{\partial u}{\partial x}} + \frac{\partial \upsilon}{\partial y}\right)$$

and similarly for the second example. They are more convenient numerically because, being <u>non-linear</u> terms, they are easier to model as *the difference in products* rather than the *product of differences*.

8.113 Repeat Example 8.7 using the *implicit* method of Eq. (8.118). Take Δt = 0.2 s and Δy = 0.01 m, which ensures that an explicit model would *diverge*. Compare your accuracy with Example 8.7.

Solution: Recall that SAE 30 oil (υ = 3.25E–4 m^2/s) was at rest at (t = 0) when the wall suddenly began moving at U = 1 m/s. Find the oil velocity at (y, t) = (3 cm, 1 s). This time σ = (3.25E–4)(0.2)/$(0.01)^2$ = **0.65** > 0.5, therefore an implicit method is required. We set up a grid with 11 nodes, going out to y = 0.1 m (N = 11), and use Eq. (8.118) to sweep all nodes for each time step. Stop at t = 1 s (j = 6):

$$u_n^{j+1} \approx \frac{u_n^j + 0.65\left(u_{n-1}^{j+1} + u_{n+1}^{j+1}\right)}{1+2(0.65)}$$

The results are shown in the table below.

j	Time	u1	u2	u3	u4	U5	u6	u7	u8	u9	u10	u11
1	0.0	1.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
2	0.2	1.000	0.310	0.096	0.030	0.009	0.003	0.001	0.000	0.000	0.000	0.000
3	0.4	1.000	0.473	0.197	0.077	0.029	0.010	0.004	0.001	0.000	0.000	0.000
4	0.6	1.000	0.568	0.283	0.129	0.055	0.023	0.009	0.003	0.001	0.000	0.000
5	0.8	1.000	0.629	0.351	0.180	0.086	0.039	0.017	0.007	0.003	0.001	0.000
6	1.0	1.000	0.671	0.406	**0.226**	0.117	0.058	0.027	0.012	0.005	0.002	0.000

This time the computed value of u_4 (y = 0.03 m) at j = 6 (t = 1 s) is **0.226**, or about **6% lower** than the exact value of *0.241*. This accuracy is comparable to the explicit method of Example 8.7 and uses twice the time step.

***8.114** The following problem is not solved in this Manual. It requires Boundary-Element-Code software. If your institution has such software (see, e.g., the computer codes in Ref. 7), this advanced exercise is quite instructive about potential flow about airfoils.

If your institution has an online potential-flow boundary-element computer code, consider flow past a symmetric airfoil, as in Fig. P8.114. The basic shape of an NACA symmetric airfoil is defined by the function [12]

$$\frac{2y}{t_{max}} \approx 1.4845\zeta^{1/2} - 0.63\zeta - 1.758\zeta^2 + 1.4215\zeta^3 - 0.5075\zeta^4$$

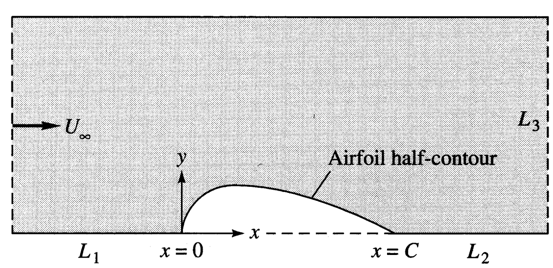


Fig. P8.114

where $\zeta = x/C$ and the maximum thickness t_{max} occurs at $\zeta = 0.3$. Use this shape as part of the lower boundary for zero angle of attack. Let the thickness be fairly large, say, t_{max} = 0.12, 0.15, or 0.18. Choose a generous number of nodes (≥60), and calculate and plot the velocity distribution V/U_∞ along the airfoil surface. Compare with the theoretical results in Ref. 12 for NACA 0012, 0015, or 0018 airfoils. If time permits, investigate the effect of the boundary lengths L_1, L_2, and L_3, which can initially be set equal to the chord length C.

8.115 Use the explicit method of Eq. (8.115) to solve Problem 4.85 *numerically* for SAE 30 oil (ν = 3.25E–4 m^2/s) with U_o = 1 m/s and $\omega = M$ rad/s, where M is the number of letters in your surname. (The author will solve it for M = 5.) When steady oscillation is reached, plot the oil velocity versus time at y = 2 cm.

Solution: Recall that Prob. 4.85 specified an oscillating wall, $u_{wall} = U_o\sin(\omega t)$. One would have to experiment to find that the "edge" of shear layer, that is, where the wall no longer influences the ambient still fluid, is about $y \approx 7$ cm. For reasonable accuracy, we could choose Δy = 0.5 cm, that is, 0.005 m, so that N = 15 is the outer "edge." For explicit calculation, we require

$$\sigma = \nu\Delta t/\Delta y^2 = (3.25E{-}4)\Delta t/(0.005)^2 < 0.5, \quad \text{or} \quad \Delta t < 0.038 \text{ s.}$$

We choose Δt = 0.0333 s, $\boldsymbol{\sigma = 0.433}$, with 1 cycle covering about 37 time steps. Use Eq. (8.115):

$$u_n^{j+1} \approx 0.433\left(u_{n-1}^j + u_{n+1}^j\right) + 0.133u_n^j \quad \textit{for } 2 \le n \le 14 \quad \textit{and} \quad u_1 = 1.0\sin(5t),\ u_{15} = 0$$

Apply this algorithm to all the internal nodes (2 < n < 14) for many (200) time steps, up to about t = 6 sec. The results for y = 2 cm, n = 5, are shown in the plot below. The amplitude has dropped to 0.18 m/s with a phase lag of 20° [Ref. 15 of Chap. 8, p. 139].

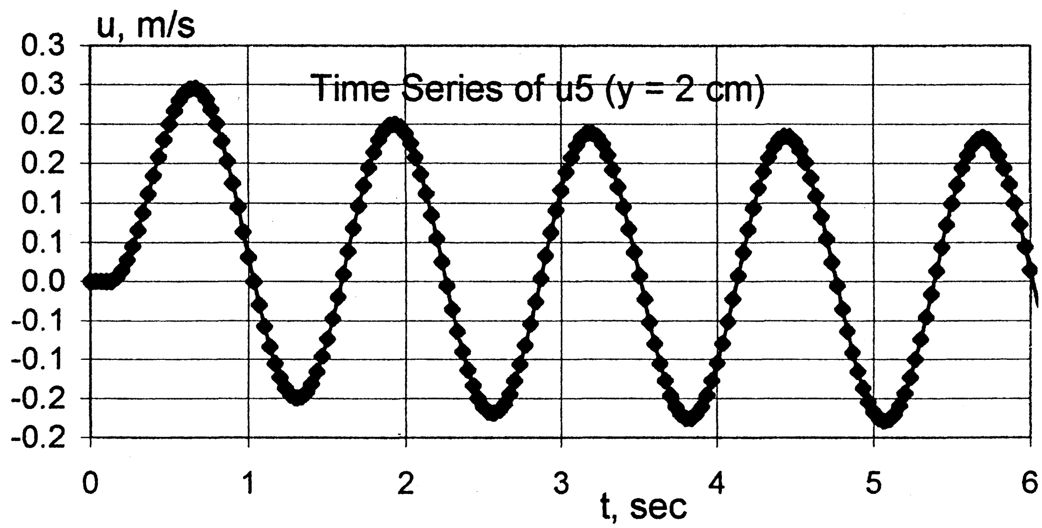


Fig. P8.115

COMPREHENSIVE PROBLEMS

C8.1 Did you know you can solve iterative CFD problems on an Excel spreadsheet? Successive relaxation of Laplace's equation is easy, since each nodal value is the average of its 4 neighbors. Calculate irrotational potential flow through a contraction as shown in the figure. To avoid "circular reference," set the Tools/Options/Calculate menu to "iteration." For full credit, attach a printout of your solution with ψ at each node.

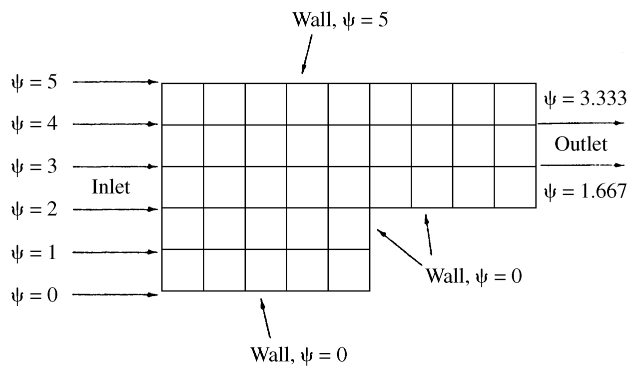


Fig. C8.1

Solution: Do exactly what the figure shows: Set bottom nodes at $\psi = 0$, top nodes at $\psi = 5$, left nodes at $\psi = 0$, 1, 2, 3, 4, 5 and right nodes at $\psi = 0$, 1.667, 3.333, and 5. Iterate with Eq. (8.115) for at least 100 iterations. Any initial guesses will do—the author chose 2.0 at all interior nodes. The final converged nodal values of stream function are shown in the table below.

i, j	1	2	3	4	5	6	7	8	9	10
1	5.000	5.000	5.000	5.000	5.000	5.000	5.000	5.000	5.000	5.000
2	4.000	3.948	3.880	3.780	3.639	3.478	3.394	3.357	3.342	3.333
3	3.000	2.911	2.792	2.602	2.298	1.879	1.742	1.694	1.675	1.667
4	2.000	1.906	1.773	1.539	1.070	0.000	0.000	0.000	0.000	0.000
5	1.000	0.941	0.856	0.710	0.445	0.000				
6	0.000	0.000	0.000	0.000	0.000	0.000				

C8.2 Use an *explicit* method, similar to but not identical to Eq. (8.115), to solve the case of SAE 30 oil starting from rest near a fixed wall. Far from the wall, the oil accelerates linearly, that is, $u_\infty = u_N = at$, where $a = 9\ m/s^2$. At t = 1 s, determine (a) the oil velocity at y = 1 cm; and (b) the instantaneous boundary-layer thickness (where $u \approx 0.99u_\infty$).

Hint: There is a non-zero pressure gradient in the outer (shear-free) stream, n = N, which must be included in Eq. (8.114).

Solution: To account for the stream acceleration as $\partial^2 u/\partial t^2 = 0$, we add a term:

$$\rho\frac{\partial u}{\partial t} = \rho a + \mu\frac{\partial^2 u}{\partial y^2},\quad \text{which changes the model of Eq. (8.115) to}$$

$$u_n^{j+1} \approx a\Delta t + \sigma\left(u_{n-1}^j + u_{n+1}^j\right) + (1-2\sigma)u_n^j$$

The added term **a∆t** keeps the outer stream accelerating linearly. For SAE 30 oil, ν = 3.25E–4 m^2/s. As in Prob. 8.115 of this Manual, choose N = 15, ∆y = 0.005 m, ∆t = 0.0333 s, σ = **0.433**, and let u_1 = 0 and u_N = u_{15} = at = 9t. All the inner nodes, 2 < n <14, are computed by the explicit relation just above. After 30 time-steps, t = 1 sec, the tabulated velocities below show that the velocity at y = 1 cm (n = 3) is $u_3 \approx$ **4.41** m/s, and the position "δ" where u = 0.99u_∞ = 8.91 m/s is at approximately **0.053** meters. *Ans*. These results are in good agreement with the known exact analytical solution for this flow.

j	Time	u1	u2	u3	u4	u5	u6	u7	u8	u9	u10	u11	u12	u13	u14	u15
31	**1.00**	0.000	2.496	**4.405**	5.832	6.871	7.607	8.114	8.453	8.673	8.811	**8.895**	8.944	8.972	8.988	9

C8.3 Model potential flow through the upper-half of the symmetric diffuser shown below. The expansion angle is θ = 18.5°. Use a *non-square* mesh and calculate and plot (a) the velocity distribution; and (b) the pressure coefficient along the centerline (the bottom boundary).

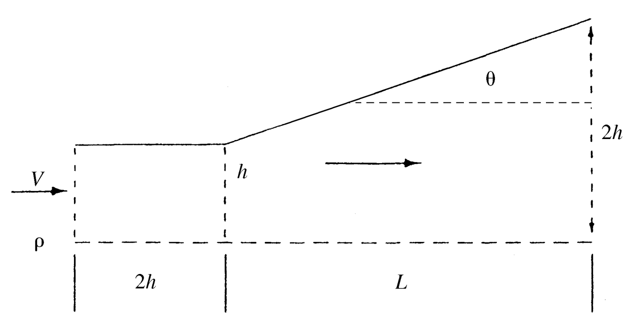


Fig. C8.3

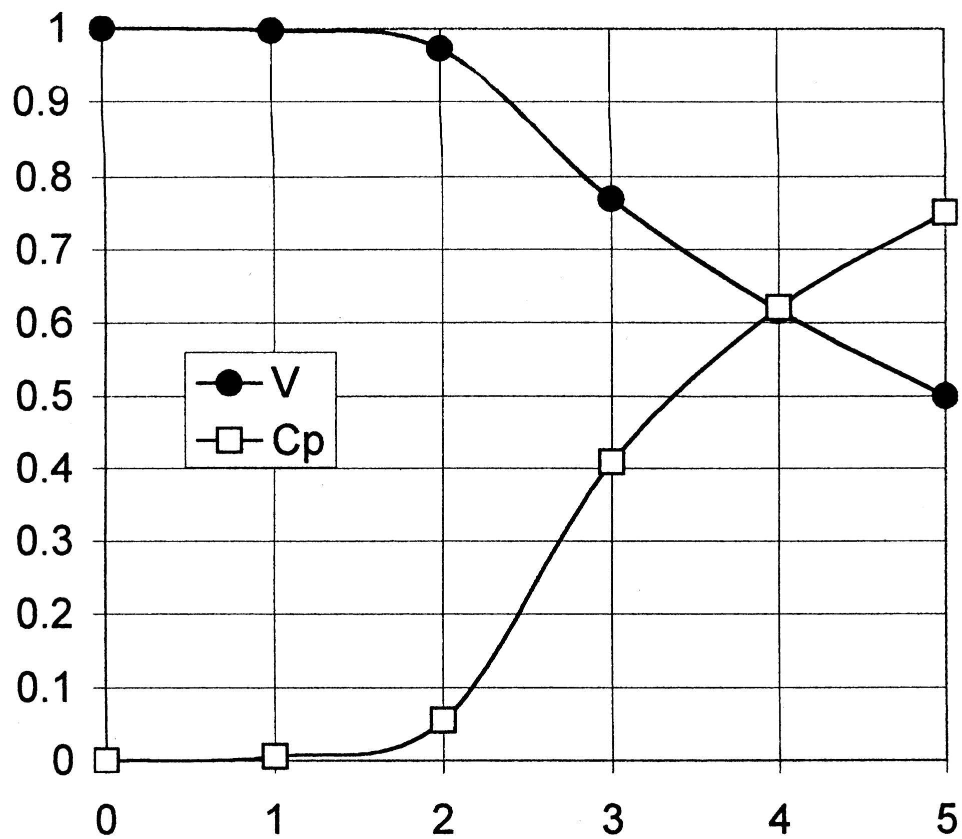
Solution: The tangent of 18.5° is 0.334, so $L \approx 3h$ and 3:1 rectangles are appropriate. If we make them h long and h/3 high, then i = 1 to 6 and j = 1 to 7. The model is given by Eq. (8.108) with $\beta = (3/1)^2 = 9$. That is,

$$2(1+9)\psi_{i,j} \approx \psi_{i-1,j} + \psi_{i+1,j} + 9(\psi_{i,j-1} + \psi_{i,j+1})$$

to be iterated over the internal nodes. For convenience, take $\psi_{top} = 10{,}000$ and $\psi_{bottom} = 0$. The iterated nodal solutions are as follows:

i, j	1	2	3	4	5	6
1						10000
2					10000	8333
3				10000	8083	6667
4	10000	10000	10000	7604	6111	5000
5	6667	6657	6546	5107	4097	3333
6	3333	3324	3240	2563	2055	1667
7	0	0	0	0	0	0

The velocities are found by taking differences: $u \approx \Delta\psi/\Delta y$ along the centerline. A plot is then made, as shown below, of velocity along the centerline (j = 7). The pressure coefficient is defined by $C_p = (p - p_{entrance})/[(1/2)\rho V^2_{entrance}]$. These are also plotted on the graph, along the centerline, using Bernoulli's equation.



Velocity and Pressure Coefficient Distribution along the Centerline of the Diffuser in Problem C8.3, Assuming unit Velocity at the Entrance.

C8.4 Use potential flow to approximate the flow of air being sucked into a vacuum cleaner through a 2-D slit attachment, as in the figure. Model the flow as a line sink of strength (–m), with its axis in the *z*-direction at height *a* above the floor. (a) Sketch the streamlines and locate any stagnation points. (b) Find the velocity V(x) along the floor in terms of *a* and *m*. (c) Define a velocity scale *U* = *m/a* and plot the pressure coefficient $C_p = (p - p_\infty)/[(1/2)\rho U^2]$ along the floor. (d) Find where C_p is a minimum—the vacuum cleaner should be most effective here. (e) Where did you *expect* the cleaner to be most effective, at x = 0 or elsewhere? (Experiment with dust later.)

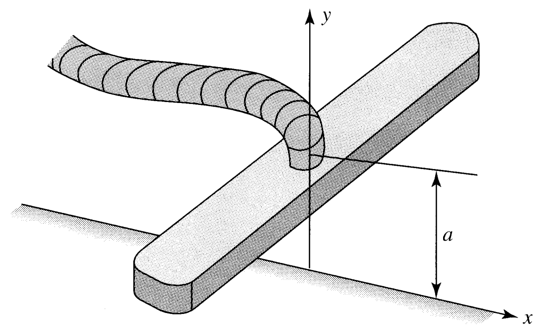
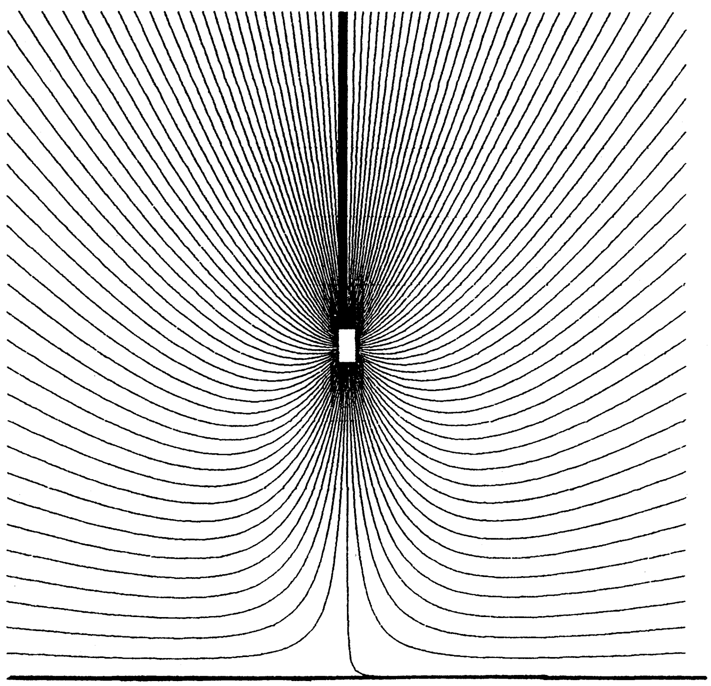


Fig. C8.4

Solution: (a) The "floor" is created by a sink at (0, +a) and an *image* sink at (0, –a), exactly like Fig. 8.17a of the text. There is one stagnation point, at the *origin*. The streamlines are shown below in a plot constructed from a MATLAB contour. *Ans.* (a)



(b) At any point *x* along the wall, the velocity *V* is the sum of image flows:

$$V = 2v_{r,sink}\cos\theta = \frac{2m}{r}\frac{x}{r} = \frac{\mathbf{2mx}}{\mathbf{x^2+a^2}} = V_{along\ wall} \quad Ans.\ (b)$$

(c) Use the Bernoulli equation to calculate pressure coefficient along the wall:

$$C_p = \frac{p - p_\infty}{(1/2)\rho U^2} = \frac{(1/2)\rho(U_\infty^2 - V^2)}{(1/2)\rho U^2} = -\frac{V^2}{U^2}; \quad C_{p,wall} = -\frac{\mathbf{4x^2a^2}}{\mathbf{(x^2+a^2)^2}} \quad \textit{Ans.}\ \text{(c)}$$

(d) The minimum wall-pressure coefficient is found by differentiation:

$$\frac{dC_p}{dx} = 0 = \frac{d}{dx}\left[\frac{-4x^2a^2}{(x^2+a^2)^2}\right] \text{ occurs at } x^2 = a^2, \quad \textit{or}: \quad \boldsymbol{x = \pm a} \quad \textit{Ans.}\ \text{(d)}$$

(e) Unexpected result! But experiments *do* show best cleaning at about $x \approx \pm a$.

C8.5 Consider three-dimensional, incompressible, irrotational flow. Use two methods to prove that the viscous term in the Navier-Stokes equation is zero: (a) using vector notation; and (b) expanding out the scalar terms using irrotationality.

Solution: (a) For irrotational flow, $\nabla \times \mathbf{V} = 0$, and $\mathbf{V} = \nabla\phi$, so the viscous term may be rewritten in terms of ϕ and then we get Laplace's equation:

$$\mu\nabla^2 V = \mu\nabla^2(\nabla\phi) = \mu\nabla(\nabla^2\phi) \equiv 0 \text{ from Laplace's equation.} \quad \textit{Ans.}\ \text{(a)}$$

(b) Expansion illustration: write out the x-term of $\nabla^2\mathbf{V}$, using irrotationality:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial \mathbf{v}}{\partial \mathbf{x}}\right) + \frac{\partial}{\partial z}\left(\frac{\partial \mathbf{w}}{\partial \mathbf{x}}\right) = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial \upsilon}{\partial y} + \frac{\partial w}{\partial z}\right) \equiv 0 \quad \textit{Ans.}\ \text{(b)}$$

Similarly, $\nabla^2\upsilon = \nabla^2 w = 0$. The viscous term <u>always vanishes for irrotational flow</u>.

C8.6 Reconsider the lift-drag data for the NACA 4412 airfoil from Prob. 8.83. (a) Again draw the polar lift-drag plot and compare qualitatively with Fig. 7.26. (b) Find the maximum value of the lift-to-drag ratio. (c) Demonstrate a straight-line construction on the polar plot which will immediately yield the maximum *L*/*D* in (b). (d) If an aircraft could use this two-dimensional wing in actual flight (no induced drag) and had a perfect pilot, estimate how far (in miles) this aircraft could glide to a sea-level runway if it lost power at 25,000 ft altitude.

Solution: (a) Simply calculate $C_L(\alpha)$ and $C_D(\alpha)$ and plot them versus each other, as shown below:

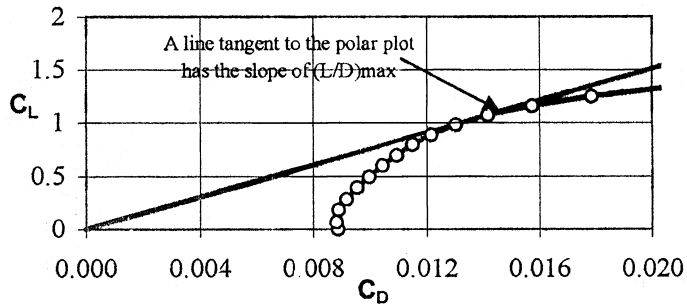


Fig. C8.6

(b, c) By calculating the ratio *L/D*, we could find a maximum value of **76** at $\alpha = 9°$. *Ans.* (b) This can be found graphically by **drawing a tangent from the origin** to the polar plot. *Ans.* (c)

(d) If the pilot could glide down at a constant angle of attack of 9°, the airplane could coast to a maximum distance of (76)(25000 ft)/(5280 ft/mi) = **360 miles**. *Ans.* (d)

C8.7 Find a formula for the stream function for flow of a doublet of strength λ at a distance a from a wall, as in Fig. C8.7. (a) Sketch the streamlines. (b) Are there any stagnation points? (c) Find the maximum velocity along the wall and its position.

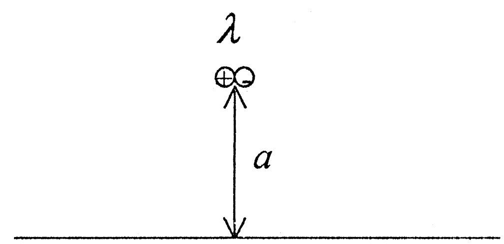
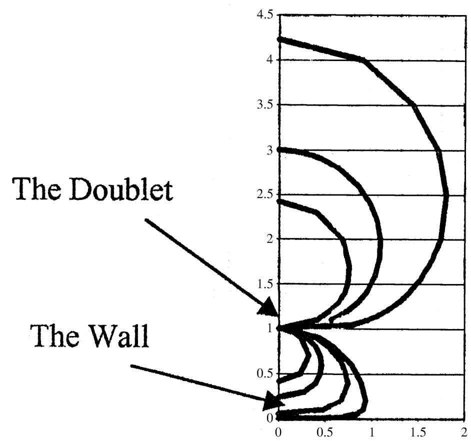


Fig. C8.7

Solution: Use an image doublet of the same strength and orientation at the $(x, y) = (0, -a)$. The stream function for this combined flow will form a "wall" at $y = 0$ between the two doublets:

$$\psi = -\frac{\lambda(y+a)}{x^2+(y+a)^2} - \frac{\lambda(y-a)}{x^2+(y-a)^2}$$

(a) The streamlines are shown on the next page for one quadrant of the doubly-symmetric flow field. They are fairly circular, like Fig. 8.8, above the doublet, but they flatten near the wall.



Problem C8.7

(b) There are **no stagnation points** in this flow field. *Ans*. (b)

(c) The velocity along the wall ($y = 0$) is found by differentiating the stream function:

$$u_{wall} = \frac{\partial\psi}{\partial y}\bigg|_{y=0} = -\frac{\lambda}{x^2+a^2} + \frac{2\lambda a^2}{(x^2+a^2)^2} - \frac{\lambda}{x^2+a^2} + \frac{2\lambda a^2}{(x^2+a^2)^2}$$

The maximum velocity occurs at $x = 0$, that is, right between the two doublets:

$$u_{w,\max} = \frac{2\lambda}{a^2} \quad \textit{Ans.}\ \text{(c)}$$