Chapter 4 • Differential Relations for a Fluid Particle

4.1 An idealized velocity field is given by the formula

$$\mathbf{V} = 4tx\mathbf{i} - 2t^2y\mathbf{j} + 4xz\mathbf{k}$$

Is this flow field steady or unsteady? Is it two- or three-dimensional? At the point $(x, y, z) = (-1, +1, 0)$, compute (a) the acceleration vector and (b) any unit vector normal to the acceleration.

Solution: (a) The flow is unsteady because time t appears explicitly in the components. (b) The flow is three-dimensional because all three velocity components are nonzero. (c) Evaluate, by laborious differentiation, the acceleration vector at (x, y, z) = (−1, +1, 0).

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = 4x + 4tx(4t) - 2t^2y(0) + 4xz(0) = 4x + 16t^2x$$

$$\frac{dv}{dt} = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -4ty + 4tx(0) - 2t^2y(-2t^2) + 4xz(0) = -4ty + 4t^4y$$

$$\frac{dw}{dt} = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = 0 + 4tx(4z) - 2t^2y(0) + 4xz(4x) = 16txz + 16x^2z$$

$$\text{or:}\quad \frac{d\mathbf{V}}{dt} = (4x + 16t^2x)\mathbf{i} + (-4ty + 4t^4y)\mathbf{j} + (16txz + 16x^2z)\mathbf{k}$$

at (x, y, z) = (−1, +1, 0), we obtain $\dfrac{d\mathbf{V}}{dt} = -4(1+4t^2)\mathbf{i} - 4t(1-t^3)\mathbf{j} + 0\mathbf{k}$ *Ans.* (c)

(d) At (–1, +1, 0) there are many unit vectors normal to d**V**/dt. One obvious one is **k**. *Ans.*

4.2 Flow through the converging nozzle in Fig. P4.2 can be approximated by the one-dimensional velocity distribution

$$u \approx V_o\left(1 + \frac{2x}{L}\right) \quad \upsilon \approx 0 \quad w \approx 0$$

(a) Find a general expression for the fluid acceleration in the nozzle. (b) For the specific case V_o = 10 ft/s and L = 6 in, compute the acceleration, in g's, at the entrance and at the exit.

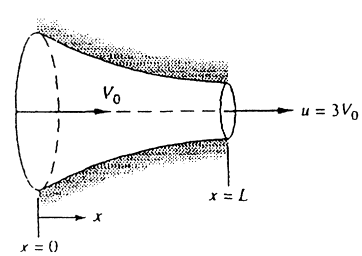


Fig. P4.2

Solution: Here we have only the single 'one-dimensional' convective acceleration:

$$\frac{du}{dt} = u\frac{\partial u}{\partial x} = \left[V_o\left(1+\frac{2x}{L}\right)\right]\frac{2V_o}{L} = \frac{\mathbf{2V_o^2}}{\mathbf{L}}\left(\mathbf{1}+\frac{\mathbf{2x}}{\mathbf{L}}\right) \quad \textit{Ans. (a)}$$

$$\textit{For } L = 6'' \textit{ and } V_o = 10\ \frac{ft}{s},\quad \frac{du}{dt} = \frac{2(10)^2}{6/12}\left(1+\frac{2x}{6/12}\right) = 400(1+4x),\ \textit{with x in feet}$$

At x = 0, du/dt = **400** ft/s^2 (12 g's); at x = L = 0.5 ft, du/dt = **1200** ft/s^2 (37 g's). *Ans.* (b)

4.3 A two-dimensional velocity field is given by

$$\mathbf{V} = (x^2 - y^2 + x)\mathbf{i} - (2xy + y)\mathbf{j}$$

in arbitrary units. At $(x, y) = (1, 2)$, compute (a) the accelerations a_x and a_y, (b) the velocity component in the direction $\theta = 40°$, (c) the direction of maximum velocity, and (d) the direction of maximum acceleration.

Solution: (a) Do each component of acceleration:

$$\frac{du}{dt} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = (x^2 - y^2 + x)(2x+1) + (-2xy - y)(-2y) = a_x$$

$$\frac{dv}{dt} = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = (x^2 - y^2 + x)(-2y) + (-2xy - y)(-2x-1) = a_y$$

At (x, y) = (1, 2), we obtain $\mathbf{a_x} = 18\mathbf{i}$ and $\mathbf{a_y} = 26\mathbf{j}$ *Ans.* (a)

(b) At (x, y) = (1, 2), $\mathbf{V} = -2\mathbf{i} - 6\mathbf{j}$. A unit vector along a 40° line would be $\mathbf{n} = \cos 40°\mathbf{i} + \sin 40°\mathbf{j}$. Then the velocity component along a 40° line is

$$V_{40°} = \mathbf{V}\cdot\mathbf{n_{40°}} = (-2\mathbf{i} - 6\mathbf{j})\cdot(\cos 40°\mathbf{i} + \sin 40°\mathbf{j}) \approx \mathbf{5.39\ units} \quad \textit{Ans. (b)}$$

(c) The maximum acceleration is $\mathbf{a_{max}} = [18^2 + 26^2]^{1/2} = \mathbf{31.6}$ units at $\angle 55.3°$ *Ans.* (c, d)

4.4 Suppose that the temperature field $T = 4x^2 - 3y^3$, in arbitrary units, is associated with the velocity field of Prob. 4.3. Compute the rate of change *dT/dt* at (x, y) = (2, 1).

Solution: For steady, two-dimensional flow, the rate of change of temperature is

$$\frac{dT}{dt} = u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = (x^2 - y^2 + x)(8x) + (-2xy - y)(-9y^2)$$

At $(x, y) = (2, 1)$, $dT/dt = (5)(16) - 5(-9) = \mathbf{125\ units}$ *Ans.*

4.5 The velocity field near a stagnation point (see Example 1.10) may be written in the form

$$u = \frac{U_o x}{L} \qquad v = \frac{-U_o y}{L} \quad U_o \text{ and } L \text{ are constants}$$

(a) Show that the acceleration vector is purely radial. (b) For the particular case $L = 1.5$ m, if the acceleration at $(x, y) = (1$ m, 1 m$)$ is 25 m/s^2, what is the value of U_o?

Solution: (a) For two-dimensional steady flow, the acceleration components are

$$\frac{du}{dt} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \left(U_o\frac{x}{L}\right)\left(\frac{U_o}{L}\right) + \left(-U_o\frac{y}{L}\right)(0) = \frac{U_o^2}{L^2}x$$

$$\frac{dv}{dt} = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = \left(U_o\frac{x}{L}\right)(0) + \left(-U_o\frac{y}{L}\right)\left(-\frac{U_o}{L}\right) = \frac{U_o^2}{L^2}y$$

Therefore the resultant $\mathbf{a} = (U_o^2/L^2)(x\mathbf{i} + y\mathbf{j}) = (U_o^2/L^2)\mathbf{r}$ (purely radial) *Ans.* (a)

(b) For the given resultant acceleration of 25 m/s^2 at (x, y) = (1 m, 1 m), we obtain

$$|a| = 25\ \frac{m}{s^2} = \frac{U_o^2}{L^2}|r| = \frac{U_o^2}{(1.5\text{ m})^2}\sqrt{2}\text{ m}, \quad \text{solve for } U_o = \mathbf{6.3}\ \frac{\mathbf{m}}{\mathbf{s}} \quad \textit{Ans. (b)}$$

4.6 Assume that flow in the converging nozzle of Fig. P4.2 has the form $\mathbf{V} = V_o(1 + 2x/L)\mathbf{i}$. Compute (a) the fluid acceleration at $x = L$; and (b) the time required for a fluid particle to travel from $x = 0$ to $x = L$.

Solution: From Prob. 4.2, the general acceleration was computed to be

$$\frac{du}{dt} = u\frac{\partial u}{\partial x} = \frac{2V_o^2}{L}\left(1 + \frac{2x}{L}\right) = \frac{\mathbf{6V_o^2}}{\mathbf{L}} \quad at\ x = L \quad \textit{Ans. (a)}$$

(b) The trajectory of a fluid particle is computed from the fact that u = dx/dt:

$$u = \frac{dx}{dt} = V_o\left(1 + \frac{2x}{L}\right), \quad or: \quad \int_0^L \frac{dx}{1 + 2x/L} = \int_0^{\Delta t} V_o\,dt,$$

$$or: \quad \Delta t_{0-L} = \frac{\mathbf{L}}{\mathbf{2V_o}}\mathbf{ln(3)} \quad \textit{Ans. (b)}$$

4.7 Consider a sphere of radius R immersed in a uniform stream U_o, as shown in Fig. P4.7. According to the theory of Chap. 8, the fluid velocity along streamline AB is given by

$$\mathbf{V} = u\mathbf{i} = U_o\left(1+\frac{R^3}{x^3}\right)\mathbf{i}$$

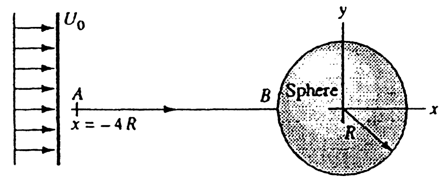


Fig. P4.7

Find (a) the position of maximum fluid acceleration along AB and (b) the time required for a fluid particle to travel from A to B.

Solution: (a) Along this streamline, the fluid acceleration is one-dimensional:

$$\frac{du}{dt} = u\frac{\partial u}{\partial x} = U_o(1+R^3/x^3)(-3U_oR^3/x^4) = -3U_oR^3(x^{-4}+R^3x^{-7}) \quad \text{for } x \le -R$$

The maximum occurs where $d(a_x)/dx = 0$, or at $x = -(7R^3/4)^{1/3} \approx \mathbf{-1.205R}$ *Ans.* (a)
(b) The time required to move along this path from A to B is computed from

$$u = \frac{dx}{dt} = U_o(1+R^3/x^3), \quad \text{or:} \quad \int_{-4R}^{-R}\frac{dx}{1+R^3/x^3} = \int_0^t U_o\,dt,$$

$$\text{or:} \quad U_o t = \left[x - \frac{R}{6}\ln\frac{(x+R)^2}{x^2-Rx+R^2} + \frac{R}{\sqrt{3}}\tan^{-1}\left(\frac{2x-R}{R\sqrt{3}}\right)\right]_{-4R}^{-R} = \infty$$

It takes **an infinite time** to actually *reach* the stagnation point, where the velocity is zero. *Ans.* (b)

4.8 When a valve is opened, fluid flows in the expansion duct of Fig. P4.8 according to the approximation

$$\mathbf{V} = \mathbf{i}U\left(1-\frac{x}{2L}\right)\tanh\frac{Ut}{L}$$

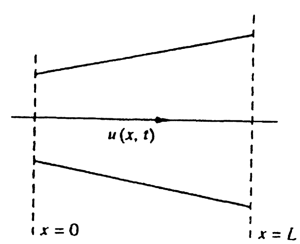


Fig. P4.8

Find (a) the fluid acceleration at $(x, t) = (L, L/U)$ and (b) the time for which the fluid acceleration at $x = L$ is zero. Why does the fluid acceleration become negative after condition (b)?

Solution: This is a one-dimensional *unsteady* flow. The acceleration is

$$a_x = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = U\left(1-\frac{x}{2L}\right)\frac{U}{L}\text{sech}^2\left(\frac{Ut}{L}\right) - U\left(1-\frac{x}{2L}\right)\left(\frac{U}{2L}\right)\tanh\left(\frac{Ut}{L}\right)$$

$$= \frac{U^2}{L}\left(1-\frac{x}{2L}\right)\left[\text{sech}^2\left(\frac{Ut}{L}\right) - \frac{1}{2}\tanh\left(\frac{Ut}{L}\right)\right]$$

At (x, t) = (L, L/U), $\mathbf{a_x} = (U^2/L)(1/2)[\text{sech}^2(1) - 0.5\tanh(1)] \approx \mathbf{0.0196U^2/L}$ *Ans.* (a)
The acceleration becomes zero when

$$\text{sech}^2\left(\frac{Ut}{L}\right) = \frac{1}{2}\tanh\left(\frac{Ut}{L}\right), \quad \text{or} \quad \frac{1}{2}\sinh\left(\frac{2Ut}{L}\right) = 2,$$

$$\text{or:} \quad \mathbf{\frac{Ut}{L} \approx 1.048} \quad \textit{Ans.}\text{ (b)}$$

The acceleration starts off positive, then goes through zero and turns negative as the negative *convective* acceleration overtakes the decaying positive *local* acceleration.

4.9 An idealized incompressible flow has the proposed three-dimensional velocity distribution

$$\mathbf{V} = 4xy^2\mathbf{i} + f(y)\mathbf{j} - zy^2\mathbf{k}$$

Find the appropriate form of the function $f(y)$ which satisfies the continuity relation.

Solution: Simply substitute the given velocity components into the incompressible continuity equation:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \frac{\partial}{\partial x}(4xy^2) + \frac{\partial f}{\partial y} + \frac{\partial}{\partial z}(-zy^2) = 4y^2 + \frac{df}{dy} - y^2 = 0$$

$$\textit{or:} \quad \frac{df}{dy} = -3y^2. \quad \text{Integrate: } f(y) = \int(-3y^2)dy = \mathbf{-y^3 + constant} \quad \textit{Ans.}$$

4.10 After discarding any constants of integration, determine the appropriate value of the unknown velocities u or v which satisfy the equation of two-dimensional incompressible continuity for:

$$\text{(a) } u = x^2y; \quad \text{(b) } v = x^2y; \quad \text{(c) } u = x^2 - xy; \quad \text{(d) } v = y^2 - xy$$

Solution: Substitute the given component into continuity and solve for the unknown component:

(a) $\dfrac{\partial u}{\partial x}+\dfrac{\partial v}{\partial y}=0=\dfrac{\partial}{\partial x}(x^2y)+\dfrac{\partial v}{\partial y};\ \dfrac{\partial v}{\partial y}=-2xy,$ ***or***: $\mathbf{v=-xy^2+f(x)}$ *Ans.* (a)

(b) $\dfrac{\partial u}{\partial x}+\dfrac{\partial v}{\partial y}=0=\dfrac{\partial u}{\partial x}+\dfrac{\partial}{\partial y}(x^2y);\ \dfrac{\partial u}{\partial x}=-x^2,$ ***or***: $\mathbf{u=-\dfrac{x^3}{3}+f(y)}$ *Ans.* (b)

(c) $\dfrac{\partial u}{\partial x}+\dfrac{\partial v}{\partial y}=0=\dfrac{\partial}{\partial x}(x^2-xy)+\dfrac{\partial v}{\partial y};\ \dfrac{\partial v}{\partial y}=-2x+y,$ ***or***: $\mathbf{v=-2xy+\dfrac{y^2}{2}+f(x)}$ *Ans.* (c)

(d) $\dfrac{\partial u}{\partial x}+\dfrac{\partial v}{\partial y}=0=\dfrac{\partial u}{\partial x}+\dfrac{\partial}{\partial y}(y^2-xy);\ \dfrac{\partial u}{\partial x}=-2y+x$ ***or***: $\mathbf{u=-2xy+\dfrac{x^2}{2}+f(y)}$ *Ans.* (d)

4.11 Derive Eq. (4.12*b*) for cylindrical coordinates by considering the flux of an incompressible fluid in and out of the elemental control volume in Fig. 4.2.

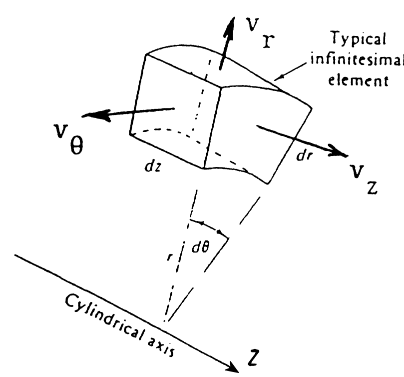


Fig. 4.2

Solution: For the differential CV shown,

$$\frac{\partial\rho}{\partial t}\,d\upsilon ol+\sum d\dot{m}_{out}-\sum d\dot{m}_{in}=0$$

$$\frac{\partial\rho}{\partial t}\left(r+\frac{dr}{2}\right)d\theta\,dr\,dz+\rho v_r r\,dz\,d\theta+\frac{\partial}{\partial r}(\rho v_r)dr(r+dr)dz\,d\theta+\rho v_\theta\,dz\,dr$$
$$+\frac{\partial}{\partial\theta}(\rho v_\theta)d\theta\,dz\,dr+\rho v_z\left(r+\frac{dr}{2}\right)d\theta\,dr+\frac{\partial}{\partial z}(\rho v_z)\left(r+\frac{dr}{2}\right)d\theta\,dr$$
$$-\rho v_r r\,dz\,d\theta-\rho v_\theta\,dz\,dr-\rho v_z\left(r+\frac{dr}{2}\right)d\theta\,dr=0$$

Cancel ($d\theta drdz$) and higher-order (4th-order) differentials such as ($drd\theta dzdr$) and, finally, divide by *r* to obtain the final result:

$$\boldsymbol{\frac{\partial\rho}{\partial t}+\frac{1}{r}\frac{\partial}{\partial r}(\rho r v_r)+\frac{1}{r}\frac{\partial}{\partial\theta}(\rho v_\theta)+\frac{\partial}{\partial z}(\rho v_z)=0}\quad \textit{Ans.}$$

4.12 Spherical polar coordinates (r, θ, ϕ) are defined in Fig. P4.12. The cartesian transformations are

$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta$$

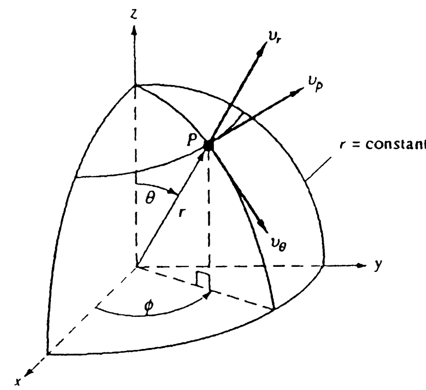


Fig. P4.12

Do not show that the cartesian incompressible continuity relation (4.12*a*) can be transformed to the spherical polar form

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\upsilon_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\upsilon_\theta\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(\upsilon_\phi) = 0$$

What is the most general form of υ_r when the flow is purely radial, that is, υ_θ and υ_ϕ are zero?

Solution: *Note to instructors*: **Do <u>not</u> assign the <u>derivation</u> part of this problem, it takes years to achieve, the writer can't do it successfully.** The problem is only meant to acquaint students with spherical coordinates. The second part is OK:

$$\text{If } \upsilon_\theta = \upsilon_\phi = 0, \quad \text{then } \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\upsilon_r) = 0, \quad \text{so, in general, } \boldsymbol{\upsilon_r = \frac{1}{r^2}\,\mathrm{fcn}(\theta,\ \phi)} \quad \textit{Ans.}$$

4.13 A two dimensional velocity field is given by

$$u = -\frac{Ky}{x^2+y^2} \qquad \upsilon = \frac{Kx}{x^2+y^2}$$

where K is constant. Does this field satisfy incompressible continuity? Transform these velocities to polar components υ_r and υ_θ. What might the flow represent?

Solution: <u>Yes</u>, continuity, $\partial u/\partial x + \partial v/\partial y = 0$, <u>is</u> satisfied. If you transform to polar coordinates, $x = r\cos\theta$ and $y = r\sin\theta$, you obtain

$$\boldsymbol{v_r = 0 \quad v_\theta = \frac{K}{r}}$$ which represents a *potential vortex* (see Section 4.10 of text). *Ans.*

4.14 For incompressible polar-coordinate flow, what is the most general form of a purely circulatory motion, $\upsilon_\theta = \upsilon_\theta(r, \theta, t)$ and $\upsilon_r = 0$, which satisfies continuity?

Solution: If $v_r = 0$, the plane polar coordinate continuity equation reduces to:

$$\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} = 0, \quad \text{or:} \quad \boldsymbol{v_\theta = \mathbf{fcn(r)}} \text{ only} \quad \textit{Ans.}$$

4.15 What is the most general form of a purely radial polar-coordinate incompressible-flow pattern, $\upsilon_r = \upsilon_r(r, \theta, t)$ and $\upsilon_\theta = 0$, which satisfies continuity?

Solution: If $v_\theta = 0$, the plane polar coordinate continuity equation reduces to:

$$\frac{1}{r}\frac{\partial}{\partial r}(rv_r) = 0, \quad \text{or:} \quad \boldsymbol{v_r = \frac{1}{r}\mathbf{fcn}(\theta)} \text{ only} \quad \textit{Ans.}$$

4.16 After discarding any constants of integration, determine the appropriate value of the unknown velocities w or v which satisfy the equation of three-dimensional incompressible continuity for:

$$\text{(a) } u = x^2yz, \quad v = -y^2x; \qquad \text{(b) } u = x^2 + 3z^2x, \quad w = -z^3 + y^2$$

Solution: Substitute into incompressible continuity and solve for the unknown component:

$$\text{(a) } \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 = \frac{\partial}{\partial x}(x^2yz) + \frac{\partial}{\partial y}(-y^2x) + \frac{\partial w}{\partial z}; \quad \frac{\partial w}{\partial z} = -2xyz + 2yx,$$

$$\textbf{\textit{or}}\textbf{:} \quad \boldsymbol{w = -xyz^2 + 2xyz} \quad \textit{Ans.} \text{ (a)}$$

$$\text{(b) } \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 = \frac{\partial}{\partial x}(x^2 + 3z^2x) + \frac{\partial v}{\partial y} + \frac{\partial}{\partial z}(-z^3 + y^2); \quad \frac{\partial v}{\partial y} = -2x - 3z^2 + 3z^2$$

$$\textbf{\textit{or}}\textbf{:} \quad \boldsymbol{v = -2xy} \quad \textit{Ans.} \text{ (b)}$$

4.17 A reasonable approximation for the two-dimensional incompressible laminar boundary layer on the flat surface in Fig. P4.17 is

$$u = U\left(\frac{2y}{\delta} - \frac{y^2}{\delta^2}\right) \quad \text{for } y \le \delta$$

where $\delta \approx Cx^{1/2}$, $C = \text{const}$

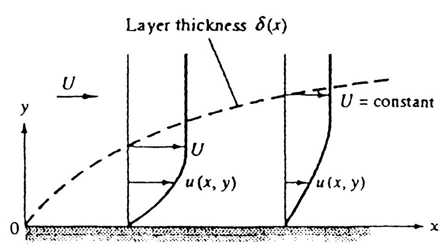


Fig. P4.17

(a) Assuming a no-slip condition at the wall, find an expression for the velocity component $v(x, y)$ for $y \le \delta$. (b) Then find the maximum value of v at the station $x = 1$ m, for the particular case of airflow, when $U = 3$ m/s and $\delta = 1.1$ cm.

Solution: The two-dimensional incompressible continuity equation yields

$$\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x} = -U\left(\frac{-2y}{\delta^2}\frac{d\delta}{dx} + \frac{2y^2}{\delta^3}\frac{d\delta}{dx}\right), \quad or: \quad v = 2U\frac{d\delta}{dx}\int_0^y \left(\frac{y}{\delta^2} - \frac{y^2}{\delta^3}\right) dy\Big|_{x=const}$$

$$or: \quad \mathbf{v = 2U\frac{d\delta}{dx}\left(\frac{y^2}{2\delta^2} - \frac{y^3}{3\delta^3}\right)}, \quad where \quad \mathbf{\frac{d\delta}{dx} = \frac{C}{2\sqrt{x}} = \frac{\delta}{2x}} \quad Ans.\ (a)$$

(b) We see that v increases monotonically with y, thus $v_{\max}$ occurs at y = δ:

$$v_{max} = v|_{y=\delta} = \frac{\mathbf{U\delta}}{\mathbf{6x}} = \frac{(3\ m/s)(0.011\ m)}{6(1\ m)} = \mathbf{0.0055\ \frac{m}{s}} \quad Ans.\ (b)$$

This estimate is within 4% of the exact $v_{\max}$ computed from boundary layer theory.

4.18 A piston compresses gas in a cylinder by moving at constant speed V, as in Fig. P4.18. Let the gas density and length at $t = 0$ be ρ_o and L_o, respectively. Let the gas velocity vary linearly from $u = V$ at the piston face to $u = 0$ at $x = L$. If the gas density varies only with time, find an expression for $\rho(t)$.

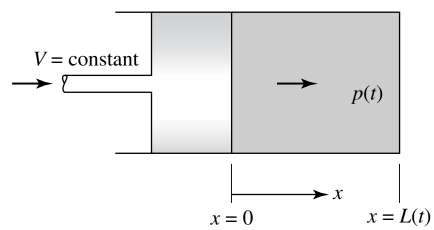


Fig. P4.18

Solution: The one-dimensional unsteady continuity equation reduces to

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = \frac{d\rho}{dt} + \rho\frac{\partial u}{\partial x}, \quad \text{where}\ \ u = V\left(1 - \frac{x}{L}\right), \quad L = L_o - Vt, \quad \rho = \rho(t)\ \text{only}$$

$$\text{Enter} \quad \frac{\partial u}{\partial x} = -\frac{V}{L} \quad \text{and separate variables:} \quad \int_{\rho_o}^{\rho} \frac{d\rho}{\rho} = V\int_o^t \frac{dt}{L_o - Vt}$$

$$\text{The solution is} \quad \ln(\rho/\rho_o) = -\ln(1 - Vt/L_o), \quad \text{or:} \quad \rho = \rho_o\left(\frac{\mathbf{L_o}}{\mathbf{L_o - Vt}}\right) \quad Ans.$$

4.19 An incompressible flow field has the cylinder components $\upsilon_\theta = Cr$, $\upsilon_z = K(R^2 - r^2)$, $\upsilon_r = 0$, where C and K are constants and $r \le R$, $z \le L$. Does this flow satisfy continuity? What might it represent physically?

Solution: We check the incompressible continuity relation in cylindrical coordinates:

$$\frac{1}{r}\frac{\partial}{\partial r}(rv_r) + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} = 0 = 0 + 0 + 0 \quad \textbf{satisfied identically} \quad \textit{Ans.}$$

This flow also satisfies (cylindrical) momentum and could represent laminar flow inside a tube of radius R whose outer wall (r = R) is rotating at uniform angular velocity.

4.20 A two-dimensional incompressible velocity field has $u = K(1 - e^{-ay})$, for $x \le L$ and $0 \le y \le \infty$. What is the most general form of $v(x, y)$ for which continuity is satisfied and $v = v_o$ at $y = 0$? What are the proper dimensions for constants K and a?

Solution: We can find the appropriate velocity v from two-dimensional continuity:

$$\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x} = -\frac{\partial}{\partial x}[K(1 - e^{-ay})] = 0, \quad \text{or:} \quad v = \text{fcn}(x) \text{ only}$$

Since $v = v_o$ at y = 0 for all x, then it must be that $\boldsymbol{v = v_o = \text{const}}$ *Ans.*

The dimensions of K are **{K} = {L/T}** and the dimensions of a are **{L^{-1}}**. *Ans.*

4.21 Air flows under steady, approximately one-dimensional conditions through the conical nozzle in Fig. P4.21. If the speed of sound is approximately 340 m/s, what is the minimum nozzle-diameter ratio D_e/D_o for which we can safely neglect compressibility effects if V_o = (a) 10 m/s and (b) 30 m/s?

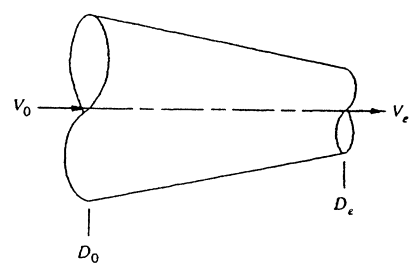


Fig. P4.21

Solution: If we apply one-dimensional continuity to this duct,

$$\rho_o V_o \frac{\pi}{4} D_o^2 = \rho_e V_e \frac{\pi}{4} D_e^2, \quad \text{or} \quad V_o \approx V_e (D_e/D_o)^2 \quad \text{if} \quad \rho_o \approx \rho_e$$

To avoid compressibility corrections, we require (Eq. 4.18) that Ma ≤ 0.3 or, in this case, the highest velocity (at the exit) should be $V_e \le 0.3(340) = 102$ m/s. Then we compute

$$(D_e/D_o)_{min} = (V_o/V_e)^{1/2} = (V_o/102)^{1/2} = \mathbf{0.31} \quad \text{if} \quad V_o = 10 \text{ m/s} \quad \textit{Ans. (a)}$$

$$= \mathbf{0.54} \quad \text{if} \quad V_o = 30 \text{ m/s} \quad \textit{Ans. (b)}$$

4.22 A flow field in the *x*-*y* plane is described by $u = U_o$ = constant, $v = V_o$ = constant. Convert these velocities into plane polar coordinate velocities, v_r and v_θ.

Solution: Each pair of components must add to give the total velocity, as seen in the sketch at right.

The geometry of the figure shows that

$$\boldsymbol{v_r = U_o \cos\theta + V_o \sin\theta;}$$
$$\boldsymbol{v_\theta = -U_o \sin\theta + V_o \cos\theta} \quad \textit{Ans.}$$

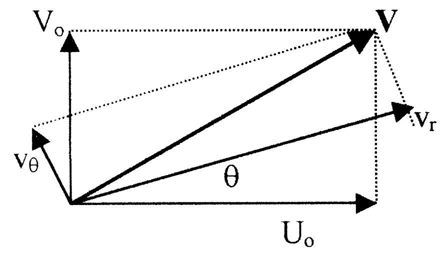


Fig. P4.22

4.23 A tank volume *V* contains gas at conditions (ρ_0, p_0, T_0). At time $t = 0$ it is punctured by a small hole of area *A*. According to the theory of Chap. 9, the mass flow out of such a hole is approximately proportional to *A* and to the tank pressure. If the tank temperature is assumed constant and the gas is ideal, find an expression for the variation of density within the tank.

Solution: This problem is a realistic approximation of the "blowdown" of a high-pressure tank, where the exit mass flow is choked and thus proportional to tank pressure. For a control volume enclosing the tank and cutting through the exit jet, the mass relation is

$$\frac{d}{dt}(m_{tank}) + \dot{m}_{exit} = 0, \quad \text{or:} \quad \frac{d}{dt}(\rho\upsilon) = -\dot{m}_{exit} = -CpA, \quad \text{where } C = \text{constant}$$

$$\text{Introduce} \quad \rho = \frac{p}{RT_o} \quad \text{and separate variables:} \quad \int_{p_o}^{p(t)} \frac{dp}{p} = -\frac{CRT_oA}{\upsilon}\int_o^t dt$$

The solution is an exponential decay of tank density: $\mathbf{p = p_o \exp(-CRT_oAt/\upsilon)}$. *Ans.*

4.24 Reconsider Fig. P4.17 in the following general way. It is known that the boundary layer thickness $\delta(x)$ increases monotonically and that there is no slip at the wall ($y = 0$). Further, $u(x, y)$ merges smoothly with the outer stream flow, where $u \approx U$ = constant outside the layer. Use these facts to prove that (a) the component $\upsilon(x, y)$ is positive everywhere within the layer, (b) υ increases parabolically with *y* very near the wall, and (c) υ is a maximum at $y = \delta$.

Solution: (a) First, if δ is continually increasing with x, then u is continually *decreasing* with x in the boundary layer, that is, $\partial u/\partial x < 0$, hence $\partial v/\partial y = -\partial u/\partial x > \mathbf{0}$ everywhere. It follows that, if $\partial v/\partial y > 0$ and $v = 0$ at $y = 0$, then $\mathbf{v(x, y) > 0}$ **for all** $\mathbf{y \le \delta}$. *Ans.* (a)

(b) At the wall, *u* must be approximately linear with y, if $\tau_w \geq 0$:

$$\text{Near wall: } u \approx y\,f(x), \quad \text{or} \quad \frac{\partial u}{\partial x} = y\frac{df}{dx}, \quad \text{where} \quad \frac{df}{dx} < 0. \quad \text{Then} \quad \frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x} = \left(\frac{df}{dx}\right)y$$

$$\text{Thus, near the wall,} \quad v \approx \left(\frac{df}{dx}\right)\int_0^y y\,dy \approx \left(\frac{df}{dx}\right)\frac{y^2}{2} \quad \textbf{Parabolic} \quad \textit{Ans.}\text{ (b)}$$

(c) At y = δ, u → U, then $\partial u/\partial x \approx 0$ there and thus $\partial v/\partial y \approx 0$, or $\mathbf{v = v_{max}}$. *Ans.* (c)

4.25 An incompressible flow in polar coordinates is given by

$$v_r = K\cos\theta\left(1 - \frac{b}{r^2}\right)$$

$$v_\theta = -K\sin\theta\left(1 + \frac{b}{r^2}\right)$$

Does this field satisfy continuity? For consistency, what should the dimensions of constants *K* and *b* be? Sketch the surface where $v_r = 0$ and interpret.

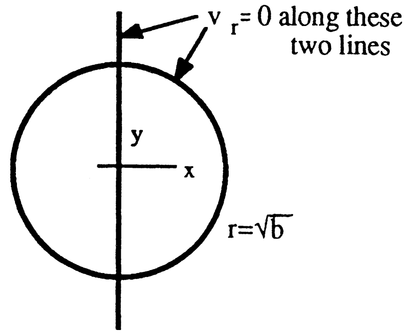


Fig. P4.25

Solution: Substitute into plane polar coordinate continuity:

$$\frac{1}{r}\frac{\partial}{\partial r}(rv_r) + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} = 0 \stackrel{?}{=} \frac{1}{r}\frac{\partial}{\partial r}\left[K\cos\theta\left(r - \frac{b}{r}\right)\right] + \frac{1}{r}\frac{\partial}{\partial \theta}\left[-K\sin\theta\left(1 + \frac{b}{r^2}\right)\right] = 0 \quad \textit{Satisfied}$$

The dimensions of *K* <u>must</u> be velocity, {K} = {L/T}, and *b* <u>must</u> be area, {b} = {L^2}. The surfaces where $v_r = 0$ are the y-axis and the circle r = √b, as shown above. The pattern represents inviscid flow of a uniform stream past a circular cylinder (Chap. 8).

4.26 Curvilinear, or streamline, coordinates are defined in Fig. P4.26, where *n* is normal to the streamline in the plane of the radius of curvature *R*. Show that Euler's frictionless momentum equation (4.36) in streamline coordinates becomes

$$\frac{\partial V}{\partial t} + V\frac{\partial V}{\partial s} = -\frac{1}{\rho}\frac{\partial p}{\partial s} + g_s \qquad (1)$$

$$-V\frac{\partial \theta}{\partial t} - \frac{V^2}{R} = -\frac{1}{\rho}\frac{\partial p}{\partial n} + g_n \qquad (2)$$

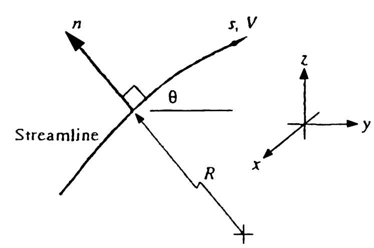


Fig. P4.26

Further show that the integral of Eq. (1) with respect to *s* is none other than our old friend Bernoulli's equation (3.76).

Solution: This a laborious derivation, really, **the problem is only meant to *acquaint* the student with streamline coordinates.** The second part is not too hard, though. Multiply the streamwise momentum equation by *ds* and integrate:

$$\frac{\partial V}{\partial t}ds + V\,dV = -\frac{dp}{\rho_2} + g_s\,ds = -\frac{dp}{\rho} - g\sin\theta\,ds = -\frac{dp}{\rho} - g\,dz$$

Integrate from 1 to 2: $\displaystyle \mathbf{\int_1^2 \frac{\partial V}{\partial t}ds + \frac{V_2^2 - V_1^2}{2} + \int_1^2 \frac{dp}{\rho} + g(z_2 - z_1) = 0}$ (Bernoulli) *Ans.*

4.27 A frictionless, incompressible steady-flow field is given by

$$\mathbf{V} = 2xy\mathbf{i} - y^2\mathbf{j}$$

in arbitrary units. Let the density be ρ_0 = constant and neglect gravity. Find an expression for the pressure gradient in the *x* direction.

Solution: For this (gravity-free) velocity, the momentum equation is

$$\rho\left(u\frac{\partial \mathbf{V}}{\partial x} + v\frac{\partial \mathbf{V}}{\partial y}\right) = -\nabla p, \quad \text{or:} \quad \rho_o[(2xy)(2y\mathbf{i}) + (-y^2)(2x\mathbf{i} - 2y\mathbf{j})] = -\nabla p$$

$$\text{Solve for} \quad \nabla p = -\rho_o(2xy^2\mathbf{i} + 2y^3\mathbf{j}), \quad \text{or:} \quad \mathbf{\frac{\partial p}{\partial x} = -\rho_o 2xy^2} \quad \textit{Ans.}$$

4.28 If *z* is "up," what are the conditions on constants *a* and *b* for which the velocity field $u = ay$, $v = bx$, $w = 0$ is an exact solution to the continuity and Navier-Stokes equations for incompressible flow?

Solution: First examine the continuity equation:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \stackrel{?}{=} 0 = \frac{\partial}{\partial x}(ay) + \frac{\partial}{\partial y}(bx) + \frac{\partial}{\partial z}(0) = 0 + 0 + 0 \quad \text{for all } a \text{ and } b$$

Given $g_x = g_y = 0$ and $w = 0$, we need only examine *x*- and *y*-momentum:

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = \rho[(ay)(0) + (bx)(a)] = -\frac{\partial p}{\partial x} + \mu(0+0)$$

$$\rho\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = -\frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) = \rho[(ay)(b) + (bx)(0)] = -\frac{\partial p}{\partial y} + \mu(0+0)$$

Solve for $\dfrac{\partial p}{\partial x} = -\rho abx$ and $\dfrac{\partial p}{\partial y} = -\rho aby$, or: $\mathbf{p = -\dfrac{\rho}{2}ab(x^2 + y^2) + const}$ *Ans.*

The given velocity field, u = ay and v = bx, is an exact solution independent of a or b. It is not, however, an "irrotational" flow.

4.29 Consider a steady, two-dimensional, incompressible flow of a newtonian fluid with the velocity field u = –2xy, v = y^2 – x^2, and w = 0. (a) Does this flow satisfy conservation of mass? (b) Find the pressure field p(x, y) if the pressure at point (x = 0, y = 0) is equal to p_a.

Solution: Evaluate and check the incompressible continuity equation:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 = -2y + 2y + 0 \equiv 0 \quad \textbf{Yes!} \quad \textit{Ans.}\ \text{(a)}$$

(b) Find the pressure gradients from the Navier-Stokes x- and y-relations:

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right), \quad \textit{or:}$$

$$\rho[-2xy(-2y) + (y^2 - x^2)(-2x)] = -\frac{\partial p}{\partial x} + \mu(0 + 0 + 0), \quad \textit{or:} \quad \frac{\partial p}{\partial x} = -2\rho(xy^2 + x^3)$$

and, similarly for the y-momentum relation,

$$\rho\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) = -\frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right), \quad \textit{or:}$$

$$\rho[-2xy(-2x) + (y^2 - x^2)(2y)] = -\frac{\partial p}{\partial y} + \mu(-2 + 2 + 0), \quad \textit{or:} \quad \frac{\partial p}{\partial y} = -2\rho(x^2 y + y^3)$$

The two gradients $\partial p/\partial x$ and $\partial p/\partial y$ may be integrated to find p(x, y):

$$p = \int \frac{\partial p}{\partial x}dx\Big|_{y=Const} = -2\rho\left(\frac{x^2y^2}{2} + \frac{x^4}{4}\right) + f(y), \quad \textit{then differentiate:}$$

$$\frac{\partial p}{\partial y} = -2\rho(x^2 y) + \frac{df}{dy} = -2\rho(x^2 y + y^3), \quad \textit{whence} \ \frac{df}{dy} = -2\rho y^3, \quad \textit{or:} \quad f(y) = -\frac{\rho}{2}y^4 + C$$

$$\textit{Thus:} \quad p = -\frac{\rho}{2}(2x^2y^2 + x^4 + y^4) + C = p_a \quad \textit{at}\ (x, y) = (0, 0), \quad \textit{or:} \quad \mathbf{C = p_a}$$

Finally, the pressure field for this flow is given by

$$\mathbf{p = p_a - \frac{1}{2}\rho(2x^2y^2 + x^4 + y^4)} \quad \textit{Ans.}\ \text{(b)}$$

4.30 Show that the two-dimensional flow field of Example 1.10 is an exact solution to the incompressible Navier-Stokes equation. Neglecting gravity, compute the pressure field p(x, y) and relate it to the absolute velocity $V^2 = u^2 + v^2$. Interpret the result.

Solution: In Example 1.10, the velocities were u = Kx, v = –Ky, w = 0, K = constant. This flow satisfies continuity identically. Let us try it in the two momentum equations:

$$\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=\rho[(Kx)K+0]=-\frac{\partial p}{\partial x}+\mu\nabla^2 u=-\frac{\partial p}{\partial x}+0,\quad \text{or:}\quad \frac{\partial p}{\partial x}=-\rho K^2 x$$

$$\rho\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)=\rho[0+(-Ky)(-K)]=-\frac{\partial p}{\partial y}+\mu\nabla^2 v=-\frac{\partial p}{\partial y}+0,\quad \text{or:}\quad \frac{\partial p}{\partial y}=-\rho K^2 y$$

Integrate the two pressure gradients to obtain

$$p=-\frac{\rho}{2}[(Kx)^2+(Ky)^2]+\text{const},\quad \text{or:}\quad \mathbf{p+\frac{1}{2}\rho(u^2+v^2)=const}\quad \textit{Ans.}$$

The given velocity is an exact solution and the pressure satisfies Bernoulli's equation.

4.31 According to potential theory (Chap. 8) for the flow approaching a rounded two-dimensional body, as in Fig. P4.31, the velocity approaching the stagnation point is given by $u = U(1 - a^2/x^2)$, where a is the nose radius and U is the velocity far upstream. Compute the value and position of the maximum viscous normal stress along this streamline. Is this also the position of maximum fluid deceleration? Evaluate the maximum viscous normal stress if the fluid is SAE 30 oil at 20°C, with $U = 2$ m/s and $a = 6$ cm.

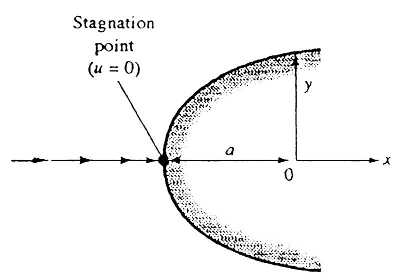


Fig. P4.31

Solution: (a) Along this line of symmetry the convective deceleration is one-dimensional:

$$a_x = u\frac{\partial u}{\partial x}=U\left(1-\frac{a^2}{x^2}\right)U\left(\frac{2a^2}{x^3}\right)=2U^2\left(\frac{a^2}{x^3}-\frac{a^4}{x^5}\right)$$

This has a maximum deceleration at $\frac{da_x}{dx}=0$, or at $\mathbf{x} = -\sqrt{(5/3)}\,a = \mathbf{-1.29a}$ *Ans.* (a)

The value of maximum deceleration at this point is $a_{x,max} = -0.372U^2/a$.

(b) The viscous normal stress along this line is given by

$$\tau_{xx} = 2\mu\frac{\partial u}{\partial x} = 2\mu\left(\frac{2a^2U}{x^3}\right) \text{ with a maximum } \quad \boldsymbol{\tau_{max} = \frac{4\mu U}{a}} \textbf{ at } \mathbf{x = -a} \quad \textit{Ans. (b)}$$

Thus maximum stress does not occur at the same position as maximum deceleration. For SAE 30 oil at 20°C, we obtain the numerical result

$$\text{SAE 30 oil,} \quad \rho = 917\ \frac{kg}{m^3}, \quad \mu = 0.29\ \frac{kg}{m{\cdot}s}, \quad \tau_{max} = \frac{4(0.29)(2.0)}{(0.06\text{ m})} \approx \mathbf{39\ Pa} \quad \textit{Ans. (b)}$$

4.32 The answer to Prob. 4.14 is $\upsilon_\theta = f(r)$ only. Do not reveal this to your friends if they are still working on Prob. 4.14. Show that this flow field is an exact solution to the Navier-Stokes equations (4.38) for only two special cases of the function $f(r)$. Neglect gravity. Interpret these two cases physically.

Solution: Given $v_\theta = f(r)$ and $v_r = v_z = 0$, we need only satisfy the θ-momentum relation:

$$\rho\left(v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}\right) = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_\theta}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2} - \frac{v_\theta}{r^2}\right],$$

$$\text{or:} \quad \rho(0+0) = -0 + \mu\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{df}{dr}\right) + 0 - \frac{f}{r^2}\right], \quad \text{or:} \quad \mathbf{f'' + \frac{1}{r}f' - \frac{1}{r^2}f = 0}$$

This is the 'equidimensional' ODE and always has a solution in the form of a *power-law*, $f = Cr^n$. The two relevant solutions for these particular coefficients are:

$\mathbf{f_1 = C_1 r}$ (solid-body rotation); $\mathbf{f_2 = C_2/r}$ (irrotational vortex) *Ans.*

4.33 From Prob. 4.15 the purely radial polar-coordinate flow which satisfies continuity is $\upsilon_r = f(\theta)/r$, where f is an arbitrary function. Determine what particular forms of $f(\theta)$ satisfy the full Navier-Stokes equations in polar-coordinate form from Eqs. (D.5) and (D.6).

Solution: To resolve this solution, we must substitute into both the r- and θ-momentum relations from Appendix D. The results, assuming $v_r = f(\theta)/r$, are:

$$\text{r-mom:} \quad \frac{\partial p}{\partial r} = \frac{\rho}{r^3}f^2 + \frac{\mu}{r^3}f''; \quad \theta\text{-momentum:} \quad \frac{\partial p}{\partial \theta} = \frac{2\mu}{r^2}f'.$$

Cross-differentiate to eliminate $\partial^2 p/\partial x\partial y$ and obtain the ODE $\quad \boldsymbol{\mu f''' + 2\rho f f' + 4\mu f' = 0}$

This ODE has two types of solution, one very simple and one very complicated:

(1) f = constant, or: $\mathbf{v_r} = \dfrac{\mathbf{const}}{\mathbf{r}}$ (a line source, as in Chap. 4) *Ans.*

(2) **Elliptic-integral solution** to the complete ODE above: these solutions, which vary in many ways with θ, represent "Jeffrey-Hamel" flow between plates. *Ans.*

For further discussion of "Jeffrey-Hamel" flow, see pp. 168–172 of Ref. 5, Chap. 4.

4.34 A proposed three-dimensional incompressible flow field has the following vector form:

$$\mathbf{V} = Kx\mathbf{i} + Ky\mathbf{j} - 2Kz\mathbf{k}$$

(a) Determine if this field is a valid solution to continuity and Navier-Stokes. (b) If $\mathbf{g} = -g\mathbf{k}$, find the pressure field $p(x, y, z)$. (c) Is the flow irrotational?

Solution: (a) Substitute this field into the three-dimensional incompressible continuity equation:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=\frac{\partial}{\partial x}(Kx)+\frac{\partial}{\partial y}(Ky)+\frac{\partial}{\partial z}(-2Kz)$$

$$= K + K - 2K = 0 \quad \textbf{Yes, satisfied}. \quad \textit{Ans.}\text{ (a)}$$

(b) Substitute into the full incompressible Navier-Stokes equation (4.38). The laborious results are:

$$\text{x} - \text{momentum:} \quad \rho(K^2x+0+0) = -\frac{\partial p}{\partial x} + \mu(0+0+0)$$

$$\text{y} - \text{momentum:} \quad \rho(0+K^2y+0) = -\frac{\partial p}{\partial y} + \mu(0+0+0)$$

$$\text{z} - \text{momentum:} \quad \rho\{0+0+(-2Kz)(-2K)\} = -\frac{\partial p}{\partial z} + \rho(-g) + \mu(0+0+0)$$

Integrate each equation for the pressure and collect terms. The result is

$$\boldsymbol{p = p(0,0,0) - \rho gz - (\rho/2)K^2(x^2 + y^2 + 4z^2)} \quad \textit{Ans.}\text{ (b)}$$

Note that the last term is identical to $(\rho/2)(u^2 + v^2 + w^2)$, in other words, Bernoulli's equation.

(c) For irrotational flow, the curl of the velocity field must be zero:

$$\nabla \times \mathbf{V} = \mathbf{i}(0-0) + \mathbf{j}(0-0) + \mathbf{k}(0-0) = \mathbf{0} \quad \textbf{Yes, irrotational}. \quad \textit{Ans.}\text{ (c)}$$

4.35 From the Navier-Stokes equations for incompressible flow in polar coordinates (App. E for cylindrical coordinates), find the most general case of purely circulating motion $\upsilon_\theta(r)$, $\upsilon_r = \upsilon_z = 0$, for flow with no slip between two fixed concentric cylinders, as in Fig. P4.35.

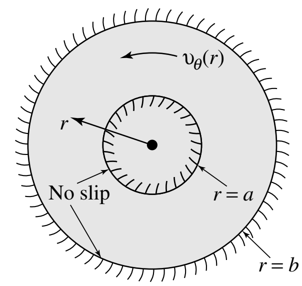


Fig. P4.35

Solution: The preliminary work for this problem is identical to Prob. 4.32 on the previous page. That is, there are two possible solutions for purely circulating motion υ_θ(r), hence

$$v_\theta = C_1 r + \frac{C_2}{r}, \quad \text{subject to } v_\theta(a) = 0 = C_1 a + C_2/a \quad \text{and} \quad v_\theta(b) = 0 = C_1 b + C_2/b$$

This requires $\mathbf{C_1 = C_2 = 0}$, or $\mathbf{v_\theta = 0}$ (no steady motion possible between fixed walls) *Ans.*

4.36 A constant-thickness film of viscous liquid flows in laminar motion down a plate inclined at angle θ, as in Fig. P4.36. The velocity profile is

$$u = Cy(2h - y) \qquad v = w = 0$$

Find the constant C in terms of the specific weight and viscosity and the angle θ. Find the volume flux Q per unit width in terms of these parameters.

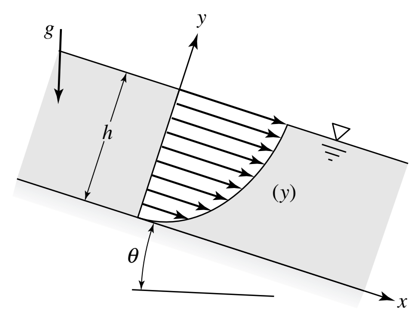


Fig. P4.36

Solution: There is atmospheric pressure all along the surface at y = h, hence $\partial p/\partial x = 0$. The x-momentum equation can easily be evaluated from the known velocity profile:

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + \rho g_x + \mu\nabla^2 u, \quad \text{or:} \quad 0 = 0 + \rho g\sin\theta + \mu(-2C)$$

$$\text{Solve for} \quad \mathbf{C = \frac{\rho g \sin\theta}{2\mu}} \quad \textit{Ans. (a)}$$

The flow rate per unit width is found by integrating the velocity profile and using C:

$$Q = \int_0^h u\,dy = \int_0^h Cy(2h - y)\,dy = \frac{2}{3}Ch^3 = \mathbf{\frac{\rho g h^3 \sin\theta}{3\mu}} \text{ per unit width} \quad \textit{Ans. (b)}$$

4.37 A viscous liquid of constant density and viscosity falls due to gravity between two parallel plates a distance 2h apart, as in the figure. The flow is fully developed, that is, w = w(x) only. There are no pressure gradients, only gravity. Set up and solve the Navier-Stokes equation for the velocity profile w(x).

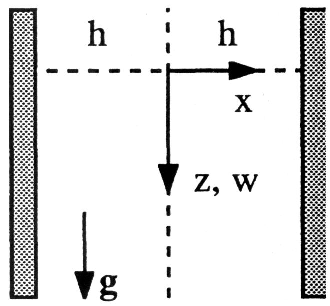


Fig. P4.37

Solution: Only the z-component of Navier-Stokes is relevant:

$$\rho\frac{dw}{dt} = 0 = \rho g + \mu\frac{d^2w}{dx^2}, \quad or: \quad w'' = -\frac{\rho g}{\mu}, \; w(-h) = w(+h) = 0 \;\; (no\text{-}slip)$$

The solution is very similar to Eqs. (4.142) to (4.143) of the text:

$$w = \frac{\boldsymbol{\rho g}}{\boldsymbol{2\mu}}(\mathbf{h}^2 - \mathbf{x}^2) \quad Ans.$$

4.38 Reconsider the angular-momentum balance of Fig. 4.5 by adding a concentrated *body couple* C_z about the *z* axis [6]. Determine a relation between the body couple and shear stress for equilibrium. What are the proper dimensions for C_z? (Body couples are important in continuous media with microstructure, such as granular materials.)

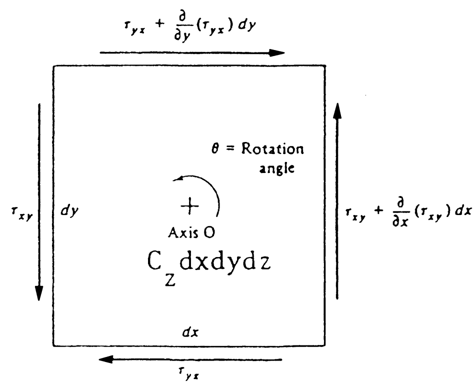


Fig. 4.5

Solution: The couple C_z has to be per unit volume to make physical sense in Eq. (4.39):

$$\left[\tau_{xy} - \tau_{yx} + \frac{1}{2}\frac{\partial\tau_{xy}}{\partial x}dx - \frac{1}{2}\frac{\partial\tau_{yx}}{\partial y}dy\right]dx\,dy\,dz + C_z dx\,dy\,dz = \frac{1}{12}\rho\,dx\,dy\,dz(dx^2 + dy^2)\frac{d^2\theta}{dt^2}$$

Reduce to third order terms and cancel (dx dy dz): $\boldsymbol{\tau_{yx} - \tau_{xy} = C_z}$ *Ans.*
The concentrated couple allows the stress tensor to have unsymmetric shear stress terms.

4.39 Problems involving viscous dissipation of energy are dependent on viscosity μ, thermal conductivity k, stream velocity U_o, and stream temperature T_o. Group these parameters into the dimensionless *Brinkman number*, which is proportional to μ.

Solution: Using the primary dimensions as mass M, length L, and time T, we obtain

$$Br = fcn(\mu,\ k,\ U_o,\ T_o) = \mu k^a U_o^b T_o^c = \left\{\frac{M}{LT}\right\}\left\{\frac{ML}{T^3\Theta}\right\}^a\left\{\frac{L}{T}\right\}^b\{\Theta\}^c = M^0L^0T^0\Theta^0$$

$$\text{Solve for}\quad a = c = -1\quad\text{and}\quad b = +2.\quad\text{Hence:}\quad \mathbf{Br} = \frac{\boldsymbol{\mu} \mathbf{U_o^2}}{\mathbf{kT_o}}\quad \textit{Ans.}$$

4.40 As mentioned in Sec. 4.11, the velocity profile for laminar flow between two plates, as in Fig. P4.40, is

$$u = \frac{4u_{max}y(h-y)}{h^2}\qquad v = w = 0$$

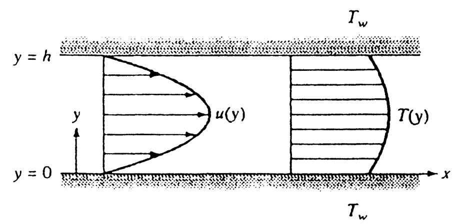


Fig. P4.40

If the wall temperature is T_w at both walls, use the incompressible-flow energy equation (4.75) to solve for the temperature distribution $T(y)$ between the walls for steady flow.

Solution: Assume T = T(y) and use the energy equation with the known u(y):

$$\rho c_p\frac{DT}{dt} = k\frac{d^2T}{dy^2} + \mu\left(\frac{du}{dy}\right)^2,\quad\text{or:}\quad \rho c_p(0) = k\frac{d^2T}{dy^2} + \mu\left[\frac{4u_{max}}{h^2}(h-2y)\right]^2,\quad\text{or:}$$

$$\frac{d^2T}{dy^2} = -\frac{16\mu u_{max}^2}{kh^4}(h^2 - 4hy + 4y^2),\quad\text{Integrate:}\quad \frac{dT}{dy} = \frac{-16\mu u_{max}^2}{kh^4}\left(h^2y - 2hy^2 + \frac{4y^3}{3} + C_1\right)$$

Before integrating again, note that dT/dy = 0 at y = h/2 (the symmetry condition), so $C_1 = -h^3/6$. Now integrate once more:

$$T = -\frac{16\mu u_{max}^2}{kh^4}\left(h^2\frac{y^2}{2} - 2h\frac{y^3}{3} + \frac{y^4}{3} + C_1 y\right) + C_2$$

If $T = T_w$ at y = 0 and at y = h, then $C_2 = T_w$. The final solution is:

$$\mathbf{T} = \mathbf{T_w} + \frac{\mathbf{8}\boldsymbol{\mu}\mathbf{u_{max}^2}}{\mathbf{k}}\left[\frac{\mathbf{y}}{\mathbf{3h}} - \frac{\mathbf{y^2}}{\mathbf{h^2}} + \frac{\mathbf{4y^3}}{\mathbf{3h^3}} - \frac{\mathbf{2y^4}}{\mathbf{3h^4}}\right]\quad \textit{Ans.}$$

4.41 The approximate velocity profile in Prob. 3.18 for steady laminar flow through a rectangular duct,

$$u = u_{max}[1 - (y/b)^2][1 - (z/h)^2]$$

satisfies the no-slip condition and gave a reasonable volume flow estimate (which was the point of Prob. 3.18). Show, however, that it does not satisfy the Navier-Stokes equation for duct flow with $\partial p/\partial x$ equal to a negative constant. Extra credit: the EXACT solution!

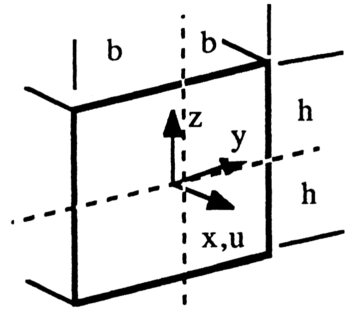


Fig. P4.41

Solution: The x-component of Navier-Stokes for fully-developed flow is

$$\mu\nabla^2 u(x, y) = \frac{\partial p}{\partial x} = const < 0 \quad \textit{(no acceleration, neglect gravity)}$$

But, in fact, the Laplacian of the above "u" approximation is NOT constant:

$$\nabla^2 u_{approx} = -\frac{2u_{max}}{b^2}\left(1 - \frac{z^2}{h^2}\right) - \frac{2u_{max}}{h^2}\left(1 - \frac{y^2}{b^2}\right) \neq \textit{constant} \quad \textit{Ans.}$$

It is *nearly* constant in the duct center (small y, z) but it is **not** exact. See Ref. 5, Ch. 4.

4.42 Suppose that we wish to analyze the rotating, partly-full cylinder of Fig. 2.23 as a *spin-up* problem, starting from rest and continuing until solid-body-rotation is achieved. What are the appropriate boundary and initial conditions for this problem?

Solution: Let $V = V(r, z, t)$. The initial condition is: $V(r, z, 0) = 0$. The boundary conditions are

Along the side walls: $v_\theta(R, z, t) = R\Omega$, $v_r(R, z, t) = 0$, $v_z(R, z, t) = 0$.

At the bottom, z = 0: $v_\theta(r, 0, t) = r\Omega$, $v_r(r, 0, t) = 0$, $v_z(r, 0, t) = 0$.

At the free surface, $z = \eta$: $p = p_{atm}$, $\tau_{rz} = \tau_{\theta z} = 0$.

4.43 For the draining liquid film of Fig. P4.36, what are the appropriate boundary conditions (a) at the bottom $y = 0$ and (b) at the surface $y = h$?

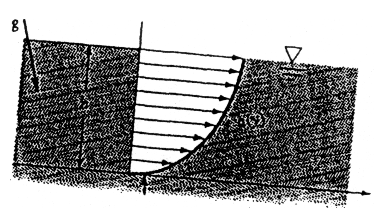


Fig. P4.36

Solution: The physically realistic conditions at the upper and lower surfaces are:

(a) at the bottom, y = 0, **no-slip:** $\mathbf{u(0) = 0}$ *Ans.* (a)

(b) At the surface, y = h, no shear stress, $\mu\dfrac{\partial u}{\partial y} = 0$, or $\dfrac{\partial \mathbf{u}}{\partial \mathbf{y}}(\boldsymbol{h}) = \mathbf{0}$ *Ans.* (b)

4.44 Suppose that we wish to analyze the sudden pipe-expansion flow of Fig. P3.59, using the full continuity and Navier-Stokes equations. What are the proper boundary conditions to handle this problem?

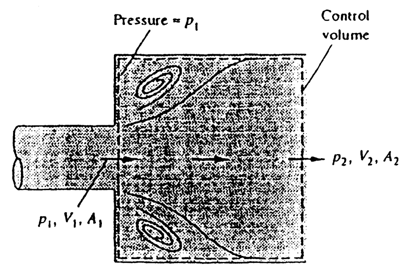


Fig. P3.59

Solution: First, at all walls, one would impose the no-slip condition: $\mathbf{u_r = u_z = 0}$ at all solid surfaces: at r = r_1 in the small pipe, at r = r_2 in the large pipe, and also on the flat-faced surface between the two.

Second, at some position upstream in the small pipe, the complete velocity distribution must be known: $\mathbf{u_1 = u_1(r)}$ at z = z_1. [Possibly the *paraboloid* of Prob. 4.34.]

Third, to be strictly correct, at some position downstream in the large pipe, the complete velocity distribution must be known: $\mathbf{u_2 = u_2(r)}$ at z = z_2. In numerical (computer) studies, this is often simplified by using a "free outflow" condition, $\partial u/\partial z = 0$.

Finally, the pressure must be specified at either the inlet or the outlet section of the flow, usually at the upstream section: $\mathbf{p = p_1(r)}$ at z = z_1.

4.45 Suppose that we wish to analyze the U-tube oscillation flow of Fig. P3.96, using the full continuity and Navier-Stokes equations. What are the proper boundary conditions to handle this problem?

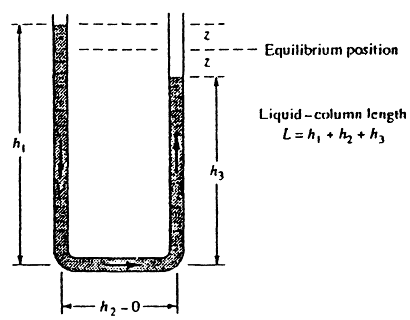


Fig. P3.96

Solution: This is an unsteady flow problem, so we need an initial condition at t = 0, z(left interface) = z_o, z(right interface) = $-z_o$: $\mathbf{u(r, z, 0) = 0}$ everywhere in the fluid column. Second, during the unsteady motion, we need boundary conditions of no-slip at the walls, $\mathbf{u = 0}$ **at** $\mathbf{r = R}$, and, if we neglect surface tension, known pressures at the two free surfaces: $\mathbf{p = p_{atmosphere}}$ at both ends. Finally, not knowing inlet or exit velocities, we would assume "free flow" at the interfaces: $\partial\mathbf{u}/\partial\mathbf{z} = \mathbf{0}.$

4.46 Fluid from a large reservoir at temperature T_o flows into a circular pipe of radius R. The pipe walls are wound with an electric-resistance coil which delivers heat to the fluid at a rate q_w (energy per unit wall area). If we wish to analyze this problem by using the full continuity, Navier-Stokes, and energy equations, what are the proper boundary conditions for the analysis?

Solution: Letting z = 0 be the pipe entrance, we can state inlet conditions: typically $u_z(r, 0) = U$ (a uniform inlet profile), $u_r(r, 0) = 0$, and $T(r, 0) = T_o$, also uniform.

At the wall, r = R, the no-slip and known-heat-flux conditions hold: $u_z(R, z) = u_r(R, z) = 0$ and $k(\partial T/\partial r) = q_w$ at (R, z) (assuming that q_w is positive for heat flow in).

At the exit, z = L, we would probably assume 'free outflow': $\partial u_z/\partial z = \partial T/\partial z = 0$.
Finally, we would need to know the pressure at one point, probably the inlet, z = 0.

4.47 Given the incompressible flow $\mathbf{V} = 3y\mathbf{i} + 2x\mathbf{j}$. Does this flow satisfy continuity? If so, find the stream function $\psi(x, y)$ and plot a few streamlines, with arrows.

Solution: With u = 3y and v = 2x, we may check $\partial u/\partial x + \partial v/\partial y = 0 + 0 = \mathbf{0}$, OK. Find the streamlines from $u = \partial\psi/\partial y = 3y$ and $v = -\partial\psi/\partial x = 2x$. Integrate to find

$$\boldsymbol{\psi = \frac{3}{2}y^2 - x^2} \quad \textit{Ans.}$$

Set ψ = 0, ±1, ±2, etc. and plot some streamlines at right: flow around corners of half-angles 39° and 51°.

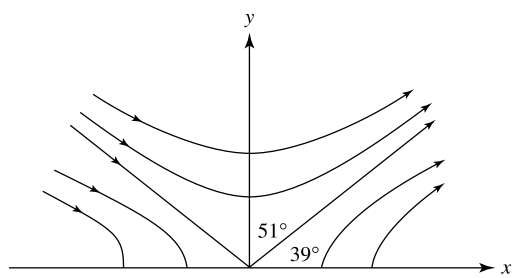


Fig. P4.47

4.48 Consider the following two-dimensional incompressible flow, which clearly satisfies continuity:

$$u = U_o = \text{constant}, \quad v = V_o = \text{constant}$$

Find the stream function $\psi(r, \theta)$ of this flow, that is, using *polar coordinates*.

Solution: In cartesian coordinates the stream function is quite easy:

$$u = \partial\psi/\partial y = U_o \quad \text{and} \quad v = -\partial\psi/\partial x = V_o \quad \text{or:} \quad \psi = U_o y - V_o x + \text{constant}$$

But, in polar coordinates, $y = r\sin\theta$ and $x = r\cos\theta$. Therefore the desired result is

$$\boldsymbol{\psi(r, \theta) = U_o r\sin\theta - V_o r\cos\theta + \text{constant}} \quad \textit{Ans.}$$

4.49 Investigate the stream function $\psi = K(x^2 - y^2)$, K = constant. Plot the streamlines in the full xy plane, find any stagnation points, and interpret what the flow could represent.

Solution: The velocities are given by

$$u = \frac{\partial\psi}{\partial y} = -2Ky; \quad v = -\frac{\partial\psi}{\partial x} = -2Kx$$

This is also stagnation flow, with the streamlines turned 45° from Prob. 4.48.

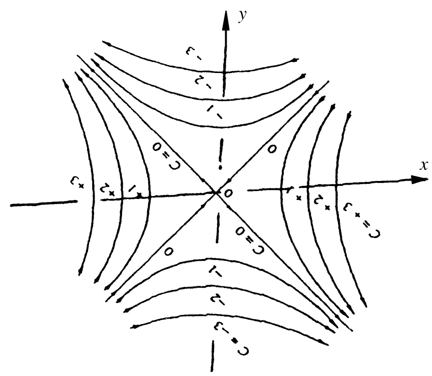


Fig. P4.49

4.50 Investigate the polar-coordinate stream function $\psi = Kr^{1/2}\sin\frac{1}{2}\theta$, K = constant. Plot the streamlines in the full xy plane, find any stagnation points, and interpret.

Solution: Simply set ψ/K = constant and plot r versus θ. This represents inviscid flow around a 180° turn. [See Fig. 8.14(e) of the text.]

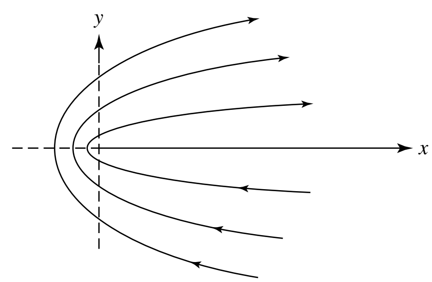


Fig. P4.50

4.51 Investigate the polar-coordinate stream function $\psi = Kr^{2/3}\sin(2\theta/3)$, K = constant. Plot the streamlines in all except the bottom right quadrant, and interpret.

Solution: Simply set ψ/K = constant and plot r versus θ. This represents inviscid flow around a 90° turn. [See Fig. 8.14(d) of the text.]

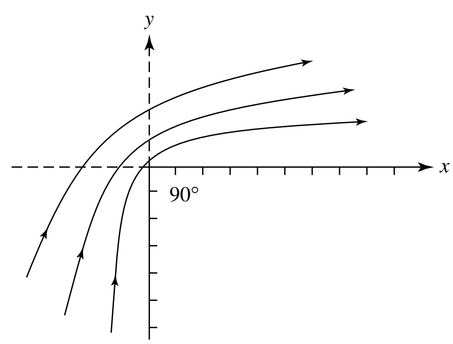


Fig. P4.51

4.52 A two-dimensional, incompressible, frictionless fluid is guided by wedge-shaped walls into a small slot at the origin, as in Fig. P4.52. The width into the paper is *b*, and the volume flow rate is *Q*. At any given distance *r* from the slot, the flow is radial inward, with constant velocity. Find an expression for the polar-coordinate stream function of this flow.

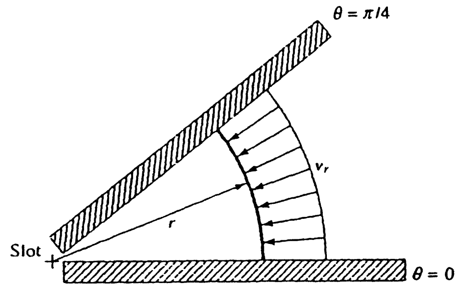


Fig. P4.52

Solution: We can find velocity from continuity:

$$v_r = -\frac{Q}{A} = -\frac{Q}{(\pi/4)rb} = \frac{1}{r}\frac{\partial\psi}{\partial\theta} \quad \text{from Eq. (4.101). Then}$$

$$\boldsymbol{\psi = -\frac{4Q}{\pi b}\theta + \text{constant}} \quad \textit{Ans.}$$

This is equivalent to the stream function for a <u>line sink</u>, Eq. (4.131).

4.53 For the fully developed laminar-pipe-flow solution of Prob. 4.34, find the axisymmetric stream function $\psi(r, z)$. Use this result to determine the average velocity $V = Q/A$ in the pipe as a ratio of u_{max}.

Solution: The given velocity distribution, $v_z = u_{max}(1 - r^2/R^2)$, $v_r = 0$, satisfies continuity, so a stream function <u>does</u> exist and is found as follows:

$$v_z = u_{max}(1 - r^2/R^2) = \frac{1}{r}\frac{\partial\psi}{\partial r}, \quad \text{solve for } \psi = u_{max}\left(\frac{r^2}{2} - \frac{r^4}{4R^2}\right) + f(z), \text{ now use in}$$

$$v_r = 0 = -\frac{1}{r}\frac{\partial\psi}{\partial z} = 0 + \frac{df}{dz}, \quad \text{thus } f(z) = \text{const}, \quad \boldsymbol{\psi = u_{max}\left(\frac{r^2}{2} - \frac{r^4}{4R^2}\right)} \quad \textit{Ans.}$$

We can find the flow rate and average velocity from the text for polar coordinates:

$$Q_{1\text{-}2} = 2\pi(\psi_2 - \psi_1), \quad \text{or:} \quad Q_{0\text{-}R} = 2\pi\left[u_{max}\left(\frac{R^2}{2} - \frac{R^4}{4R^2}\right) - u_{max}(0 - 0)\right] = \frac{\pi}{2}R^2 u_{max}$$

$$\text{Then} \quad V_{avg} = Q/A_{pipe} = [(\pi/2)R^2 u_{max}/(\pi R^2)] = \boldsymbol{\frac{1}{2}u_{max}} \quad \textit{Ans.}$$

4.54 An incompressible stream function is defined by

$$\psi(x, y) = \frac{U}{L^2}(3x^2y - y^3)$$

where U and L are (positive) constants. Where in this chapter are the streamlines of this flow plotted? Use this stream function to find the volume flow Q passing through the rectangular surface whose corners are defined by $(x, y, z) = (2L, 0, 0)$, $(2L, 0, b)$, $(0, L, b)$, and $(0, L, 0)$. Show the direction of Q.

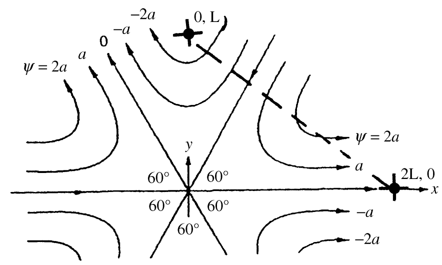


Fig. E4.7

Solution: This flow, with velocities $u = \partial\psi/\partial y = 3U/L^2(x^2 - y^2)$, and $v = -\partial\psi/\partial x = -6xyU/L^2$, is identical to Example 4.7 of the text, with "a" $= 3U/L^2$. The streamlines are plotted in Fig. E4.7. The volume flow <u>per unit width</u> between the points (2L, 0) and (0, L) is

$$Q/b = \psi(2L, 0) - \psi(0, L) = \frac{U}{L^2}(0-0) - \frac{U}{L^2}[3(0)^2L - L^3] = UL, \quad \text{or:} \quad \mathbf{Q = ULb} \quad \textit{Ans.}$$

Since ψ at the *lower* point (2L, 0) is larger than at the *upper* point (0, L), the flow through this diagonal plane is to the <u>left</u>, as per Fig. 4.9 of the text.

4.55 In spherical polar coordinates, as in Fig. P4.12, the flow is called *axisymmetric* if $\upsilon_\theta \equiv 0$ and $\partial/\partial\phi \equiv 0$, so that $\upsilon_r = \upsilon_r(r, \theta)$ and $\upsilon_\theta = \upsilon_\theta(r, \theta)$. Show that a stream function $\psi(r, \theta)$ exists for this case and is given by

$$\upsilon_r = \frac{1}{r^2 \sin\theta}\frac{\partial\psi}{\partial\theta} \qquad \upsilon_\theta = -\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r}$$

This is called the *Stokes stream function* [5, p. 204].

Solution: From Prob. 4.12 with zero velocity υ_ϕ, the continuity equation is

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\upsilon_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\upsilon_\theta \sin\theta) = 0, \quad \text{or:} \quad \frac{\partial}{\partial r}(r^2\upsilon_r\sin\theta) + \frac{\partial}{\partial\theta}(r\upsilon_\theta\sin\theta) = 0$$

Compare this to a stream function cross-differentiated form $\quad \dfrac{\partial}{\partial r}\left(\dfrac{\partial\psi}{\partial\theta}\right) + \dfrac{\partial}{\partial\theta}\left(-\dfrac{\partial\psi}{\partial r}\right) = 0$

It follows that: $\quad \boldsymbol{\upsilon_r}\textbf{(Stokes)} = \dfrac{\mathbf{1}}{\mathbf{r^2\sin\theta}}\dfrac{\partial\psi}{\partial\theta}; \quad \boldsymbol{\upsilon_\theta}\textbf{(Stokes)} = \dfrac{\mathbf{-1}}{\mathbf{r\sin\theta}}\dfrac{\partial\psi}{\partial \mathbf{r}} \quad$ *Ans.*

4.56 Investigate the velocity potential $\phi = Kxy$, K = constant. Sketch the potential lines in the full xy plane, find any stagnation points, and sketch in by eye the orthogonal streamlines. What could the flow represent?

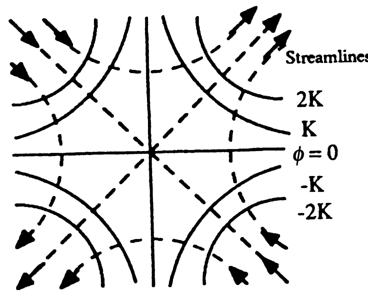


Fig. P4.56

Solution: The potential lines, ϕ = constant, are hyperbolas, as shown. The streamlines, sketched in as normal to the ϕ lines, are also hyperbolas. The pattern represents plane stagnation flow (Prob. 4.48) turned at 45°.

4.57 A two-dimensional incompressible flow field is defined by the velocity components

$$u = 2V\left(\frac{x}{L} - \frac{y}{L}\right) \qquad v = -2V\frac{y}{L}$$

where V and L are constants. If they exist, find the stream function and velocity potential.

Solution: First check continuity and irrotationality:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{2V}{L} - \frac{2V}{L} = 0 \quad \boldsymbol{\psi\ exists;}$$

$$\nabla x\mathbf{V} = \mathbf{k}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \mathbf{k}\left(0 + \frac{2V}{L}\right) \neq 0 \quad \boldsymbol{\phi\ does\ not\ exist}$$

To find the stream function ψ, use the definitions of u and v and integrate:

$$u = \frac{\partial \psi}{\partial y} = 2V\left(\frac{x}{L} - \frac{y}{L}\right), \quad \therefore \quad \psi = 2V\left(\frac{xy}{L} - \frac{y^2}{2L}\right) + f(x)$$

$$\text{Evaluate} \quad \frac{\partial \psi}{\partial x} = \frac{2Vy}{L} + \frac{df}{dx} = -v = \frac{2Vy}{L}$$

$$\text{Thus} \quad \frac{df}{dx} = 0 \quad \text{and} \quad \boldsymbol{\psi = V\left(\frac{2xy}{L} - \frac{y^2}{L}\right) + const} \quad \textit{Ans.}$$

4.58 Show that the incompressible velocity potential in plane polar coordinates $\phi(r,\theta)$ is such that

$$\upsilon_r = \frac{\partial \phi}{\partial r} \qquad \upsilon_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta}$$

Further show that the angular velocity about the z axis in such a flow would be given by

$$2\omega_z = \frac{1}{r}\frac{\partial}{\partial r}(r\upsilon_\theta) - \frac{1}{r}\frac{\partial}{\partial\theta}(\upsilon_r)$$

Finally show that ϕ as defined satisfies Laplace's equation in polar coordinates for incompressible flow.

Solution: All of these things are quite true and easy to show from their definitions. *Ans.*

4.59 Consider the two-dimensional incompressible velocity potential $\phi = xy + x^2 - y^2$. (a) Is it true that $\nabla^2\phi = 0$, and, if so, what does this mean? (b) If it exists, find the stream function $\psi(x, y)$ of this flow. (c) Find the equation of the streamline which passes through $(x, y) = (2, 1)$.

Solution: (a) First check that $\nabla^2\phi = 0$, which means that **incompressible continuity is satisfied**.

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0 + 2 - 2 = 0 \quad \textbf{Yes}$$

(b) Now use ϕ to find u and v and then integrate to find ψ.

$$u = \frac{\partial\phi}{\partial x} = y + 2x = \frac{\partial\psi}{\partial y}, \quad \text{hence} \quad \psi = \frac{y^2}{2} + 2xy + f(x)$$

$$v = \frac{\partial\phi}{\partial y} = x - 2y = -\frac{\partial\psi}{\partial x} = -2y - \frac{df}{dx}, \quad \text{hence} \quad f(x) = -\frac{x^2}{2} + const$$

The final stream function is thus $\boldsymbol{\psi = \frac{1}{2}(y^2 - x^2) + 2xy + const}$ *Ans.* (b)

(c) The streamline which passes through $(x, y) = (2, 1)$ is found by setting ψ = a constant:

$$At\ (x, y) = (2,\ 1), \quad \psi = \frac{1}{2}(1^2 - 2^2) + 2(2)(1) = -\frac{3}{2} + 4 = \frac{5}{2}$$

Thus the proper streamline is $\boldsymbol{\psi = \frac{1}{2}(y^2 - x^2) + 2xy = \frac{5}{2}}$ *Ans.* (c)

4.60 Liquid drains from a small hole in a tank, as shown in Fig. P4.60, such that the velocity field set up is given by $\upsilon_r \approx 0$, $\upsilon_z \approx 0$, $\upsilon_\theta = \omega R^2/r$, where $z = H$ is the depth of the water far from the hole. Is this flow pattern rotational or irrotational? Find the depth z_0 of the water at the radius $r = R$.

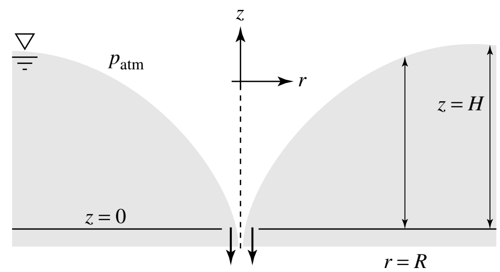


Fig. P4.60

Solution: From App. D, the angular velocity is

$$\omega_z = \frac{1}{r}\frac{\partial}{\partial r}(rv_\theta) - \frac{1}{r}\frac{\partial}{\partial \theta}(v_\theta) = 0 \text{ (IRROTATIONAL)}$$

Incompressible continuity is valid for this flow, hence Bernoulli's equation holds at the surface, where $p = p_{atm}$, both at infinity and at $r = R$:

$$p_{atm} + \frac{1}{2}\rho V^2_{r=\infty} + \rho g H = p_{atm} + \frac{1}{2}\rho V^2_{r=R} + \rho g z_o$$

Introduce $V_{r=\infty} = 0$ and $V_{r=R} = \omega R$ to obtain $\mathbf{z_o = H - \dfrac{\omega^2 R^2}{2g}}$ *Ans.*

4.61 Investigate the polar-coordinate velocity potential $\phi = Kr^{1/2}\cos\frac{1}{2}\theta$, $K =$ constant. Plot the potential lines in the full *xy* plane, sketch in by eye the orthogonal streamlines, and interpret.

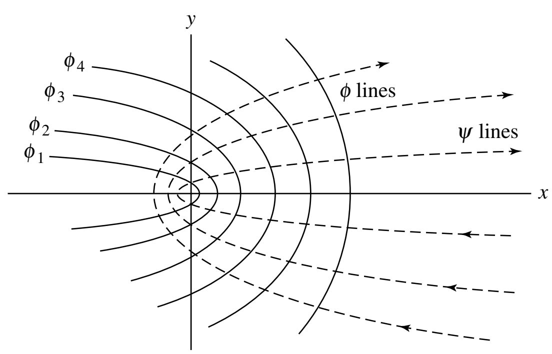


Fig. P4.61

Solution: These are the ϕ lines associated with the 180°-turn streamlines from Prob. 4.50.

4.62 Show that the linear Couette flow between plates in Fig. 1.6 has a stream function but no velocity potential. Why is this so?

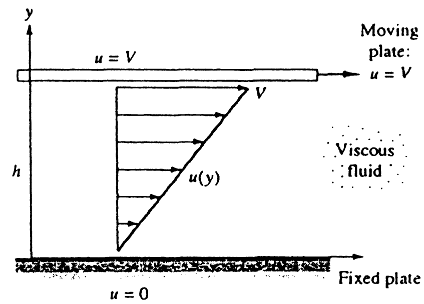


Fig. 1.6

Solution: Given u = Vy/h, v = 0, check continuity:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y} \stackrel{?}{=} 0 = 0+0 \quad \textbf{(Satisfied therefore } \boldsymbol{\psi} \textbf{ exists).} \text{ Find } \psi \text{ from}$$

$$u = \frac{Vy}{h} = \frac{\partial \psi}{\partial y}, \quad v = 0 = -\frac{\partial \psi}{\partial x}, \quad \text{solve for } \boldsymbol{\psi = \frac{V}{2h}y^2 + \textbf{const}} \quad \textit{Ans.}$$

Now check irrotationality:

$$2\omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \stackrel{?}{=} 0 = 0 - \frac{V}{h} \neq 0! \ \textbf{(Rotational, } \boldsymbol{\phi} \textbf{ does not exist.)} \quad \textit{Ans.}$$

4.63 Find the two-dimensional velocity potential $\phi(r,\theta)$ for the polar-coordinate flow pattern $\upsilon_r = Q/r$, $\upsilon_\theta = K/r$, where Q and K are constants.

Solution: Relate these velocity components to the polar-coordinate definition of ϕ:

$$v_r = \frac{Q}{r} = \frac{\partial \phi}{\partial r}, \quad v_\theta = \frac{K}{r} = \frac{1}{r}\frac{\partial \phi}{\partial \theta}; \quad \text{solve for } \boldsymbol{\phi = Q\ln(r) + K\theta + \textbf{const}} \quad \textit{Ans.}$$

4.64 Show that the velocity potential ϕ(r, z) in axisymmetric cylindrical coordinates (see Fig. 4.2 of the text) is defined by the formulas:

$$\upsilon_r = \frac{\partial \phi}{\partial r} \qquad \upsilon_z = \frac{\partial \phi}{\partial z}$$

Further show that for incompressible flow this potential satisfies Laplace's equation in (r, z) coordinates.

Solution: All of these things are quite true and are easy to show from their definitions. *Ans.*

4.65 A two-dimensional incompressible flow is defined by

$$u = -\frac{Ky}{x^2 + y^2} \qquad \upsilon = \frac{Kx}{x^2 + y^2}$$

where K = constant. Is this flow irrotational? If so, find its velocity potential, sketch a few potential lines, and interpret the flow pattern.

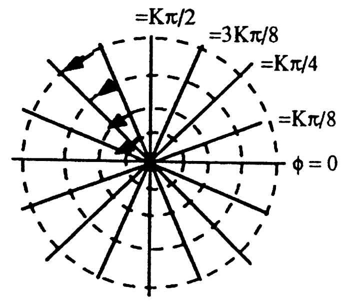


Fig. P4.65

Solution: Evaluate the angular velocity:

$$2\omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \frac{K}{x^2+y^2} - \frac{2Kx^2}{(x^2+y^2)^2} + \frac{K}{x^2+y^2} - \frac{2Ky^2}{(x^2+y^2)^2} = \mathbf{0\ (Irrotational)} \quad \textit{Ans.}$$

Introduce the definition of velocity potential and integrate to get $\phi(x, y)$:

$$u = \frac{\partial \phi}{\partial x} = -\frac{Ky}{x^2+y^2}; \quad v = \frac{\partial \phi}{\partial y} = \frac{Kx}{x^2+y^2}, \quad \text{solve for } \boldsymbol{\phi = K\tan^{-1}\left(\frac{y}{x}\right) = K\theta} \quad \textit{Ans.}$$

The ϕ lines are plotted above. They represent a <u>counterclockwise line vortex</u>.

4.66 A plane polar-coordinate velocity potential is defined by

$$\phi = \frac{K\cos\theta}{r} \quad K = \text{const}$$

Find the stream function for this flow, sketch some streamlines and potential lines, and interpret the flow pattern.

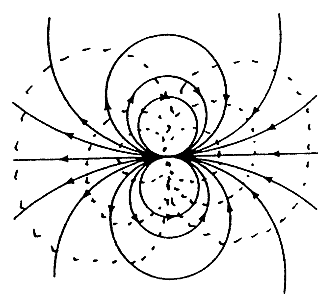


Fig. P4.66

Solution: Evaluate the velocities and thence find the stream function:

$$v_r = \frac{\partial \phi}{\partial r} = -\frac{K\cos\theta}{r^2} = \frac{1}{r}\frac{\partial \psi}{\partial \theta}; \quad v_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = -\frac{K\sin\theta}{r^2} = -\frac{\partial \psi}{\partial r},$$

$$\text{solve} \quad \psi = -\frac{\mathbf{K\sin\theta}}{\mathbf{r}} \quad \textit{Ans.}$$

The streamlines and potential lines are shown above. This pattern is a <u>line doublet</u>.

4.67 A stream function for a plane, irrotational, polar-coordinate flow is

$$\psi = C\theta - K\ln r \quad C \text{ and } K = \text{const}$$

Find the velocity potential for this flow. Sketch some streamlines and potential lines, and interpret the flow pattern.

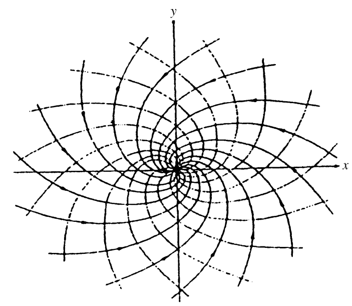


Fig. 4.14

Solution: If this problem is given *early* enough (before Section 4.10 of the text), the students will discover this pattern for themselves. It is a line source plus a line vortex, a tornado-like flow, Eq. (4.134) and Fig. 4.14 of the text. Find the velocity potential:

$$v_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = \frac{C}{r} = \frac{\partial\phi}{\partial r}; \quad v_\theta = -\frac{\partial\psi}{\partial r} = \frac{K}{r} = \frac{1}{r}\frac{\partial\phi}{\partial\theta}, \quad \text{solve } \boldsymbol{\phi = C\ln(r) + K\theta} \quad \textit{Ans.}$$

The streamlines and potential lines are plotted above for *negative* C (a line sink).

4.68 Find the stream function and plot some streamlines for the combination of a line source *m* at $(x, y) = (0, +a)$ and an equal line source placed at $(0, -a)$.

Solution: In the spirit of Eq. (4.133), we add two *sources* together:

$$\psi = \text{Source @ (0, a)} + \text{Source @ (0,–a)}$$

$$= m\tan^{-1}\left(\frac{y-a}{x}\right) + m\tan^{-1}\left(\frac{y+a}{x}\right)$$

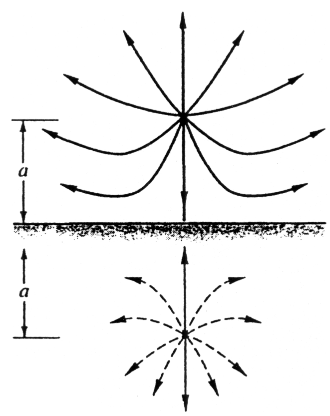


Fig. P4.68

Use the identity $\tan^{-1}\alpha + \tan^{-1}\beta = \tan^{-1}\left(\frac{\alpha+\beta}{1-\alpha\beta}\right)$ to get

$$\boldsymbol{\psi = m\tan^{-1}\left(\frac{2xy}{x^2 - y^2 + a^2}\right)} \quad \textit{Ans.}$$

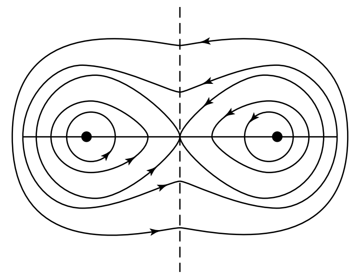
The latter form uses the trig identity $\tan^{-1}\alpha + \tan^{-1}\beta = \tan^{-1}[(\alpha+\beta)/(1-\alpha\beta)]$. If we plot lines of constant ψ (streamlines), we find the source-flow *image* pattern shown above.

4.69 Find the stream function and plot some streamlines for the combination of a counterclockwise line vortex K at $(x, y) = (+a, 0)$ and an equal line vortex placed at $(-a, 0)$.

Solution: The combined stream function is

$$\psi = -K\ln r_1 - K\ln r_2 = -K\ln[(x-a)^2 + y^2]^{1/2} - K\ln[(x+a)^2 + y^2]^{1/2}$$

Plotting this, for various K = constant, reveals the "cat's-eye" pattern shown at right.



4.70 Take the limit of ϕ for the source-sink combination, Eq. (4.133), as strength m becomes large and distance a becomes small, so that (ma) = constant. What happens?

Solution: Given $\phi = \frac{1}{2}m\ln[\{(x+a)^2 + y^2\}/\{(x-a)^2 + y^2\}]$, divide [] by $(x + a)^2$ and use the series form $\ln[(1 + \varepsilon)/(1 - \varepsilon)] = 2\varepsilon + 2\varepsilon^2/3 + \cdots$ the result is the *line doublet*:

$$\phi_{doublet} = \mathbf{lim}\big|_{am=0}\,(\phi_{source+sink}) = \frac{2amx}{x^2 + y^2} = \frac{\lambda\cos\theta}{\mathbf{r^2}},\quad \lambda = 2am \quad \textit{Ans.}$$

4.71 Find the stream function and plot some streamlines for the combination of a counterclockwise line vortex K at $(x, y) = (+a, 0)$ and an opposite (clockwise) line vortex placed at $(-a, 0)$.

Solution: The combined stream function is

$$\psi = -K\ln r_1 + K\ln r_2 = -K\ln[(x-a)^2 + y^2]^{1/2} + K\ln[(x+a)^2 + y^2]^{1/2}$$

Plotting this, for various K = constant, reveals the swirling "vortex-pair" pattern shown at right. It is equivalent to an "image" vortex pattern, as in Fig. 8.17(b) of the text.

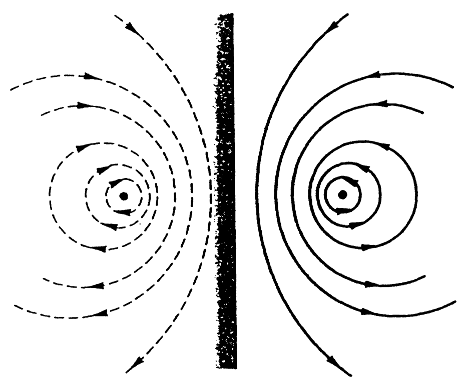


Fig. P4.71

4.72 A coastal power plant takes in cooling water through a vertical perforated manifold, as in Fig. P4.72. The total volume flow intake is 110 m^3/s. Currents of 25 cm/s flow past the manifold, as shown. Estimate (a) how far downstream and (b) how far normal to the paper the effects of the intake are felt in the ambient 8-m-deep waters.

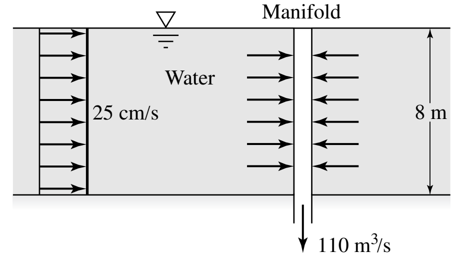
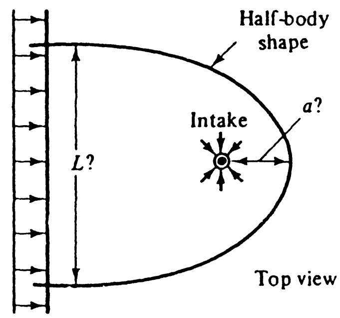


Fig. P4.72

Solution: A top view of the flow is shown at right. The sink strength is

$$m = \frac{Q}{2\pi b} = \frac{110\ m^3/s}{2\pi(8\ m)} = 2.19\ \frac{m^2}{s}$$

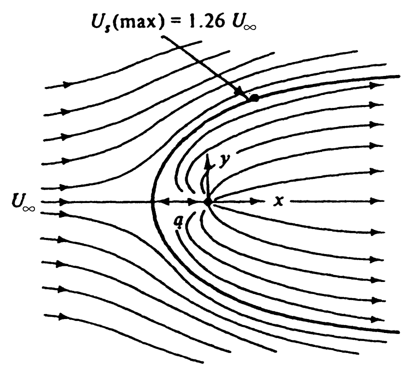


Then the appropriate lengths are:

$$a = \frac{m}{U} = \frac{2.19\ m^2/s}{0.25\ m/s} = \mathbf{8.75\ m} \quad \textit{Ans.}\ (a)$$

$$L = 2\pi a = 2\pi(8.75) = \mathbf{55\ m} \quad \textit{Ans.}\ (b)$$

4.73 A two-dimensional Rankine half-body, 8 cm thick, is placed in a water tunnel at 20°C. The water pressure far up-stream along the body centerline is 120 kPa. What is the nose radius of the half-body? At what tunnel flow velocity will cavitation bubbles begin to form on the surface of the body?



Solution: The nose radius is given by

$$a = \frac{L}{2\pi} = \frac{8\ cm}{2\pi} = \mathbf{1.27\ cm} \quad \textit{Ans.}$$

At 20°C the vapor pressure of water is 2337 Pa. Maximum velocity occurs, as shown, on the upper surface at $\theta \approx 63°$, where $z \approx 2.04a \approx 2.6$ cm and $V \approx 1.26U_\infty$. Write the Bernoulli equation between upstream and V, assuming the surface pressure is vaporizing:

$$p_\infty + \frac{\rho}{2}U_\infty^2 + \rho g z_\infty \approx p_{vap} + \frac{\rho}{2}V_{max}^2 + \rho g z_{surface},$$

$$\text{or:}\quad 120000+\frac{998}{2}U_\infty^2+0=2337+\frac{998}{2}(1.26U_\infty)^2+9790(0.026),$$

$$\text{solve}\quad \mathbf{U_\infty \approx 20\ \frac{m}{s}}\quad \textit{Ans.}$$

4.74 A small fish pond is approximated by a half-body shape, as shown in Fig. P4.74. Point O, which is 0.5 m from the left edge of the pond, is a water source delivering about 0.63 m^3/s per meter of depth into the paper. Find the point B along the axis where the water velocity is approximately 25 cm/s.

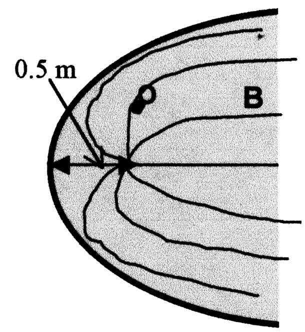


Fig. P4.74

Solution: We are given $a = 0.5$ m and $Q = 0.63\ m^3/(s)$, hence the source strength from Eq. (4.131) is $m = Q/(2\pi b) = (0.63\ m^3/s)/[2\pi(1\ m)] = 0.1003\ m^2/s$. The stream velocity is thus

$$U = m/a = (0.1003\ m^2/s)/(0.5\ m) = 0.2005\ m/s$$

Along line OB, the velocity is purely radial. We find point B from the known velocity:

$$V_B = 0.25\ \text{m/s} = U+\frac{m}{r_{OB}} = 0.2005+\frac{0.1003}{r_{OB}},$$

$$\text{hence}\quad \mathbf{r_{OB} = 2.03\ m}\quad \textit{Ans.}$$

4.75 Find the stream function and plot some streamlines for the combination of a line source $2m$ at $(x, y) = (+a, 0)$ and a line source m at $(-a, 0)$. Are there any stagnation points in the flow field?

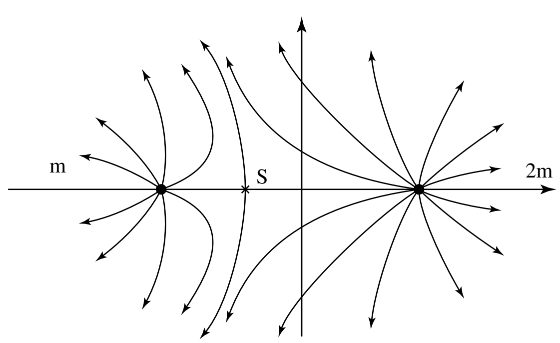


Fig. P4.75

Solution: The combined stream function is

$$\psi = 2m\tan^{-1}\left(\frac{y}{x-a}\right) + m\tan^{-1}\left(\frac{y}{x+a}\right)$$

The streamlines, viewed from up close, look like the drawing on the previous page. There is one stagnation point, where $2m/r_1 = m/r_2$, or at $(x, y) = (-a/3, 0)$. Viewed from afar, they look like the radial streamlines of a single source of strength *3m*.

4.76 Air flows at 1.2 m/s along a flat wall when it meets a jet of air issuing from a slot at A. The jet volume flow is 0.4 m^3/s per m of width into the paper. If the jet is approximated as a line source, (a) locate the stagnation point S. (b) How far vertically will the jet flow extend?

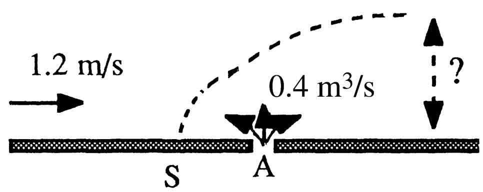
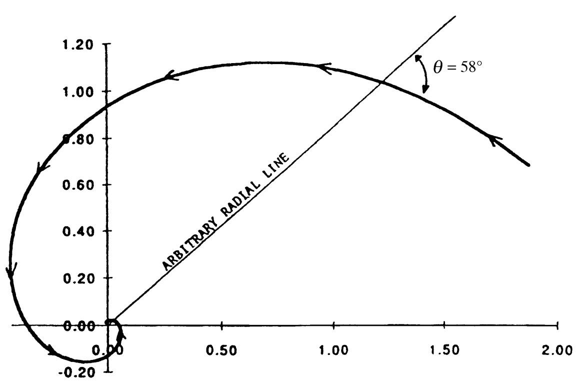


Fig. P4.76

Solution: The equivalent source strength is $m = 0.4/\pi = 0.127\ m^2/s$. Then, as in Figure 4.15 of the text, the stagnation point S is at $a = m/U = 0.127/1.2 =$ **0.106 m from A**. The effective 'half-body,' shown as a dashed line in the figure, extends out to a distance equal to $\pi a = \pi(0.106) =$ **0.333 m above the wall**. *Ans.*

4.77 A tornado is simulated by a line sink $m = -1000\ m^2/s$ plus a line vortex $K = +1600\ m^2/s$. Find the angle between any streamline and a radial line, and show that it is independent of both r and θ. If this tornado forms in sea-level standard air, at what radius will the local pressure be equivalent to 29 inHg?



Solution: The combined stream function is

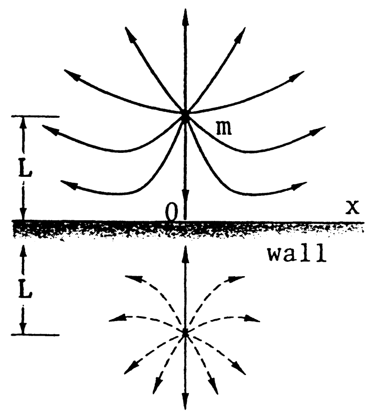
$$\psi = -K\ln(r) - m\theta, \text{ with the angle } \phi \text{ given by}$$

$$\tan\phi = |v_\theta/v_r| = \frac{K/r}{m/r} = \frac{K}{m} = \mathbf{1.6} \text{ independent of r, } \theta$$

The desired angle is $\phi = \mathbf{tan^{-1}(1.6) \approx 58°}$ *Ans.*

Local pressure = 29″Hg = 98 kPa at V = 75 m/s, or **r = 25 meters.** *Ans.*

4.78 We wish to study the flow due to a line source of strength *m* placed at position (*x*, *y*) = (0, +*L*), above the plane horizontal wall *y* = 0. Using Bernoulli's equation, find (a) the point(s) of minimum pressure on the plane wall and (b) the magnitude of the maximum flow velocity along the wall.



Solution: The "wall" is produced by an image source at (0, –L), as in Prob. 4.68. Along the wall, y = 0, v = 0, U = m/L,

$$u = 2u_{\text{one source}} = \frac{2UL}{(x^2+L^2)^{1/2}} \cdot \frac{x}{(x^2+L^2)^{1/2}} = \frac{2ULx}{x^2+y^2}$$

By differentiation, the maximum velocity occurs at $x = L$, or $u_{max} = U = \frac{m}{L}$ *Ans.* (b)

By Bernoulli's equation, this is also the point of <u>minimum pressure</u>, at (x, y) = **(L, 0)**:

$$p_{min} = p(0,0) - \frac{1}{2}\rho u_{max}^2 = \mathbf{p_o - \frac{1}{2}\rho(m/L)^2 \quad at \quad (\pm L, 0)} \quad \textit{Ans. (a)}$$

4.79 Study the combined effect of the two viscous flows in Fig. 4.16. That is, find *u*(*y*) when the upper plate moves at speed *V* and there is also a constant pressure gradient (*dp/dx*). Is superposition possible? If so, explain why. Plot representative velocity profiles for (a) zero, (b) positive, and (c) negative pressure gradients for the same upper-wall speed *V*.

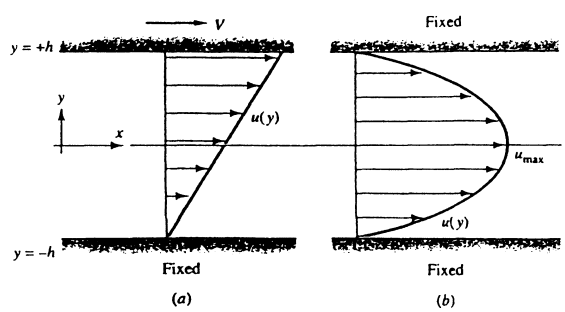


Fig. 4.16

Solution: The combined solution is

$$u = \frac{V}{2}\left(1+\frac{y}{h}\right)+\frac{h^2}{2\mu}\left(-\frac{dp}{dx}\right)\left(1-\frac{y^2}{h^2}\right)$$

The superposition is quite valid because the convective acceleration is zero, hence what remains is linear: $\nabla p = \mu\nabla^2\mathbf{V}$. Three representative velocity profiles are plotted at right for various (dp/dx).

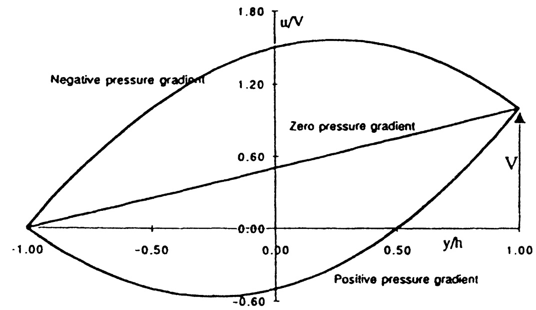
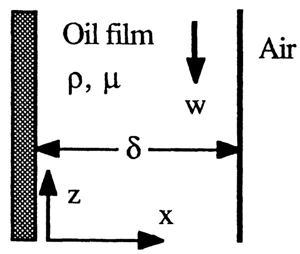


Fig. P4.79

4.80 An oil film drains steadily down the side of a vertical wall, as shown. After an initial development at the top of the wall, the film becomes independent of z and of constant thickness. Assume that w = w(x) only that the atmosphere offers no shear resistance to the film. (a) Solve Navier-Stokes for w(x). (b) Suppose that film thickness and [$\partial w/\partial x$] at the wall are measured. Find an expression which relates μ to this slope [$\partial w/\partial x$].



Solution: First, there is no pressure gradient $\partial p/\partial z$ because of the constant-pressure atmosphere. The Navier-Stokes z-component is $\mu d^2w/dx^2 = \rho g$, and the solution requires w = 0 at x = 0 and (dw/dx) = 0 (no shear at the film edge) at x = δ. The solution is:

$$\boldsymbol{w = \frac{\rho g x}{2\mu}(x - 2\delta)} \quad \textit{Ans.}\text{ (a)} \quad \text{NOTE: } w \text{ is negative (down)}$$

The wall slope is $\dfrac{dw}{dx}\Big|_{wall} = -\dfrac{\rho g\delta}{\mu}$, *or rearrange:* $\boldsymbol{\mu = -\dfrac{\rho g\delta}{[dw/dx]_{wall}}}$ *Ans.* (b)

4.81 Modify the analysis of Fig. 4.17 to find the velocity v_θ when the inner cylinder is fixed and the outer cylinder rotates at angular velocity Ω_0. May this solution be *added* to Eq. (4.146) to represent the flow caused when both inner and outer cylinders rotate? Explain your conclusion.

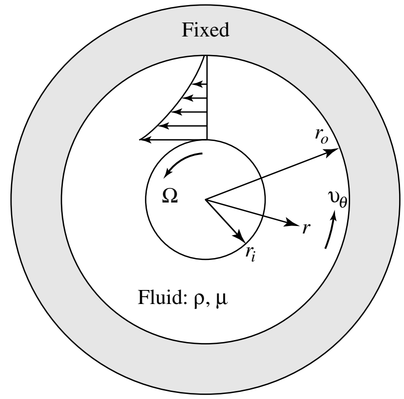


Fig. 4.17

Solution: We apply new boundary conditions to Eq. (4.145) of the text:

$$v_\theta = C_1 r + C_2/r;$$

$$\text{At } r = r_i, \quad v_\theta = 0 = C_1 r_i + C_2/r_i$$

$$\text{At } r = r_o, \quad v_\theta = \Omega_o r_o = C_1 r_o + C_2/r_o$$

Solve for C_1 and C_2. The final result: $\mathbf{v}_\theta = \mathbf{\Omega_o r_o}\left(\dfrac{\mathbf{r/r_i - r_i/r}}{\mathbf{r_o/r_i - r_i/r_o}}\right)$ *Ans.*

This solution may indeed be added to the inner-rotation solution, Eq. (4.146), because the convective acceleration is zero and hence the Navier-Stokes equation is *linear*.

4.82 A solid circular cylinder of radius *R* rotates at angular velocity Ω in a viscous incompressible fluid which is at rest far from the cylinder, as in Fig. P4.82. Make simplifying assumptions and derive the governing differential equation and boundary conditions for the velocity field v_θ in the fluid. Do not solve unless you are obsessed with this problem. What is the steady-state flow field for this problem?

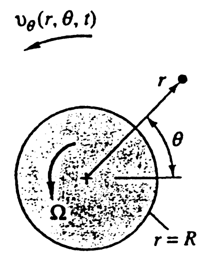


Fig. P4.82

Solution: We assume purely circulating motion: $v_z = v_r = 0$ and $\partial/\partial\theta = 0$. Thus the remaining variables are $v_\theta = \text{fcn}(r, t)$ and $p = \text{fcn}(r, t)$. Continuity is satisfied identically, and the θ-momentum equation reduces to a partial differential equation for v_θ:

$$\frac{\partial \mathbf{v}_\theta}{\partial \mathbf{t}} = \frac{\mu}{\rho}\left[\frac{\mathbf{1}}{\mathbf{r}}\frac{\partial}{\partial \mathbf{r}}\left(\mathbf{r}\frac{\partial \mathbf{v}_\theta}{\partial \mathbf{r}}\right) - \frac{\mathbf{v}_\theta}{\mathbf{r}^2}\right] \quad \text{subject to } v_\theta(R, t) = \Omega R \quad \text{and} \quad v_\theta(\infty, t) = 0 \quad \textit{Ans.}$$

I am not obsessed with this problem so will not attempt to find a solution. However, at large times, or $t = \infty$, the steady state solution is $\mathbf{v}_\theta = \mathbf{\Omega R^2/r}$. *Ans.*

4.83 The flow pattern in bearing lubrication can be illustrated by Fig. P4.83, where a viscous oil (ρ, μ) is forced into the gap $h(x)$ between a fixed slipper block and a wall moving at velocity U. If the gap is thin, $h \ll L$, it can be shown that the pressure and velocity distributions are of the form $p = p(x)$, $u = u(y)$, $v = w = 0$. Neglecting gravity, reduce the Navier-Stokes equations (4.38) to a single differential equation for $u(y)$. What are the proper boundary conditions? Integrate and show that

$$u = \frac{1}{2\mu}\frac{dp}{dx}(y^2 - yh) + U\left(1 - \frac{y}{h}\right)$$

where $h = h(x)$ may be an arbitrary slowly varying gap width. (For further information on lubrication theory, see Ref. 16.)

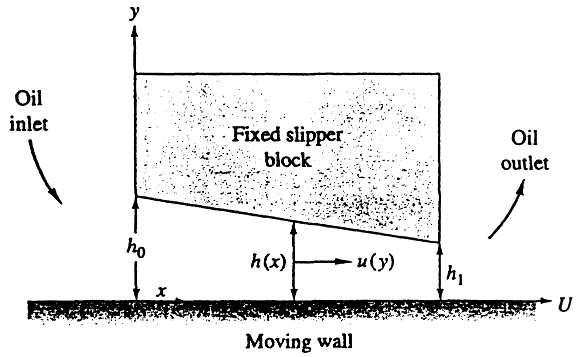


Fig. P4.83

Solution: With u = u(y) and p = p(x) only in the gap, the *x*-momentum equation becomes

$$\rho\frac{du}{dt} = 0 = -\frac{dp}{dx} + \mu\frac{\partial^2 u}{\partial y^2}, \quad \text{or:} \quad \frac{d^2u}{dy^2} = \frac{1}{\mu}\frac{dp}{dx} = \text{constant}$$

$$\text{Integrate twice:} \quad u = \frac{1}{\mu}\frac{dp}{dx}\frac{y^2}{2} + C_1 y + C_2, \quad \text{with } u(0) = U \quad \text{and} \quad u(h) = 0$$

With C_1 and C_2 evaluated, the solution is exactly as listed in the problem statement:

$$\mathbf{u = \frac{1}{2\mu}\frac{dp}{dx}(y^2 - yh) + U\left(1 - \frac{y}{h}\right)} \quad \textit{Ans.}$$

4.84 Consider a viscous film of liquid draining uniformly down the side of a vertical rod of radius *a*, as in Fig. P4.84. At some distance down the rod the film will approach a terminal or *fully developed* draining flow of constant outer radius *b*, with $\upsilon_z = \upsilon_z(r)$, $\upsilon_\theta = \upsilon_r = 0$. Assume that the atmosphere offers no shear resistance to the film motion. Derive a differential equation for υ_z, state the proper boundary conditions, and solve for the film velocity distribution. How does the film radius *b* relate to the total film volume flow rate *Q*?

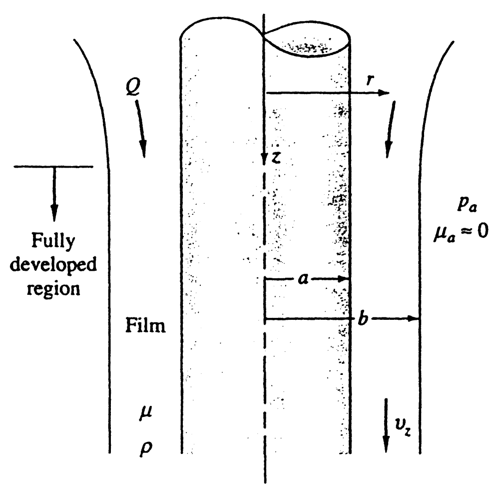


Fig. P4.84

Solution: With $v_z = \text{fcn}(r)$ only, the Navier-Stokes *z*-momentum relation is

$$\rho\frac{dv_z}{dt} = 0 = -\frac{\partial p}{\partial z} + \rho g + \mu\nabla^2 v_z,$$

$$\text{or:}\quad \frac{1}{r}\frac{d}{dr}\left(r\frac{dv_z}{dr}\right) = -\frac{\rho g}{\mu},\quad \text{Integrate twice:}\quad v_z = -\frac{\rho g r^2}{4\mu} + C_1\ln(r) + C_2$$

The proper B.C. are: $u(a) = 0$ (no-slip) and $\mu\frac{\partial v_z}{\partial r}(b) = 0$ (no free-surface shear stress)

The final solution is $\mathbf{v_z = \frac{\rho g b^2}{2\mu}\ln\left(\frac{r}{a}\right) - \frac{\rho g}{4\mu}(r^2 - a^2)}$ *Ans.*

The flow rate is $Q = \int_a^b v_z 2\pi r\, dr = \boldsymbol{\frac{\pi\rho g a^4}{8\mu}(-3\sigma^4 - 1 + 4\sigma^2 + 4\sigma^4\ln\sigma)},$

where $\sigma = \frac{b}{a}$ *Ans.*

4.85 A flat plate of essentially infinite width and breadth oscillates sinusoidally in its own plane beneath a viscous fluid, as in Fig. P4.85. The fluid is at rest far above the plate. Making as many simplifying assumptions as you can, set up the governing differential equation and boundary conditions for finding the velocity field *u* in the fluid. Do not solve (if you *can* solve it immediately, you might be able to get exempted from the balance of this course with credit).

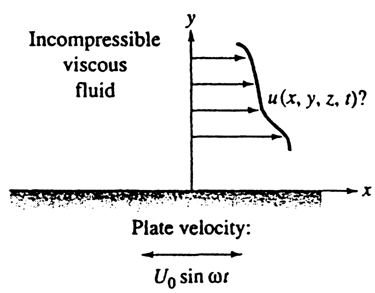


Fig. P4.85

Solution: Assume $u = u(y, t)$ and $\partial p/\partial x = 0$. The x-momentum relation is

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + \rho g_x + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right),$$

$$\text{or:}\quad \rho\left(\frac{\partial u}{\partial t} + 0 + 0\right) = 0 + 0 + \mu\left(0 + \frac{\partial^2 u}{\partial y^2}\right),\quad \text{or, finally:}$$

$$\boldsymbol{\frac{\partial u}{\partial t} = \frac{\mu}{\rho}\frac{\partial^2 u}{\partial y^2}}\quad \text{subject to: } \mathbf{u(0,t) = U_o\sin(\omega t)}\quad \textbf{and}\quad \mathbf{u(\infty,t) = 0.}\quad \textit{Ans.}$$

4.86 SAE 10 oil at 20°C flows between parallel plates 8 mm apart, as in Fig. P4.86. A mercury manometer, with wall pressure taps 1 m apart, registers a 6-cm height, as shown. Estimate the flow rate of oil for this condition.

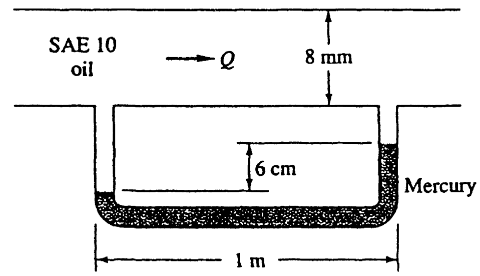


Fig. P4.86

Solution: Assuming laminar flow, this geometry fits Eqs. (4.143, 144) of the text:

$$V_{avg} = \frac{2}{3}u_{max} = \left(\frac{dp}{dx}\right)\frac{h^2}{3\mu}, \quad \text{where } h = \text{plate half-width} = 4\,\text{mm}$$

For SAE 10W oil, take $\rho = 870$ kg/m^3 and $\mu = 0.104$ kg/m·s. The manometer reads

$$\Delta p = (\rho_{Hg} - \rho_{oil})g\Delta h = (13550 - 870)(9.81)(0.06) \approx 7463 \text{ Pa} \quad \text{for } \Delta x = L = 1\text{m}$$

$$\text{Then} \quad V = \frac{\Delta p}{\Delta x}\frac{h^2}{3\mu} = \left(\frac{7463 \text{ Pa}}{1 \text{ m}}\right)\frac{(0.004)^2}{3(0.104)} \approx 0.383 \ \frac{\text{m}}{\text{s}}$$

The flow rate per unit width is $Q = VA = (0.383)(0.008) \approx \mathbf{0.00306\ \frac{m^3}{s\cdot m}}$ *Ans.*

4.87 SAE 30W oil at 20°C flows through the 9-cm-diameter pipe in Fig. P4.87 at an average velocity of 4.3 m/s. (a) Verify that the flow is laminar. (b) Determine the volume flow rate in m^3/h. (c) Calculate the expected reading *h* of the mercury manometer, in cm.

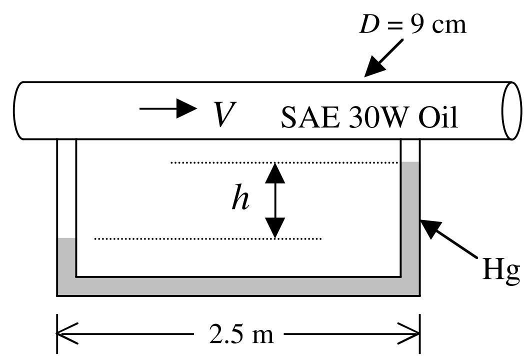


Fig. P4.87

Solution: (a) Check the Reynolds number. For SAE 30W oil, from Appendix A.3, ρ = 891 kg/m^3 and μ = 0.29 kg/(m·s). Then

$Re_d = \rho Vd/\mu = (891 \text{ kg/m}^3)(4.3 \text{ m/s})(0.09 \text{ m})/[0.29 \text{ kg/(m·s)}] = 1190 < 2000$ **Laminar** *Ans.* (a)

(b) With average velocity known, the volume flow follows easily:

$$Q = AV = [(\pi/4)(0.09\text{ m})^2](4.3\text{ m/s})(3600\text{ s/h}) = \mathbf{98.5\ m^3/h} \quad \textit{Ans. (b)}$$

(c) The manometer measures the pressure drop over a 2.5 m length of pipe. From Eq. (4.147),

$$V = 4.3\ \frac{m}{s} = \frac{\Delta p}{L}\frac{R^2}{8\mu} = \frac{\Delta p}{2.5\ m}\frac{(0.045\ m)^2}{8(0.29\ kg/m{\cdot}s)},\quad \textit{solve for}\ \ \Delta p = 12320\ Pa$$

$$\Delta p_{mano} = 12320 = (\rho_{merc} - \rho_{oil})gh = (13550 - 891)(9.81)h,\quad \text{Solve}\ \ \mathbf{h = 0.099\ m}\quad \textit{Ans.}\ \text{(c)}$$

4.88 The viscous oil in Fig. P4.88 is set into steady motion by a concentric inner cylinder moving axially at velocity *U* inside a fixed outer cylinder. Assuming constant pressure and density and a purely axial fluid motion, solve Eqs. (4.38) for the fluid velocity distribution $v_z(r)$. What are the proper boundary conditions?

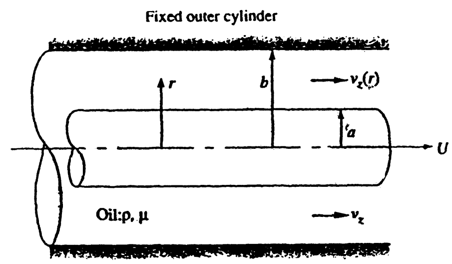


Fig. P4.88

Solution: If v_z = fcn(r) only, the *z*-momentum equation (Appendix E) reduces to:

$$\rho\frac{dv_z}{dt} = -\frac{\partial p}{\partial z} + \rho g_z + \mu\nabla^2 v_z,\quad \text{or:}\quad 0 = 0 + 0 + \frac{\mu}{r}\frac{d}{dr}\left(r\frac{dv_z}{dr}\right)$$

The solution is $v_z = C_1 \ln(r) + C_2$, subject to $v_z(a) = U$ and $v_z(b) = 0$

Solve for $C_1 = U/\ln(a/b)$ and $C_2 = -C_1 \ln(b)$

The final solution is: $$\mathbf{v_z = U\frac{\ln(r/b)}{\ln(a/b)}}\quad \textit{Ans.}$$

4.89 Modify Prob. 4.88 so that the outer cylinder also moves to the *left* at constant speed *V*. Find the velocity distribution $v_z(r)$. For what ratio *V/U* will the wall shear stress be the same at both cylinder surfaces?

Solution: We merely modify the boundary conditions for the known solution in 4.88:

$v_z = C_1 \ln(r) + C_2$, subject to $v_z(a) = U$ and $v_z(b) = -V$

Solve for $C_1 = (U + V)/\ln(a/b)$ and $C_2 = U - (U + V)\ln(a)/\ln(a/b)$

The final solution is $$\mathbf{v_z = U + (U + V)\frac{\ln(r/a)}{\ln(a/b)}}\quad \textit{Ans.}$$

The shear stress $\tau = \mu(U + V)/[r \ln(a/b)]$ and is <u>never equal at both walls for any ratio of V/U</u> unless the clearance is vanishingly small, that is, unless a ≈ b. *Ans.*

4.90 SAE 10W oil at 20°C flows through a straight horizontal pipe. The pressure gradient is a constant 400 Pa/m. (a) What is the appropriate pipe diameter D in cm if the Reynolds number Re_D of the flow is to be exactly 1000? (b) For case *a*, what is the flow rate Q in m^3/h?

Solution: For SAE 10W oil, from Appendix A.3, $\rho = 870\ kg/m^3$ and $\mu = 0.104$ kg/(m·s). (a) Relate V_{avg} to pressure drop from Eq. (4.147) and set the Reynolds number equal to 1000:

$$Re_D = \frac{\rho}{\mu}VD = \frac{\rho}{\mu}\left[\frac{\Delta p}{L}\frac{R^2}{8\mu}\right]D = 1000 = \frac{870\ kg/m^3}{0.104\ k/m{\cdot}s}\left[400\frac{Pa}{m}\left\{\frac{(D/2)^2}{8(0.104)}\right\}\right]D$$

$$= 1.005E6\ D^3$$

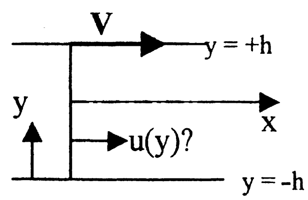
$$\text{Solve for}\quad \mathbf{D \approx 0.100\ m}\quad \textit{Ans. (a)}$$

(b) Use your value for D to calculate

$$V = (\Delta p/L)(D/2)^2/8\mu = (400)(0.050)^2/[8(0.104)] = 1.20\ m/s.$$

$$\text{Then}\quad Q = AV = [(\pi/4)(0.100\ m)^2](1.20\ m/s)(3600\ s/h) = \mathbf{34\ m^3/h}\quad \textit{Ans. (b)}$$

4.91 Consider 2-D incompressible steady Couette flow between parallel plates with the upper plate moving at speed V, as in Fig. 4.16a. Let the fluid be *nonnewtonian*, with stress given by



$$\tau_{xx} = a\left(\frac{\partial u}{\partial x}\right)^c \qquad \tau_{yy} = a\left(\frac{\partial v}{\partial y}\right)^c \qquad \tau_{xy} = \tau_{yx} = \frac{a}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^c, \quad a \text{ and } c \text{ are constants}$$

Make all the same assumptions as in the derivation of Eq. (4.140). (a) Find the velocity profile $u(y)$. (b) How does the velocity profile for this case compare to that of a newtonian fluid?

Solution: (a) Neglect gravity and pressure gradient. If u = u(y) and v = 0 at both walls, then continuity specifies that v = 0 everywhere. Start with the *x*-momentum equation:

$$\rho\left(\cancel{u\frac{\partial u}{\partial x}} + \cancel{v\frac{\partial u}{\partial y}}\right) = \cancel{\rho g_x} - \cancel{\frac{\partial p}{\partial x}} + \cancel{\frac{\partial \tau_{xx}}{\partial x}} + \frac{\partial \tau_{xy}}{\partial y}$$

Many terms drop out because v = 0 and τ_{xx} and $\partial u/\partial x = 0$ (because *u* does not vary with *x*). Thus we only have

$$\frac{\partial \tau_{xy}}{\partial y} = \frac{d}{dy}\left[\frac{a}{2}\left(\frac{du}{dy}\right)^c\right] = 0, \quad or: \quad \frac{du}{dy} = constant, \quad u = C_1 y + C_2$$

The boundary conditions are no-slip at both walls:

$$u(y=-h) = 0 = C_1(-h) + C_2; \quad u(y=+h) = V = C_1(+h) + C_2, \quad solve \;\; C_1 = \frac{V}{2h}, \; C_2 = \frac{V}{2}$$

The final solution for the velocity profile is:

$$\boldsymbol{u(y) = \frac{V}{2h}y + \frac{V}{2}} \quad Ans.\ (a)$$

This is **exactly the same** as Eq. (4.140) for the newtonian fluid! *Ans.* (b)

4.92 A tank of area A_o is draining in laminar flow through a pipe of diameter *D* and length *L*, as shown in Fig. P4.92. Neglecting the exit-jet kinetic energy and assuming the pipe flow is driven by the hydrostatic pressure at its entrance, derive a formula for the tank level *h*(*t*) if its initial level is h_o.

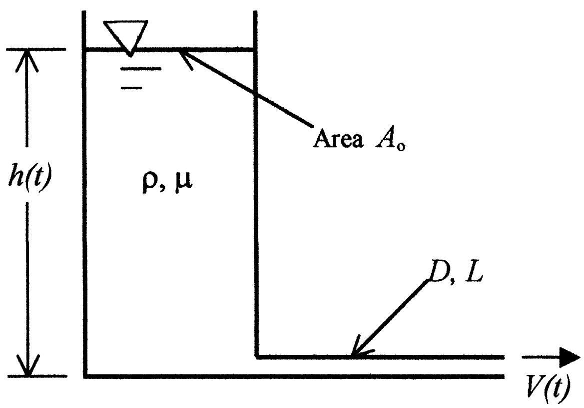


Fig. P4.92

Solution: For laminar flow, the flow rate out is given by Eq. (4.147). A control volume mass balance shows that this flow out is balanced by a tank level decrease:

$$Q_{out} = \frac{\pi D^4}{128\mu}\frac{\Delta p}{L} = -A\frac{dh}{dt} \quad \text{where } \Delta p \approx \rho g h(t)$$

Thus we can separate the variables and integrate to find the tank level change:

$$\int_{h_o}^{h}\frac{dh}{h} = -\int_0^t \frac{\pi D^4 \rho g}{128\mu L A_o}dt, \quad or: \quad \boldsymbol{h = h_o \exp\left[-\frac{\pi D^4 \rho g}{128\mu L A_o}t\right]} \quad Ans.$$

4.93 A number of straight 25-cm-long microtubes, of diameter *d*, are bundled together into a "honeycomb" whose total cross-sectional area is 0.0006 m^2. The pressure drop from entrance to exit is 1.5 kPa. It is desired that the total volume flow rate be 1 m^3/h of water at 20°C. (a) What is the appropriate microtube diameter? (b) How many microtubes are in the bundle? (c) What is the Reynolds number of each microtube?

Solution: For water at 20°C, ρ = 998 kg/m^3 and μ = 0.001 kg/m·s. Each microtube of diameter *D* sees the same pressure drop. If there are N tubes,

$$Q = \frac{1}{3600}\ \frac{m^3}{s} = NQ_{tube} = N\frac{\pi D^4 \Delta p}{128\mu L} = N\frac{\pi D^4(1500\,Pa)}{128(0.001\,kg/m\cdot s)(0.25\,m)} = 1.47E5\,N\ D^4$$

$$\text{At the same time,}\quad N = A_{bundle}/A_{tube} = \frac{0.0006\,m^2}{(\pi/4)D^2}$$

Combine to find $D^2 = 2.47E{-}6\ m^2$ or **D = 0.00157 m** *and* **N = 310** *Ans.*(a, b)

With *D* known, compute $V = Q/A_{\text{bundle}} = Q_{\text{tube}}/A_{\text{tube}}$ = 0.462 m/s and

$$Re_D = \rho VD/\mu = (998)(0.462)(0.00157)/(0.001) = \mathbf{724}\ \text{(laminar)}\quad \textit{Ans.}\ \text{(c)}$$

FUNDAMENTALS OF ENGINEERING EXAM PROBLEMS: Answers

Chapter 4 is not a favorite of the people who prepare the FE Exam. Probably not a single problem from this chapter will appear on the exam, but if some did, they might be like these:

FE4.1 Given the steady, incompressible velocity distribution $\mathbf{V} = 3x\mathbf{i} + Cy\mathbf{j} + 0\mathbf{k}$, where C is a constant, if conservation of mass is satisfied, the value of C should be

(a) 3 (b) 3/2 (c) 0 (d) –3/2 **(e) –3**

FE4.2 Given the steady velocity distribution $\mathbf{V} = 3x\mathbf{i} + 0\mathbf{j} + Cy\mathbf{k}$, where C is a constant, if the flow is irrotational, the value of C should be

(a) 3 (b) 3/2 **(c) 0** (d) –3/2 (e) –3

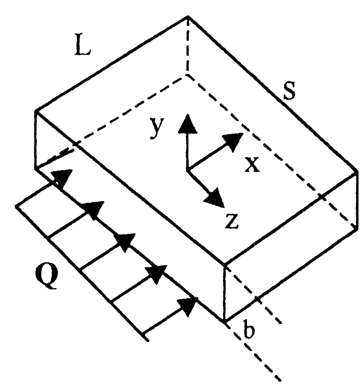
FE4.3 Given the steady, incompressible velocity distribution $\mathbf{V} = 3x\mathbf{i} + Cy\mathbf{j} + 0\mathbf{k}$, where C is a constant, the shear stress τ_{xx} at the point (x, y, z) is given by

(a) 3μ (b) $(3x + Cy)\mu$ **(c) 0** (d) $C\mu$ (e) $(3 + C)\mu$

COMPREHENSIVE PROBLEMS

C4.1 In a certain medical application, water at room temperature and pressure flows through a rectangular channel of length L = 10 cm, width s = 1 cm, and gap thickness b = 0.3 mm. The volume flow is sinusoidal, with amplitude Q_o = 0.5 ml/s and frequency f = 20 Hz, that is, $Q = Q_o\sin(2\pi f\,t)$.
(a) Calculate the maximum Reynolds number Re = Vb/ν, based on maximum average velocity and gap thickness. Channel flow remains *laminar* for Re < 2000, otherwise it will be *turbulent*. Is this flow laminar or turbulent?
(b) Assume quasi-steady flow, that is, solve as if the flow were steady at any given Q(t). Find an expression for streamwise velocity u as a function of y, μ, dp/dx, and b, where dp/dx is the pressure gradient required to drive the flow through the channel at flow rate Q. Also estimate the maximum magnitude of velocity component u.
(c) Find an analytic expression for flow rate Q(t) as a function of dp/dx.
(d) Estimate the wall shear stress τ_w as a function of Q, f, μ, b, s, and time t.
(e) Finally, use the given numbers to estimate the wall shear amplitude, τ_{wo}, in Pa.

Solution: (a) Maximum flow rate is the amplitude, Q_o = 0.5 ml/s, hence average velocity V = Q/A:



$$V = \frac{Q}{bs} = \frac{0.5E\text{–}6\ m^3/s}{(0.0003\,m)(0.01\,m)} = 0.167\,m/s$$

$$\mathrm{Re}_{\max} = \frac{Vb}{\nu} = \frac{(0.167)(0.0003)}{(0.001/998)}$$

$$= \mathbf{50\ (laminar)} \quad \textit{Ans.}\ \text{(a)}$$

(b, c) The quasi-steady analysis is just like Eqs. (4.142–144) of the text, with "h" = b/2:

$$\boldsymbol{u = \frac{-1}{2\mu}\frac{dp}{dx}\left(\frac{b^2}{4} - y^2\right),\quad u_{\max} = \frac{-1}{2\mu}\frac{dp}{dx}\frac{b^2}{4},\quad Q_{\max} = \frac{2}{3}u_{\max}bs = \frac{-sb^3}{12\mu}\frac{dp}{dx}} \quad \textit{Ans.}\ \text{(b, c)}$$

$$\text{(d) Wall shear:}\quad \tau_{wall} = \mu\left|\frac{du}{dy}\right|_{wall} = \boldsymbol{\frac{b}{2}\frac{dp}{dx} = \frac{6\mu Q}{sb^2} = \frac{6\mu Q_o}{sb^2}\sin(2\pi f\,t)} \quad \textit{Ans.}\ \text{(d)}$$

(e) For our given numerical values, the amplitude of wall shear stress is:

$$\tau_{wo} = \frac{6\mu Q_o}{sb^2} = \frac{6(0.001)(0.5E\text{–}6)}{(0.01)(0.0003)^2} = \mathbf{3.3\,Pa} \quad \textit{Ans.}\ \text{(e)}$$

C4.2 A belt moves upward at velocity V, dragging a film of viscous liquid of thickness *h*, as in Fig. C4.2. Near the belt, the film moves upward due to no-slip. At its outer edge, the film moves downward due to gravity. Assuming that the only non-zero velocity is $v(x)$, with zero shear stress at the outer film edge, derive a formula for (a) $v(x)$; (b) the average velocity V_{avg} in the film; and (c) the wall velocity V_C for which there is no net flow either up or down. (d) Sketch $v(x)$ for case (c).

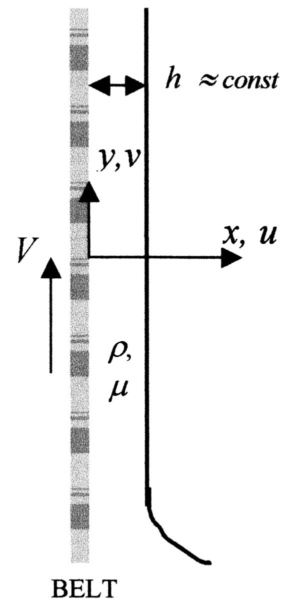


Fig. C4.2

Solution: (a) The assumption of parallel flow, $u = w = 0$ and $v = v(x)$, satisfies continuity and makes the *x*- and *z*-momentum equations irrelevant. We are left with the *y*-momentum equation:

$$\rho\left(\cancel{u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}}\right)=-\cancel{\frac{\partial p}{\partial y}}-\rho g+\mu\left(\frac{\partial^2 v}{\partial x^2}+\cancel{\frac{\partial^2 v}{\partial y^2}}+\cancel{\frac{\partial^2 v}{\partial z^2}}\right)$$

There is no convective acceleration, and the pressure gradient is negligible due to the free surface. We are left with a second-order linear differential equation for $v(x)$:

$$\frac{d^2v}{dx^2}=\frac{\rho g}{\mu}\quad \text{Integrate: } \frac{dv}{dx}=\frac{\rho g}{\mu}x+C_1\quad \text{Integrate again: } v=\frac{\rho g}{\mu}\frac{x^2}{2}+C_1x+C_2$$

At the free surface, $x = h$, $\tau = \mu(dv/dx) = 0$, hence $C_1 = -\rho gh/\mu$. At the wall, $v = V = C_2$. The solution is

$$\boldsymbol{v = V - \frac{\rho g h}{\mu}x + \frac{\rho g}{2\mu}x^2}\quad \textit{Ans. (a)}$$

(b) The average velocity is found by integrating the distribution $v(x)$ across the film:

$$v_{avg}=\frac{1}{h}\int_0^h v(x)\,dx=\frac{1}{h}\left[Vx-\frac{\rho g h x^2}{2\mu}+\frac{\rho g x^3}{6\mu}\right]_0^h=\boldsymbol{V-\frac{\rho g h^2}{3\mu}}\quad \textit{Ans. (b)}$$

(c) Since $hv_{avg} \equiv Q$ per unit depth into the paper, there is no net up-or-down flow when

$$\boldsymbol{V = \rho g h^2/(3\mu)}\quad \textit{Ans. (c)}$$

(d) A graph of case (c) is shown below. *Ans.* (d)