Chapter 5

Solutions

5.1

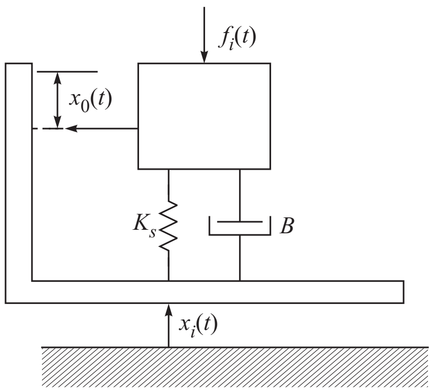


Fig. 1

$$f_i - K_s\, x_0 - BD\, x_0 = MD^2\, (x_0 - x_i) \tag{1}$$
$$(MD^2 + BD + K_s)x_0 = f_i + MD^2\, x_i \tag{2}$$

$$x_0 = \frac{1}{MD^2 + BD + K_s} f_i + \frac{MD^2}{MD^2 + BD + K_s} x_i \tag{3}$$

If the input acceleration $\ddot{x}_i(t)$ can be measured, the output signal can be subtracted appropriately to eliminate this error. This can be easily done while using impedance heads, as they measure force and acceleration together.

5.2 We need to calculate the bending moment at each of the gage locations and then use the property of bridge circuit to calculate the net effect of all the gages.

$$T = FL_f \tag{1}$$

Bending moment at location L is given by

$$T_L = T - L \times F \tag{2}$$

$$T_L = \frac{T(L_f - L)}{L_f} \tag{3}$$

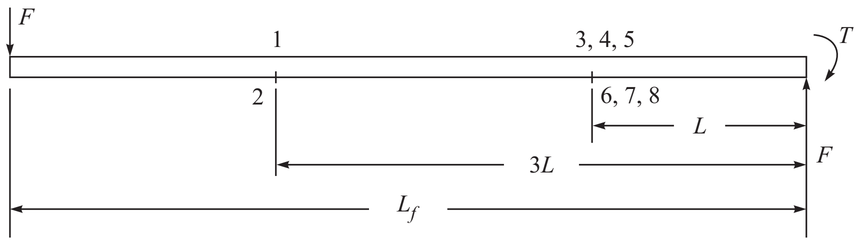


Fig. 1

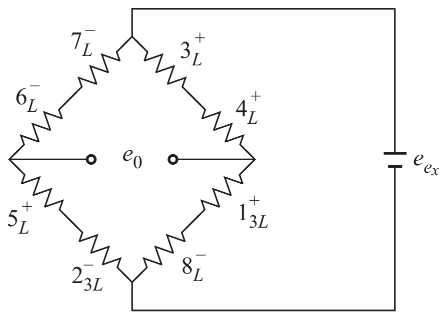


Fig. 2

$$T_{3L} = \frac{T(L_f - 3L)}{L_f} \tag{4}$$

Note: The usual rule of bridge circuits is that adjacent gages "fight" and opposite gages "cooperate"

As shown in Fig. 2, the signs, depending on whether the gages are in compression or tension are noted along with their locations.

Let us take the arm containing gages 3 and 4 as the reference.

		Net voltage	Location
3_L^+		+	L
4_L^+		+	L
-6_L^-	adjacent	+	L
-7_L^-		+	L
-8_L^-	adjacent	+	L
-1_{3L}^+		−	$3L$
$+5_L$	opposite	+	L
$+2_{3L}^-$		−	$3L$

6 gages at location 'L' are positive

2 gages at location $3L$ are negative

$$e_0 = K\left\{\frac{6T(L_f - L)}{L_f} - \frac{2T(L_f - 3L)}{L_f}\right\} \tag{5}$$

where K is a constant of the bridge.

5.3

$$M_1 g - V_1 \times \rho_{air} \times g = M_2 g - V_2 \times \rho_{air} \times g$$

M_1 : Mass of the object to be weighed
V_1 : Volume of the object
ρ_{air} : Density of air = 1.23 kg/m^3
M_2 : Mass of the standard weight
V_2 : Volume of the standard weight

$$M_1 g = M_2 g + (V_1 - V_2)\, \rho_{air} \times g \tag{1}$$

$$M_1 = M_2 + (V_1 - V_2) \times \rho_{air} \tag{2}$$

$$M_1 = 0.5 + (160 - 50) \times 10^{-6} \times 1.23$$
$$= 0.5001353 \text{ kg}$$

Therefore, the correction is negligible.

5.4

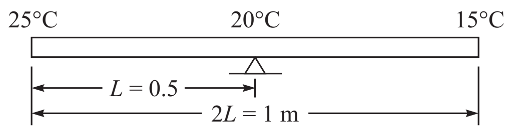


Fig. 1

Left and the right half must be considered separately. The temperature at the pivot is 20°C

For the right half $$T(x) = 5\left(1 - \frac{x}{L}\right) + 15 \tag{1}$$

where x is measured from the pivot.

$$\Delta T(x) = 5\left(1 - \frac{x}{L}\right) \tag{2}$$

$$\Delta L = \int_0^L \alpha \Delta T(x)\,dx = 5\alpha \int_0^L \left(1 - \frac{x}{L}\right) dx \tag{3}$$

$$\Delta L = \frac{5}{2}\alpha L \tag{4}$$

α: Coefficient of thermal expansion

for the left beam

$$\Delta T(x) = 5\left(2 - \frac{x}{L}\right)$$

where x is measured from the left end.

$$\Delta L = \int_0^L 5\alpha\left(2 - \frac{x}{L}\right) dx = 5\alpha \int_0^L \left(2 - \frac{x}{L}\right) dx = \frac{15\alpha L}{2}$$

$\therefore$ They are unequal by $\dfrac{15\alpha L}{2} - \dfrac{5}{2}\alpha L = 5\alpha L$

5.5 Since the center of mass is designed to be slightly below the pivot point, the balance is essentially a pendulum. For small oscillations, a pendulum is a second-order dynamic system with small damping.

5.6

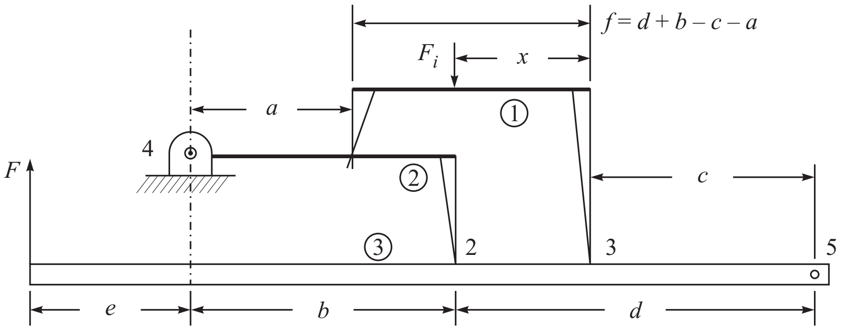


Fig. 1

Objective: To prove that F does not vary when the point x, where the input force F_i is applied moves.

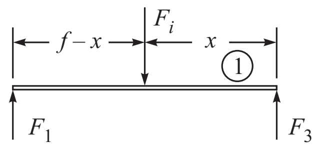


Fig. 2

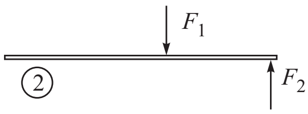


Fig. 3

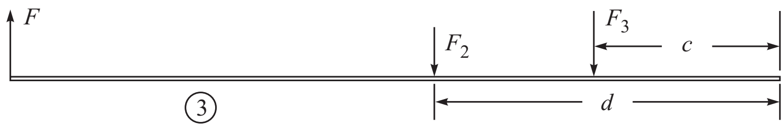


Fig. 4

from Fig. 2, $$F_i = F_1 + F_3 \tag{1}$$

from Fig. 3, $$F_1 \times a = F_2 \times b \quad \Rightarrow \quad F_2 = F_1 \frac{a}{b} \tag{2}$$

from Fig. 1. $$F_1 \times f = F_i\, x \tag{3}$$

from Fig. 2. $$F_1(f - x) = F_3\, x$$

$$F_1 = F_3\left(\frac{x}{f-x}\right) \qquad (4)$$

$$F_3 = F_i - F_1 \qquad (5)\ \{\text{from eq. } 1\}$$

$$F_3 = F_i - F_3\frac{x}{f-x}$$

$$\rightarrow F_3 = \left(\frac{f-x}{f}\right)F_i \qquad (6)$$

$$F_1 = F_i - F_3 \qquad (7)\ \{\text{from eq. } 1\}$$

$$F_1 = F_i\left\{1-\frac{f-x}{f}\right\}$$

$$F_1 = F_i\frac{x}{f} \qquad (8)$$

$$\rightarrow F_2 = F_i\frac{x}{f}\frac{a}{b} \qquad (9)\ \{\text{from eq. (2) and (8)}\}$$

The sum of the moments $F_2d + F_3c$ must be constant if F has to be independent of the position of F_i

$$F_2d + F_3c = F_i\ \frac{x}{f}\frac{ad}{b} + \left(\frac{f-x}{f}\right)F_ic$$

$$= \frac{F_i}{f}\left\{\frac{adx}{b} + (f-x)c\right\} \qquad (10)$$

Given $$\frac{a}{b} = \frac{c}{d} \qquad (11)$$

from (10) and (11), $$F_2d + F_3c = \frac{F_i}{f}\left\{\frac{c}{d}\times dx + (f-x)c\right\}$$

$$= F_ic = \text{constant}$$

5.7

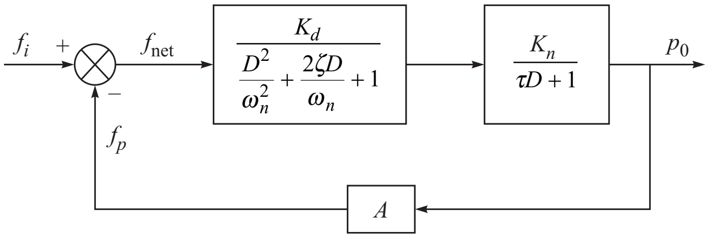
$$M = 0.5 \text{ kg (Mass)}$$
$$A = 20\text{g (acceleration)}$$
$$\text{Force} = \text{Mass} \times \text{acceleration}$$
$$= 0.5 \times 20 \times 9.81 = 98.1 \text{ N}$$

If $F_f = 5$ N is the friction,

$= 103.1$ N $\quad F = 98.1 + 5$

$$\text{Error} = \frac{103.1 - 98.1}{103.1} = 4.84\%$$

5.8



$$G(D) = \frac{K_n K_d}{\left(\frac{D^2}{\omega_n^2} + \frac{2\zeta D}{\omega_n} + 1\right)(\tau D + 1)} \quad H = \text{A} \tag{1}$$

$$\frac{p_0}{f_i} = \frac{G(D)}{1 + G(D)\,H} = \frac{\dfrac{K_n K_d}{\left(\frac{D^2}{\omega_n^2} + \frac{2\zeta D}{\omega_n} + 1\right)(\tau D + 1)}}{1 + \dfrac{K_n K_d A}{\left(\frac{D^2}{\omega_n^2} + \frac{2\zeta D}{\omega_n} + 1\right)(\tau D + 1)}} \tag{2}$$

$$\frac{p_0}{f_i} = \frac{K_n K_d}{\frac{1}{\omega_n^2}(D^2 + 2\zeta\omega_n D + \omega_n^2)(\tau D + 1) + K_n K_d A} \tag{3}$$

Denominator of Eq. (3)

$$\left(\frac{\tau}{\omega_n^2}\right)D^3 + \left(\frac{2\zeta\tau}{\omega_n} + \frac{1}{\omega_n^2}\right)D^2 + \left(\tau + \frac{2\zeta}{\omega_n}\right)D + 1 + K_n K_d A \tag{4}$$

Routh's stability criterion can be applied to Eq. (4) for specific values.

5.9 Newton's law should be applied to the motor and slide

$$f_m - K_s(x_m - x_s) = M_m D^2 x_m \tag{1}$$

f_m: Magnetic force on the motor
K_s: Stiffness of the load cell
x_m: Displacement of the motor

x_s : Displacement of the slide
M_m : Mass of the motor

$$K_s(x_m - x_s) = M_s\, D^2\, x_s \tag{2}$$

M_s: Mass of the slide.

Rewriting Eqs. (1) and (2) in the matrix form

$$\begin{bmatrix} M_m D^2 & 0 \\ 0 & M_s D^2 \end{bmatrix}\begin{Bmatrix} x_m \\ x_s \end{Bmatrix} + \begin{bmatrix} K_s & -K_s \\ -K_s & K_s \end{bmatrix}\begin{Bmatrix} x_m \\ x_s \end{Bmatrix} = \begin{Bmatrix} f_m \\ 0 \end{Bmatrix} \tag{3}$$

or

$$\begin{bmatrix} M_m D^2 + K_s & -K_s \\ -K_s & M_s D^2 + K_s \end{bmatrix}\begin{Bmatrix} x_m \\ x_s \end{Bmatrix} = \begin{Bmatrix} f_m \\ 0 \end{Bmatrix} \tag{4}$$

Solving Eq. (4)

$$x_m = \frac{1}{M_m M_s D^4 + K_s D^2 (M_m + M_s)} \times (M_s\, D^2 + K_s) f_m \tag{5}$$

$$x_s = \frac{1}{M_m M_s D^4 + K_s D^2 (M_m + M_s)} \times K_s\, f_m \tag{6}$$

from Eqs. (5) and (6)

$$x_m - x_s = \frac{f_m M_s D^2}{M_m M_s D^4 + K_s D^2 (M_m + M_s)}$$

$$e_0 = K_n\, (x_m - x_s)\, K_s \tag{7}$$

where K_n = volts/N

$$\frac{f_m}{e_0}(D) = \frac{\cancel{f_m}\,(M_m M_s D^4 + K_s D^2 (M_m + M_s))}{K_n \cancel{f_m}\, M_s D^2 K_s}$$

$$\frac{f_m}{e_0}(D) = \frac{D^2 (D^2\, M_m\, M_s + K_s\, (M_m + M_s))}{K_n\, D^2\, M_s\, K_s} \tag{8}$$

$$\frac{f_m}{e_0}(D) = \left(\frac{M_s + M_m}{M_s}\right) \cdot \frac{1}{K_n}\left\{\frac{D^2}{\omega_n^2} + 1\right\} \tag{9}$$

If $M_m << M_s$

$$\frac{f_m}{e_0}(D) = \frac{1}{K_n}\left\{\frac{D^2}{\omega_n^2} + 1\right\} \tag{10}$$

f_m leads e_0 because 'f'$_m$ happens first.

$$\frac{f_m}{e_0}(j\omega) = \frac{1}{K_n}\left\{1-\left(\frac{\omega}{\omega_n}\right)^2\right\} \tag{11}$$

The frequency response is shown in Fig. 1.
The band width is $0.2\,\omega_n$ for 5% accuracy

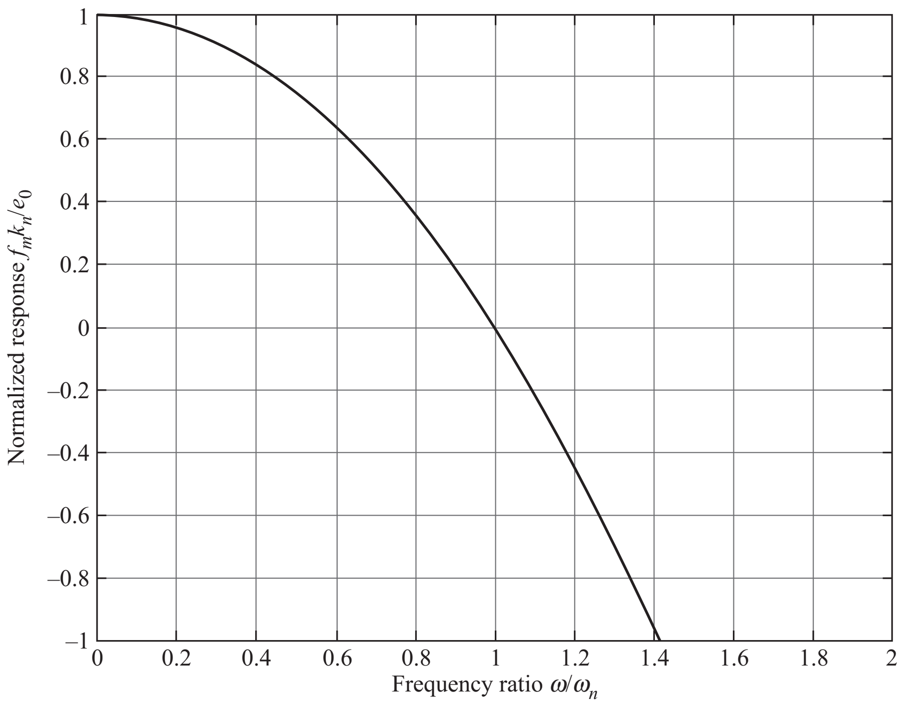
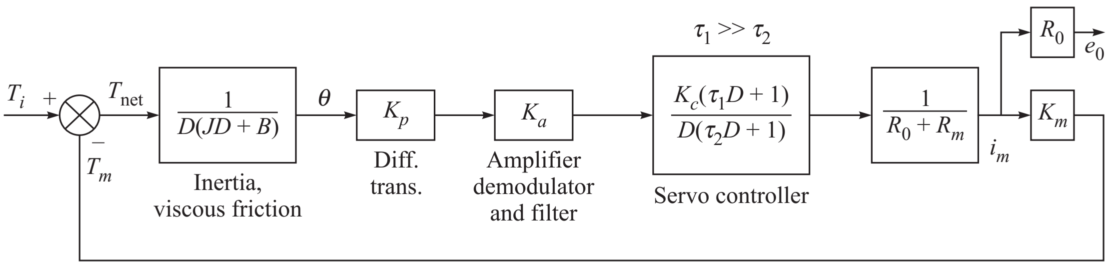


Fig. 1

5.10



Feedback torque sensor (Fig. 5.15).

$$i_m = \frac{e_0}{R_0} \tag{1}$$

$$T_m = i_m K_m = \frac{e_0}{R_0} K_m \tag{2}$$

$$\left(T_i - \frac{K_m e_0}{R_0}\right)\left\{\frac{1/B}{D\left(\frac{J}{B}\right)D+1}\right\}\left\{K_p K_a K_c\left(\frac{\tau_1 D+1}{D}\right)\frac{R_0}{R_0+R_m}\right\} = e_0$$

{ τ_2 is neglected for this problem}

Gathering terms in T_i and e_0

$$\left(\frac{\tau_m}{K_1}D^3 + \frac{1}{K_1}D^2 + \tau_1 D + 1\right)e_0 = \frac{R_0}{K_m}(\tau_1 D + 1)\, T_i \tag{3}$$

$$\tau_m = \frac{J}{B}; \quad K_1 = \frac{K_p K_a K_c K_m}{B(R_0 + R_m)}$$

The characteristic equation is given by the left hand side of Eq. (3)

$$\frac{\tau_m}{K_1}D^3 + \frac{1}{K_1}D^2 + \tau_1 D + 1 = 0$$

Applying Routh criterion

$\tau_1 > \tau_m$ is the condition for stability. The above approximate analysis is useful only for moderate loop gains.

5.11 To prove that $\theta = 0$ for any constant value of T_i. If T_i is a constant, right hand side of Eq. (3) of Prob. 5.10 becomes

$$\frac{R_0}{K_m} T_i \tag{1}$$

Since there is an integrator in the servo controller loop, θ should be zero to get a constant output from it.

5.12

(i) Axial stresses affect all 4 gages identically, so the bridge is not sensitive to axial loads.

(ii) Bending can occur in any plane that contains the shaft axis. Such a moment can always be resolved into components in the plane of a gage-pair and perpendicular to this plane. The component in the perpendicular plane puts the gage-pair on the neutral axis, giving no strain. The in-plane component produces compression in one gage of the pair and equal tension in the other. Since these two gages are in adjacent bridge legs, the net contribution from them is zero. Due to the above two reasons, the torque sensor is insensitive to axial and bending loads.

5.13 To prove that for equivalent strain/torque sensitivity, a square shaft is stiffer in bending than a round one.

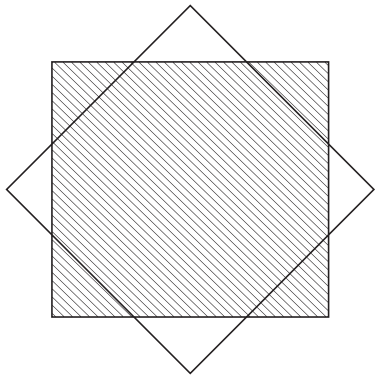


Fig. 1

For a torsion square shaft, the maximum stress occurs at the midpoint of each side 'a' given by

$$\sigma_{\max} = \frac{4.81T}{a^3} \tag{1}$$

where a is the length of each side and T, torque.

For a round member

$$\sigma_{\max} = \frac{0.637T}{r^3} \tag{2}$$

where r is the radius.

for the same maximum stress, equating (1) and (2)

$$r = a\left(\frac{0.637}{4.81}\right)^{1/3} = 0.510\text{ a} \tag{3}$$

$$\left.\begin{array}{lll} \text{For the shaded region} & I = \dfrac{a^4}{12} = 0.083a^4 & (4) \\ \text{For the unshaded region} & I = \dfrac{a^4}{6} & (5) \end{array}\right\} \text{square section}$$

For a circle $\quad I = 0.785\ r^4 \quad$ (6)

from Eq. (3) $\quad I = 0.785 \times 0.510^4\ a^4 = 0.0531a^4 \quad$ (7)

From Eqs. (4) and (7), since the square shaft has higher moment of inertia for the same torque sensitivity than the round shaft, it is stiffer in bending.

5.14

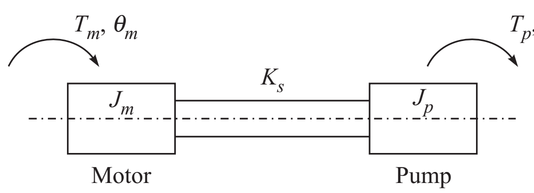


Fig. 1

$$\begin{bmatrix} J_m & 0 \\ 0 & J_p \end{bmatrix}\begin{Bmatrix} \ddot{\theta}_m \\ \ddot{\theta}_p \end{Bmatrix} + \begin{bmatrix} K_s & -K_s \\ -K_s & K_s \end{bmatrix}\begin{Bmatrix} \theta_m \\ \theta_p \end{Bmatrix} = \begin{Bmatrix} T_m \\ T_p \end{Bmatrix} \tag{1}$$

$$\begin{bmatrix} J_m D^2 + K_s & -K_s \\ -K_s & J_p D^2 + K_s \end{bmatrix}\begin{Bmatrix} \theta_m \\ \theta_p \end{Bmatrix} = \begin{Bmatrix} T_m \\ T_p \end{Bmatrix} \tag{2}$$

$$\begin{Bmatrix} \theta_m \\ \theta_p \end{Bmatrix} = \frac{1}{\det}\begin{bmatrix} J_p D^2 + K_s & K_s \\ K_s & J_m D^2 + K_s \end{bmatrix}\begin{Bmatrix} T_m \\ T_p \end{Bmatrix} \tag{3}$$

$$\begin{aligned} \det &= (J_m D^2 + K_s)(J_p D^2 + K_s) - K_s^2 \\ &= J_m J_p D^4 + D^2(J_m K_s + J_p K_s) \\ &= J_m J_p D^4 + K_s D^2(J_m + J_p) \\ &= D^2 J_m J_p \left\{D^2 + \frac{K_s(J_m + J_p)}{J_m J_p}\right\} \end{aligned} \tag{4}$$

$$\begin{aligned} \theta_m &= \{(J_p D^2 + K_s) T_m + K_s T_p\}/\det \\ \theta_p &= \{K_s T_m + (J_m D^2 + K_s)T_p\}/\det \end{aligned} \tag{5}$$

$$\frac{e_0}{K_T}(D) = K_s(\theta_m - \theta_p) \tag{6}$$

$$\frac{K_s\left\{(J_p D^2 + K_s)T_m + K_s T_p - K_s T_m - (J_m D^2 + K_s)T_p\right\}}{D^2 J_m J_p \left\{D^2 + \dfrac{K_s(J_m + J_p)}{J_m J_p}\right\}}$$

By physical considerations $T_p = -T_m$

$$\frac{e_0}{K_T}(D) = \frac{K_T}{\dfrac{D^2}{\omega_n^2} + 1} \tag{7}$$

where $$\omega_n = \sqrt{\frac{K_s}{\dfrac{J_m J_p}{J_m + J_p}}}$$

For 5% error,

Eq. (7) has flat response upto $w = 0.2\,\omega_n$

$\therefore$ $$\omega_n = 5\omega = 5 \times 60 = 300 \text{ Hz}$$

$$K_T = 110 \text{ N–m/rad} \quad J_m = 0.0016 \text{ kg–m}^2 = J_p$$

$$(300 \times 2\pi)^2 = \frac{K_s}{\dfrac{0.0016^2}{2 \times 0.0016}}$$

$$K_s = 2842 \text{ N–m/rad}$$

5.15 Figure 5.23

Let F be the force in Newtons and ω angular velocity in rad/s

Sensitivity of the tacho generator

$$K_T = \frac{6V}{1000\,RPM} = \frac{6}{\frac{1000}{60}\times 2\pi} = 0.0573 \text{ V/rad/s} \quad (1)$$

$$e_{ex} = K_T\,\omega = (0.0573\,\omega)\text{V} \quad (2)$$

Sensitivity of load cell

$$K_L = 0.011 \text{ mV/Force/}e_{ex} \quad (3)$$

Output

$$e = 0.011 \text{ mV} \times F \times 0.0573\,\omega \quad (4)$$

$$F\,\omega = 1586.5\,e\{\text{mV}\} \quad (5)$$

$$P = T\omega = Fl\omega \quad (6)$$

$$P = 1586.5 \times 0.3\ e \quad (7)$$

$$P = 476\ e \quad (8)$$

Calibration factor = 476 ω/mV

5.16

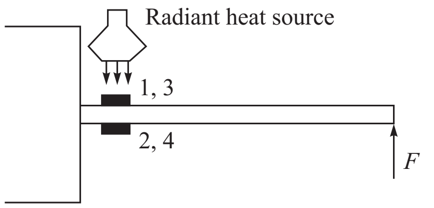


Fig. 1

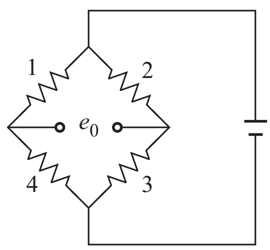


Fig. 2

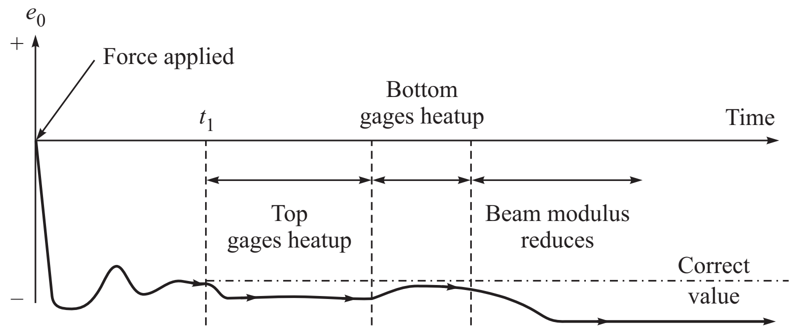


Fig. 3

As per the bridge configuration, increase in resistance of gages 1 and 3 causes an increase in output voltage, where as increase in 2 and 4 decreases the voltage.

At time $t = 0$, when F is applied, 1 and 3 are subject to compression and 2 and 4 are subjected to tension. Therefore, there is a net negative output, 4 times larger than for a single gage. The output settles down to a correct value.

At t_1, heating is applied, the top gages will first heat up assuming the dominance of differential expansion, 1 and 3 have decreased resistance. Therefore, e_0 becomes more negative.

As bottom gages also heat up, temperature compensation takes place and e_0 comes back to the correct value (almost). Since the bottom gages are always slightly cooler, e_0 is slightly more negative than the correct value.

After somtime, the entire beam gets heated up, resulting in reduction of modulus. This increases strain in all the gages, which results in a constant offset from the true value

5.17

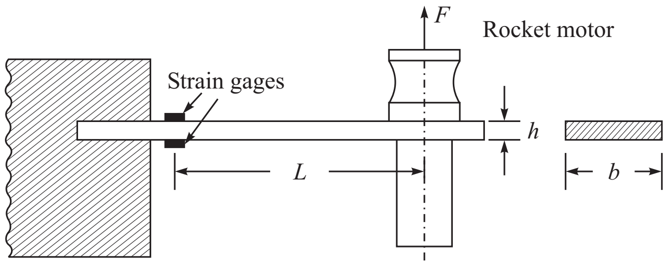


Fig. 1

Let h = thickness, b = breadth, L = length of the cantilever beam

L = Point of force application to the centreline of the strain gage element

$$\text{Stress } \sigma = \frac{6FL}{bh^2} \tag{1}$$

$$F = \text{force}$$

$$\text{Stiffness} = K_s = \frac{4bh^3 E}{L^3} \tag{2}$$

ω_n = 5: ω_n = natural frequency rad/s

ω: Maximum excitation frequency

= 100 Hz (given)

M = mass of the rocket engine (much larger than the beam)

= 10 kg (given)

$K_s = M\omega_n^2 = M\omega^2 = -5 \times (2\pi \times 500)^2 = 49.34$ MN/m

From Eqs. (1) and (2) $\left(\text{noting } \dfrac{\sigma}{\varepsilon} = E\right)$

$$K_s = \frac{24Fh}{\varepsilon L^2} \tag{3}$$

Let $\varepsilon = 1500$ μs $F = 250$ N (given)

(max. strain)

From Eq. (3)
$$\frac{h}{L^2} = \frac{49.34 \times 10^6 \times 1500 \times 10^{-6}}{24 \times 250} \left\{= \frac{K_s\,\varepsilon}{24F}\right\} \tag{4}$$

$$\frac{h}{L^2} = 12.335$$

If $\quad L = 0.250$ m $\quad h = 12.335 \times 0.250^2 = 0.77$ m (too large)

Therefore, K_s is reduced 5 times by reducing its natural frequency 5 times

$$K_s = 5 \times (2\pi \times 100)^2 = 1.97 \text{ MN/m}$$

From Eq. (4)

$$\frac{h}{L^2} = \frac{1.97 \times 10^6 \times 1500 \times 10^{-6}}{24 \times 250} = 0.4925$$

For $\quad L = 0.250, \quad h = 0.4925 \times 0.250^2 = 30.78$ mm

h is still large. Therefore ε is reduced to 300 μs

$$\frac{h}{L^2} = \frac{1.97 \times 10^6 \times 300 \times 10^{-6}}{24 \times 250} = 0.0985$$

If $\quad L = 0.250 \quad h = 0.0985 \times 0.250^2 = 6$ mm (acceptable)
If $\quad L = 0.375$ m $\quad h = 0.0985 \times 0.375^2 = 13.85$ mm

For reducing ω_n five times, we will have to add a damper
For reducing maximum strain, amplifier gain has to be increased

From Eq. (1) $$\frac{\sigma}{\varepsilon} = E = \frac{6FL}{\varepsilon b h^2} \tag{5}$$

From Eqs. (5)

$$b = \left\{\frac{6FL}{\varepsilon h^2}\right\}\frac{1}{E} = \frac{6 \times 250 \times 0.375}{300 \times 10^{-6} \times (13.85 \times 10^{-3})^2} = \frac{9.77 \times 10^9}{E}$$

For steel $\quad = 200$ GPa

$$b = \frac{9.77 \times 10^9}{200 \times 10^9} = 49 \text{ mm \{enough space to mount strain gages\}}$$

For the strain gages, the bridge factor = 4

$$\frac{e_0}{E_{ex}} = \left\{\frac{R_4}{(R_1 + R_4)^2}\Delta R_1\right\} \times 4 = \frac{\Delta R}{R} = \text{gage factor} \times \text{strain}$$

$$= 2.1 \times 300 \times 10^{-6} = 0.63 \text{ mV/V}$$

For $\quad E_{ex} = 10$ volts, $e_0 = 0.63 \times 10 = 6.3$ mV

$$\text{Sensitivity} = \frac{6.3\,\text{mV}}{250\,\text{N}} = 0.0252 \text{ mV/N}$$

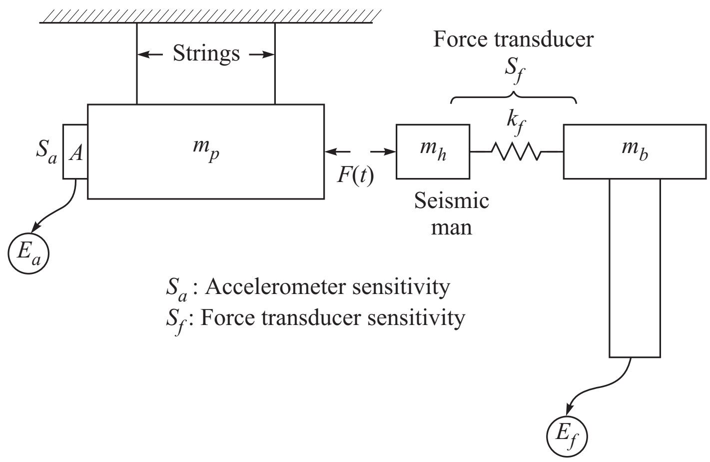
2.5 N changes will have to be detected for 1 mm (say) of the recorder. So, 4 mV are required at the recorder input.

$$e_0 \text{ for } 2.5\text{ N} = 0.0252 \times 2.5 = 0.063 \text{ mV}$$

$$\text{Amplification required} = \frac{4}{0.063} = 64 \text{ (acceptable).}$$

Use the second-order system theory to design the damper. If necessary, relax the accuracy to 10%

5.18



m_p is the mass of the pendulum, which is hung using strings. Accelerometer 'A' is mounted on the other face of the pendulum. When struck with the impact hammer, it indicates E_a volts.

The impact hammer has a force transducer of seismic mass 'm_h' and spring stiffness k_f. It is attached to the hammer body of mass 'm_b'. When the impact hammer strikes the pendulum, voltage E_f is indicated. S_a is the sensitivity of the accelerometer and S_f is the sensitivity of the force transducer.

$$S_a = 0.5 \text{ mV/m/s}^2$$

Slope of $\dfrac{E_f}{E_a}$ curve 25 m/s²/N

Case (i)

$$m_{h_1} = 0.1 \text{ kg}, \; m_{b_1} = 0.5 \text{ kg}$$
$$S_f = ?$$

$$\frac{S_f}{S_a} \propto \frac{E_f}{E_a} \qquad \frac{S_f}{S_a} = 25$$

$$S_f = S_a \times \left(\frac{S_f}{S_a}\right) = 25 \times 0.5 = 12.5 \text{ mV/N}$$

$$E_f = \left\{ \frac{m_{b_1}}{m_{h_1} + m_{b_1}} \right\} \frac{S_z}{k_f} \; F(t)$$

$$\frac{S_z}{k_f} = \frac{E_f / F(t)}{\dfrac{m_{b_1}}{m_{h_1} + m_{b_1}}} = \frac{S_f}{\dfrac{m_{b_1}}{m_{h_1} + m_{b_1}}} = \frac{12.5}{\dfrac{0.5}{0.1 + 0.5}}$$

$$= 15 \text{ mV/N/N/m} \quad \{\text{constant for the impact hammer}\}$$

Case (ii)

$$m_{h_2} = 0.3 \text{ kg} \quad m_{b_2} = 0.8 \text{ kg}$$

$$S_f = \frac{m_{b_2}}{m_{h_2} + m_{b_2}} \times \frac{S_z}{k_f}$$

$$= \left\{ \frac{0.8}{0.3 \times 0.8} \right\} \times 15 = 10.9 \text{ mV/N}$$

5.19

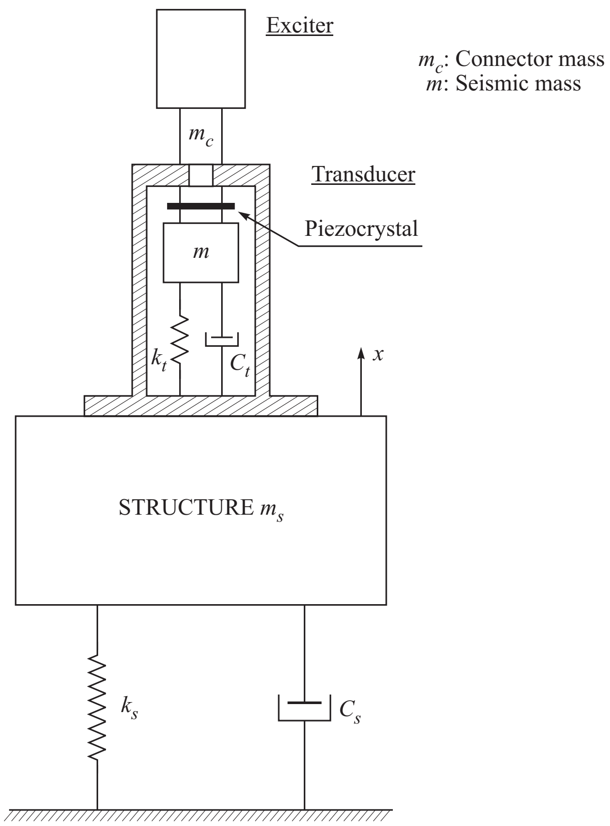


Fig. 1

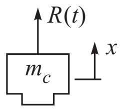


Fig. 2

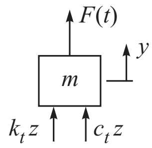


Fig. 3

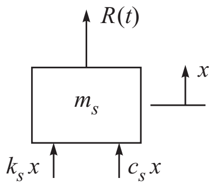


Fig. 4

Note: (1) $R(t)$ is applied on the connector mass by the exciter and the same force is experienced by the structure also

(2) z is very small, $y \approx x$, z is proportional to $F(t)$

(3) $F(t)$ is the force at the interface of the connector mass and the seismic mass at which the piezo crystal is located

Applying Newtons law to the seismic mass

$$m\ddot{z} + c_t\,\dot{z} + k_t\,z = R(t) - (m + m_c)\,\ddot{x} = F(t) \tag{1}$$

Applying Newtons law to the structure

$$m_s\ddot{x} + c_s\dot{x} + k_s\,x = R(t) \tag{2}$$

Let

$$R(t) = \bar{R}\,e^{j\omega t} \tag{3}$$

$$x(t) = \bar{X}\,e^{j\omega t} \tag{4}$$

From Eqs. (3) and (4), Eq. (2) becomes

$$[(-\,m_s\,\omega^2 + k_s) + j\omega c_s]\bar{X} = \bar{R} \tag{5}$$

$$\bar{X} = \frac{\bar{R}}{(k_s - m_s\,\omega^2) + j\omega\,c_s} \tag{6}$$

From Eqs. (4) and (6)

$$x(t) = \frac{\bar{R}}{(k_s - m_s\,\omega^2) + j\omega\,c_s}\;e^{j\omega t} \tag{7}$$

$$\ddot{x}(t) = \frac{-\bar{R}\,\omega^2}{(k_s - m_s\,\omega^2) + j\omega\,c_s}\,e^{j\omega t} \tag{8}$$

From Eqs. (1), (3) and (8)

$$\bar{F} = \bar{R}\left\{1 + \frac{(m + m_c)\,\omega^2}{(k_s - m_s\,\omega^2) + j\,c_s\,\omega}\right\} \tag{9}$$

From Eq. (9), the error in force measurement is given by

$$\bar{\varepsilon} = \frac{(m + m_c)\,\omega^2}{(k_s - m_s\,\omega^2) + j\,c_s\,\omega} \tag{10}$$

(a) given $m_c = 2$ kg $\quad m = 0.1$ kg

$$m_s = 100 \text{ kg}$$
$$k_s = 5 \text{ MN/m} \quad k_t = 3 \text{ GN/m}$$
$$\zeta_s = 3\%$$

$$\zeta_s = \frac{c_s}{2\sqrt{k_s\,m_s}} \tag{11}$$

$$c_s = 2\sqrt{k_s\,m_s\,\zeta_s} \tag{12}$$

$$= 2\sqrt{5 \times 10^{-6} \times 100} \times 0.03$$

$$1342 \text{ N–s/m} \tag{13}$$

$$\bar{\varepsilon} = \frac{2.1\,\omega^2}{(5 \times 10^6 - 100\,\omega^2) + j\,1342\,\omega} \tag{14}$$

Equation (11) is plotted for various values of ω_1 from 0 to $2\pi 100$ rad/s, as shown in Fig. 5 the magnitude of error is less than 1% upto 20 Hz and the phase is ≈ zero upto 20 Hz.

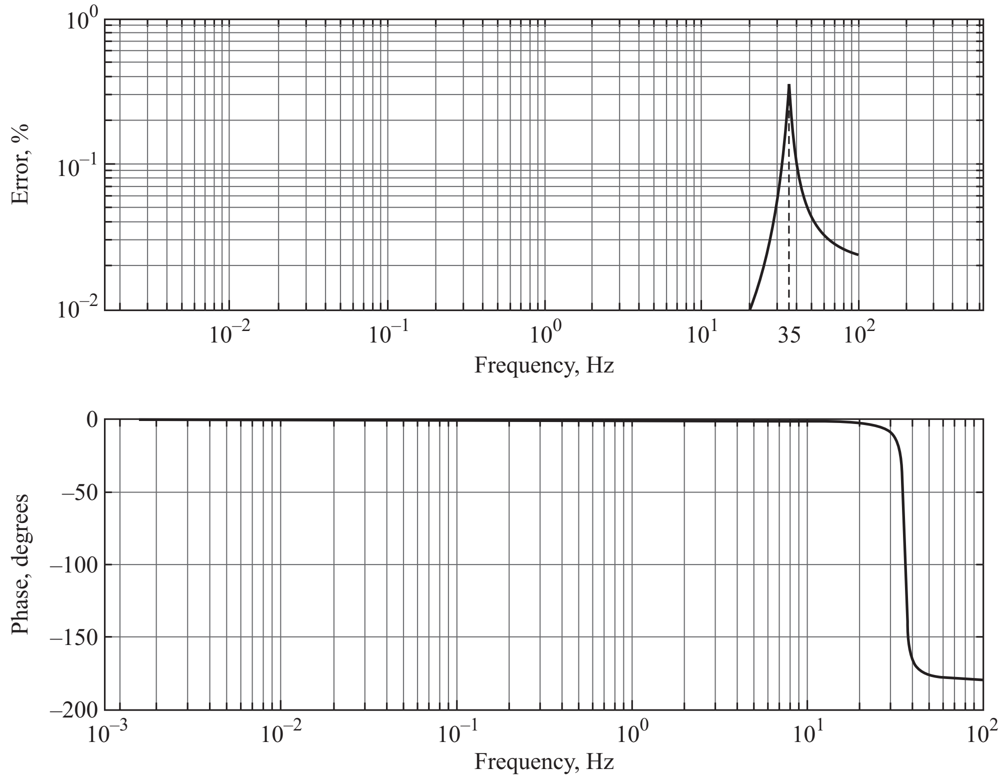
(b) $f = 20$ Hz

$$|\bar{\varepsilon}| = 0.97\% \quad \angle\bar{\varepsilon} = -3°$$

$$f = 70 \text{ Hz}$$

$$|\bar{\varepsilon}| = 2.83\% \quad \angle\bar{\varepsilon} = -178°$$

$|\varepsilon|_{max} = 35\%$ at $f_s = 35$ Hz (resonant frequency of the structure)



10^0
10^{-1}
10^{-2}
Error, %
10^{-2}
10^{-1}
10^0
10^1
35
10^2
Frequency, Hz
0
–50
–100
–150
–200
Phase, degrees
10^{-3}
10^{-2}
10^{-1}
10^0
10^1
10^2
Frequency, Hz

Fig. 5