Chapter 4

Solutions

4.1

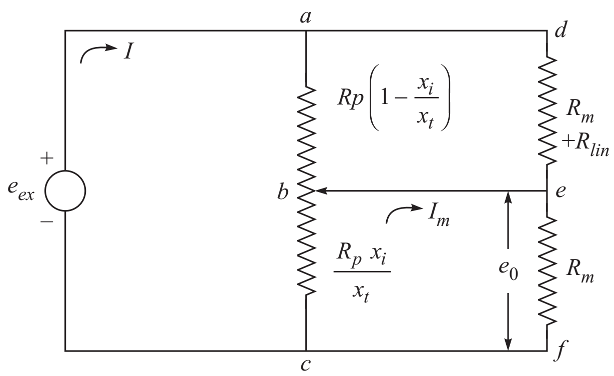


Fig. 1 [Fig. 4.5(b)]

R_P : Potentiometer resistance
x_i : Position of the wiper from the zero position
x_t : Total travel of the potentiometer

Figure 4.5 (b) is a modification of Fig. 4.5 (a) to improve the non linearity of the potentiometer. Basically we expect the ratio $\frac{e_0}{e_{ex}}$ to vary in a linear manner for various positions of the wiper. Fig. 4.5 (a) of the text shows that, as the ratio R_p/R_m keeps on increasing, the nonlineality gets worse. Adding a resistor $R_m + R_{lin}$ in parallel does reduce the overall nonlinearity, as shown in Fig. 4.5 (b). This can also be verified using MATLAB simulation.

The main objective of this problem is to derive an equation for the circuit of Fig. 4.5 (b) that is similar to Eq. (4.1). To simplify the equation it is assumed here that $R_{lin} = 0$, so we have only R_m in the arm '*de*'

For any position 'x_i' of the wiper, the resistance of the arm '*bc*' is given by

$$R_{bc} = \frac{R_p\, x_i}{x_t} \qquad (1)$$

The resistance of the arm '*ab*' is given by

$$R_{ab} = R_p \left(1 - \frac{x_i}{x_t}\right) \tag{2}$$

For the purpose of solving the above circuit of Fig. 4.5 (b), (a, d), (b, e), (c, f) are common points. First step is to find the equivalent resistance for the entire circuit, determining the total current 'I', current through the meter 'I_m', which gives e_0.

$$R_{ab}^{eq} = \frac{R_p\left(1 - \frac{x_i}{x_t}\right) R_m}{R_p\left(1 - \frac{x_i}{x_t}\right) + R_m} \tag{3}$$

$$R_{be}^{eq} = \frac{R_p \frac{x_i}{x_t} R_m}{R_p \frac{x_i}{x_t} + R_m} \tag{4}$$

Equations (3) can be simplified as

$$R_{ab}^{eq} = \frac{R_m\left(1 - \frac{x_i}{x_t}\right)}{1 - \frac{x_i}{x_t} + \frac{R_m}{R_p}} \tag{5}$$

Equation (4) can be simplified as

$$R_{bc}^{eq} = \frac{R_m \frac{x_i}{x_t}}{\frac{x_i}{x_t} + \frac{R_m}{R_p}} \tag{6}$$

The total resistance of the circuit is given by

$$R = R_{ab}^{eq} + R_{bc}^{eq} \tag{7}$$

The total current supplied by the battery at the wiper position x_i is given by

$$I = \frac{e_{ex}}{R} \tag{8}$$

The current 'I' splits into two and flows through the arms '*ab*' and '*de*' unites into I at the wiper point and once again splits into two to flow through the arms '*bc*' and '*ef*'

The current I_m through the meter is given by

$$I_m = I\ R_p \frac{x_i}{x_t} \Big/ R_m + R_p \frac{x_i}{x_t} \tag{9}$$

The output voltage is given by

$$e_0 = I_m\ R_m \tag{10}$$

From Eq. (6) through (9), Eq. (10) becomes

$$e_0 = \frac{e_{ex}\ R_m\ R_p \dfrac{x_i}{x_t} \Big/ \left(R_m + R_p \dfrac{x_i}{x_t} \right)}{\dfrac{R_m \left(1 - \dfrac{x_i}{x_t} \right)}{1 - \dfrac{x_i}{x_t} + \dfrac{R_m}{R_p}} + \dfrac{R_m \dfrac{x_i}{x_t}}{\dfrac{x_i}{x_t} + \dfrac{R_m}{R_p}}} \tag{11}$$

$$\frac{e_0}{e_{ex}} = \frac{\dfrac{\cancel{R_m}\ \cancel{R_p} \dfrac{x_i}{x_t}}{\cancel{R_p} \left(\dfrac{R_m}{R_p} + \dfrac{x_i}{x_t} \right)}}{\cancel{R_m} \left\{ \dfrac{1 - \dfrac{x_i}{x_t}}{1 - \dfrac{x_i}{x_t} + \dfrac{R_m}{R_p}} + \dfrac{\dfrac{x_i}{x_t}}{\dfrac{x_i}{x_t} + \dfrac{R_m}{R_p}} \right\}} \tag{12}$$

$$\frac{e_0}{e_{ex}} = \frac{\dfrac{\dfrac{x_i}{x_t}}{\cancel{\left(\dfrac{R_m}{R_p} + \dfrac{x_i}{x_t} \right)}}}{\dfrac{\left(1 - \dfrac{x_i}{x_t} \right) \left(\dfrac{x_i}{x_t} + \dfrac{R_m}{R_p} \right) + \left(1 - \dfrac{x_i}{x_t} + \dfrac{R_m}{R_p} \right) \dfrac{x_i}{x_t}}{\left(1 - \dfrac{x_i}{x_t} + \dfrac{R_m}{R_p} \right) \cancel{\left(\dfrac{x_i}{x_t} + \dfrac{R_m}{R_p} \right)}}} \tag{13}$$

$$\frac{e_0}{e_{ex}} = \frac{\dfrac{x_i}{x_t} \left(1 - \dfrac{x_i}{x_t} + \dfrac{R_m}{R_p} \right)}{\dfrac{x_i}{x_t} - \left(\dfrac{x_i}{x_t} \right)^2 + \dfrac{R_m}{R_p} - \dfrac{x_i}{x_t} \dfrac{R_m}{R_p} + \dfrac{x_i}{x_t} - \left(\dfrac{x_i}{x_t} \right)^2 + \dfrac{R_m}{R_p} \dfrac{x_i}{x_t}} \tag{14}$$

$$\frac{e_0}{e_{ex}} = \frac{\left(1 - \frac{x_i}{x_t} + \frac{R_m}{R_p}\right)}{2\left(1 - \frac{x_i}{x_t}\right) + \frac{R_m}{R_p} \cdot \frac{x_t}{x_i}} \tag{15}$$

A plot of Eq. (15) shows that nonlinearity has decreased by introducing a parallel resistor.

4.2

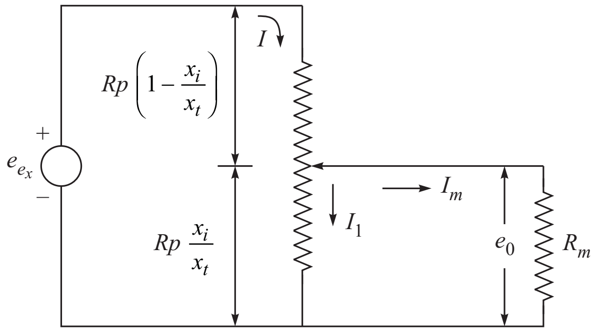


Fig. 1 [Fig. 4.5(a)]

Objective : Derivation of Eq. (4.1)

e_{ex} : Input excitation voltage, V

R_p : Total potentiometer resistance, Ω

R_m : Meter resistance, Ω

x_i : Position of the wiper *w.r.t* to zero e_0 position, m

x_t : Total length of travel of the wiper, m

$R_p \frac{x_i}{x_t}$: Resistance of the potentiometer upto wiper position, Ω

$R_p\left(1 - \frac{x_i}{x_t}\right)$: Resistance of the potentiometer after the wiper position

The circuit of Fig. 1 can be simplified as

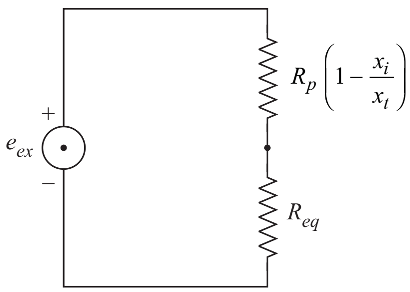


Fig. 2

R_{eq} of Fig. 2 is given by

$$R_{eq} = \frac{R_p \dfrac{x_i}{x_t} R_m}{R_p \dfrac{x_i}{x_t} + R_m} = \frac{\dfrac{x_i}{x_t} R_m}{\dfrac{x_i}{x_t} + \dfrac{R_m}{R_p}} \tag{1}$$

The total resistance of the circuit shown in Fig. 2 is given by

$$R = R_p \left(1 - \frac{x_i}{x_t}\right) + \frac{\dfrac{x_i}{x_t} R_m}{\dfrac{x_i}{x_t} + \dfrac{R_m}{R_p}} \tag{2}$$

$$R = R_p \left\{ \left(1 - \frac{x_i}{x_t}\right) + \frac{\dfrac{x_i}{x_t} \dfrac{R_m}{R_p}}{\dfrac{x_i}{x_t} + \dfrac{R_m}{R_p}} \right\} \tag{3}$$

Refer to Fig. 1. The total current I, at any wiper position, splits into I, and I_m at the wiper. I_m is the current that flows through the meter

I is given by

$$I = \frac{e_{ex}}{R} \tag{4}$$

I_m is given by

$$I_m = \frac{I R_p \dfrac{x_i}{x_t}}{R_p \dfrac{x_i}{x_t} + R_m} = \frac{I \dfrac{x_i}{x_t}}{\dfrac{x_i}{x_t} + \dfrac{R_m}{R_p}} \tag{5}$$

The output voltage e_0, indicated by the meter is given by

$$e_0 = I_m R_m = \frac{I R_m \dfrac{x_i}{x_t}}{\dfrac{x_i}{x_t} + \dfrac{R_m}{R_p}} \tag{6}$$

From Eqs (3), (4) and (6)

$$e_0 = \frac{e_{ex}}{R_p \left\{ \left(1 - \dfrac{x_i}{x_t}\right) + \dfrac{\dfrac{x_i}{x_t} \dfrac{R_m}{R_p}}{\dfrac{x_i}{x_t} + \dfrac{R_m}{R_p}} \right\}} \times \frac{R_m \dfrac{x_i}{x_t}}{\left\{ \dfrac{x_i}{x_t} + \dfrac{R_m}{R_p} \right\}} \tag{7}$$

$$\frac{e_0}{e_{ex}} = \frac{\dfrac{R_m}{R_p}\dfrac{x_i}{x_t}}{\dfrac{\left\{\left(\dfrac{x_i}{x_t}+\dfrac{R_m}{R_p}\right)\left(1-\dfrac{x_i}{x_t}\right)+\dfrac{x_i}{x_t}\dfrac{R_m}{R_p}\right\}}{\cancel{\left\{\dfrac{x_i}{x_t}+\dfrac{R_m}{R_p}\right\}}}\times\cancel{\left\{\dfrac{x_i}{x_t}+\dfrac{R_m}{R_p}\right\}}} \tag{8}$$

$$\frac{e_0}{e_{ex}} = \frac{\dfrac{R_m}{R_p}\dfrac{x_i}{x_t}}{\dfrac{x_i}{x_t}\left(1-\dfrac{x_i}{x_t}\right)+\dfrac{R_m}{R_p}-\cancel{\dfrac{R_m}{R_p}\dfrac{x_i}{x_t}}+\cancel{\dfrac{R_m}{R_p}\dfrac{x_i}{x_t}}} \tag{9}$$

$$\frac{e_0}{e_{ex}} = \frac{1}{\dfrac{R_p}{R_m}\left(1-\dfrac{x_i}{x_t}\right)+\dfrac{x_t}{x_i}} \qquad \text{[Eq. (4.1)]} \tag{10}$$

4.3

$R_m = 10000\ \Omega$
Nonlinearity = 1%
$P = 5$ W
$R_p = 100, 200, 300 \ldots 10000\ \Omega$

For minimizing nonlinearity $\dfrac{R_m}{R_p} = 15$ (1)

$$R_p = \frac{10000}{15} = 667\ \Omega$$

So we can use either $R_p = 600\ \Omega$ or $700\ \Omega$
Let us use $R_p = 600\ \Omega$

Max $e_{ex} = \sqrt{PR_p} = \sqrt{5\times 600} = 54.7$ V (2)

Since $\dfrac{e_{0\max}}{e_{ex}} = 1$, 54.7 V has to be distributed over 360°

$$\text{Sensitivity} = \frac{e_{ex}}{360} = \frac{54.7}{360} = 0.152 \text{ V/degree} \tag{3}$$

Since the value of R_p is just an approximation, actual nonlinearity must be checked by plotting.

4.4 When the pot resistance changes, the excitation voltage is still the same, assuming a well regulated voltage. Assuming that the potentiometer length does not change due to thermal expansion, the sensitivity is still not affected. If the change in resistance due to thermal inputs is not uniform, then there will be errors. In addition, the change in potentiometer resistance can change the nonlinearity of the instrument. Since the change of resistance due to temperature is negligible, the overall effect is negligible.

4.5

$$l = 250 \text{ mm} \quad e_{ex} = 100 \text{ V}$$

$$\frac{e_{ex}}{l} = \frac{100}{250} = \frac{0.4 \text{ V}}{\text{mm}} = \frac{0.4 \text{ mV}}{\mu\text{m}}$$

Resolution of the potentiometer = 25 µm

Irrespective of the sensitivity of the oscilloscope, the potentiometer cannot measure less than 25 µm. Therefore, it cannot measure 2.5 µm.

4.6

$$f_{n_1} = 30 \text{ Hz} \quad \text{for} \quad M + M_p$$
$$f_{n_2} = 25 \text{ Hz} \quad \text{for} \quad M + M_p + M_p = M + 2M_p$$

$$f_{n_1} = \frac{1}{2\pi}\sqrt{\frac{K}{M + M_p}} \tag{1}$$

$$f_{n_2} = \frac{1}{2\pi}\sqrt{\frac{K}{M + 2M_p}} \tag{2}$$

$$\frac{f_{n_1}}{f_{n_2}} = \sqrt{\left(\frac{K}{M + M_p}\right)\left(\frac{M + 2M_p}{K}\right)} = \frac{30}{25} \tag{3}$$

$$M\left\{\left(\frac{30}{25}\right)^2 - 1\right\} = M_p\left\{2 - \left(\frac{30}{25}\right)^2\right\} \tag{4}$$

$$0.44M = 0.56M_p \quad M = \frac{0.56}{0.44} \times 0.01 \tag{5}$$

$$M = 0.0127 \text{ kg.}$$

$$K = (2\pi f_{n_1})^2 (M + M_p) = (2\pi\, 30)^2 (0.01 + 0.0127) = 806.5 \text{ N/m} \tag{6}$$

4.7 Objective: reducing cross sensitivity

From Eq. (4.12)

$$dR = \frac{\rho\, dL(1 + 2v)}{A} + \frac{Ld\rho}{A} \tag{1}$$

The change in resistance is therefore inversely proportional to the area section of the strain gage wire. Since the strain gage elements in the direction of intended strain measurement have to be sensitive to strain, those that run perpendicular to the intended strain direction must be insensitive.

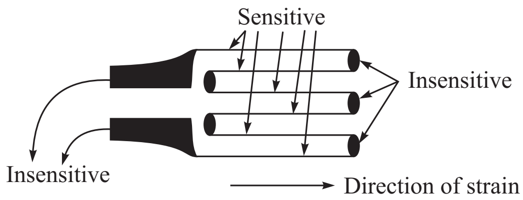


Fig. 1

Therefore, as shown in Fig. 1, those elements that run parallel to the strain direction are thin, and therefore sensitive to strain; those elements that run in a different direction to that of strain are very thick and therefore insensitive to strain in any other direction. Thus, cross sensitivity is minimized.

4.8

$$E_{\text{noise, rms}} = \sqrt{4kTR\Delta f V} \qquad (1)\ (4.22)$$

$k = 1.38 \times 10^{-23}$ J/K (Boltzmann's constant)

$T = 300\ K$

$R = 120\ \Omega$

$\Delta f = 10000$ Hz (given)

$$E_{\text{noise, rms}} = \sqrt{4 \times 1.38 \times 10^{-23} \times 120 \times 10000} = 8.14 \text{ nanovolts (nV)} \qquad (2)$$

$$\text{For 7 kPa stress, signal to noise ratio} = \frac{0.13 \times 10^{-6}}{8.14 \times 10^{-9}} = 15.97 \qquad (3)$$

4.9

$R_1 = R_2 = R_3 = R_4 = 120\ \Omega$

$P = 0.25$ W for each strain gage

At the null balance $\quad e_{ex} = I\ (R_1 + R_2) = I\ (R_3 + R_4) \qquad (1)$

I is the current through R_1, R_2 or R_3, R_4

$$e_{ex}\, I = I^2(R_1 + R_2) = I^2(R_3 + R_4) = 2P \qquad (2)$$

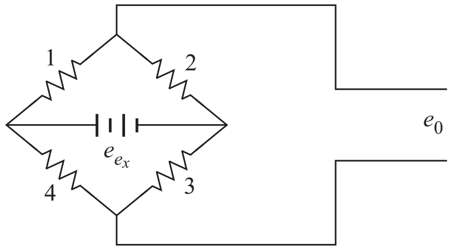
P is the power dissipation capacity of each strain gage.

From Eq. (2) $\quad I^2 = \dfrac{2P}{R_1 + R_2} = \dfrac{2 \times 0.25}{120 + 120} \qquad (3)$

$I = 0.0456$ A

From Eq. (2) $\quad e_{ex} = 2P/I = \dfrac{2 \times 0.25}{0.0456} = 10.96$ V $\qquad (4)$

4.10



Part of this problem has already been discussed in the example problem on page 172.

(a) E: Young's modulus = 200 GPa

R_1 is a strain gage mounted on a steel specimen; R_2, R_3, R_4 are resistors; $R = R_1 = R_2 = R_3 = R_4 = 120\ \Omega$

I_{max} : Maximum allowable current = 0.03 A

This current flows through, R_1 and R_2; R_3 and R_4

e_{ex} = Excitation voltage = $I_{max}(R + R) = 0.03 \times 2 \times 120 = 7.2$ V

(b) σ: Stress = 7 MPa

$$\varepsilon : \text{Strain} = \frac{\sigma}{E} = \frac{7 \times 10^6}{200 \times 10^9} = 35\mu\varepsilon$$

$$\Delta R = (GF) \times \varepsilon \times R \qquad (1)\ (4.17)$$

GF: Gage factor = 2

$$\Delta R = 2 \times 35 \times 10^{-6} \times 120 = 8.4\ \text{m}\,\Omega$$

$$e_0 = e_{ex} \frac{1}{4R} \Delta R \qquad (2)\ (4.18)$$

(Since there is only one active strain gage, bridge factor = 1)

$$e_0 = \frac{7.2 \times 8.4 \times 10^{-3}}{4 \times 120} = 0.126 \text{ mV for 7 MPa} \qquad (3)$$

(c) $\Delta T = 55°$C: Temperature difference between the strain gage and other resistors

$$\alpha_{\text{steel}} = 12 \times 10^{-6} \text{ mm/mm°C}$$

$$\alpha_{\text{Advance alloy}} = 27 \times 10^{-6} \text{ mm/mm°C}$$

$$\varepsilon_{T_1} \text{ (differential expansion)} = \Delta T\,(\alpha_{\text{steel}} - \alpha_{\text{advance}})$$

$$\varepsilon_{T_1} \text{ (differential expansion)} = 55(12 \times 10^{-6} - 27 \times 10^{-6}) \qquad (4)$$

$$= -825\mu\varepsilon$$

$$\Delta R_{T_1} = GF \times \varepsilon_{T_1} \times R$$

$$= 2 \times (-825 \times 10^{-6}) \times 120 = -0.198\ \Omega \qquad (5)$$

ΔR_{T_2}: Due to increase in temp = $\Delta T \times R \times \rho_{adv}$

ρ_{adv} = Temperature coeffcient of resistance of the strain gage material = $11 \times 10^{-6}\ \Omega/\Omega°C$

$$\Delta R_{T_2} = 55 \times 120 \times 11 \times 10^{-6} = 0.0726\ \Omega \quad (6)$$

Net change of resistance due to the effect of temperature, $\Delta R_T = \Delta R_{T_1} + \Delta R_{T_2} = -0.198 + 0.0726$

$$= -0.1254\ \Omega$$

From Eq. 4.17, the equivalent strain due to temperature change $\Delta T = 55°C$ is given by

$$\varepsilon_T = \frac{\Delta RT}{GF \times R} = \frac{-0.1254}{2 \times 120} = 522.5\mu\varepsilon \quad (7)$$

Equivalent stress $\sigma_T = E\varepsilon_T = 200 \times 10^9 \times 522.5 \times 10^{-6}$ (8)

$$= 104.5\ \text{MPa}$$

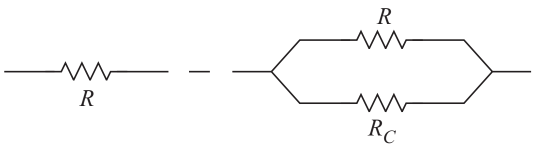
Therefore, when only one strain gage is used without temperature compensation, significant errors are introduced.

(d) Shunt resistance to obtain a resistance change equivalent to 70 MPa

$$\varepsilon = \frac{70 \times 10^6}{200 \times 10^9} = 350\ \mu\varepsilon$$

$$\Delta R = 2 \times 350 \times 10^{-6} \times 120 \text{ (eq. 4.17)}$$
$$= 0.084\ \Omega$$

ΔR =



$$\Delta R = R - \frac{R R_c}{R + R_c} = \frac{R^2 + R R_c - R R_c}{R + R_c}$$

$$\Delta R = \frac{R^2}{R + R_c}; \; R_c = \frac{R^2 - \Delta R \times R}{\Delta R} = \frac{120^2 - 0.084 \times 120}{0.084}$$

$$R_c = 171309\ \Omega$$

4.11

$$\frac{e_0}{e_i}(D) = \frac{R_m (M_2 - M_1) D}{\left[(M_1 - M_2)^2 + L_p L_s\right] D^2 + \left[L_p (R_s + R_m) + L_s R_p\right] D + (R_s + R_m) R_p} \quad (4.31)$$

Let
$$(M_1 - M_2)^2 + L_p L_s = M \quad (1)$$
$$L_p + (R_s + R_m) + R_p = C \quad (2)$$
$$(R_s + R_m) R_p = K \quad (3)$$

From Eq. (1), (2), (3), Eq. (4.31) can be written as

$$\frac{e_0}{e_i}(D) = \frac{R_m(M_2 - M_1)D}{MD^2 + CD + K} \tag{4}$$

Since $D = j\omega$, the numerator of Eq. (4) always has a phase angle of + 90°.

The denominator of Eq. (4) is CD at the undamped natural frequency $\omega_n = \sqrt{\dfrac{K}{M}}$

Therefore Eq. (4) has a zero phase shift at the undamped natural frequency

$$\omega_n = \sqrt{\frac{K}{M}} = \sqrt{\frac{(R_s + R_m)R_p}{(M_1 - M_2)^2 + L_p L_s}} \tag{5}$$

4.12 First let us derive Eq. (4.31) using Eq. (4.29) and (4.30)
Rearranging Eq. (4.29)

$$(R_p + L_p D)i_p - (M_1 - M_2)\, Di_s = e_{ex} \tag{1}$$

$$(M_1 - M_2)\, Di_p + (R_s + R_m + L_s D)i_s = 0 \tag{2}$$

$$\begin{bmatrix} R_p + L_p D & -(M_1 - M_2)D \\ (M_1 - M_2)D & R_s + R_m + L_s D \end{bmatrix} \begin{Bmatrix} i_p \\ i_s \end{Bmatrix} = \begin{Bmatrix} e_{ex} \\ 0 \end{Bmatrix} \tag{3}$$

$$\begin{Bmatrix} i_p \\ i_s \end{Bmatrix} = \frac{1}{\det}\begin{bmatrix} R_s + R_m + L_s D & (M_1 - M_2)D \\ -(M_1 - M_2)D & R_p + L_p D \end{bmatrix} \begin{Bmatrix} e_{ex} \\ 0 \end{Bmatrix} \tag{4}$$

$$\det = (R_p + L_p D)(R_s + R_m + L_s D) + (M_1 - M_2)^2 D^2 \tag{5}$$

$$i_s = \frac{(M_2 - M_1)D \times e_{ex}}{\det} \tag{6}$$

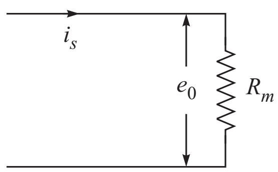


Fig. 1

The output voltage e_0 is given by $i_s R_m$

$$\frac{e_0}{e_{ex}} = \frac{R_m(M_2 - M_1)D}{(M_1 - M_2)^2 D^2 + \left[L_p(R_s + R_m) + L_s R_p\right]D + (R_s + R_m)R_p} \tag{7}$$

Equation (7) is same as Eq. 4.31.
Now, let us come back to the modifications required for this problem
(a) Introducing a parallel resistance to R_m

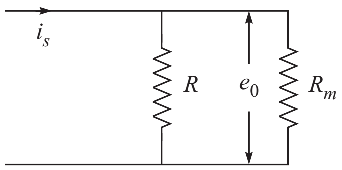


Fig. 2

This is one of the easiest modifications. The new resistance $R_{eq} = \dfrac{R\,R_m}{R + R_m}$ (8)

Replacing R_m by R_{eq} of Eq. (8) in Eq. (7) will directly give e_0/e_{ex} for the above change

$$\frac{e_0}{e_{ex}} = \frac{R_{eq}\,(M_2 - M_1)D}{(M_1 - M_2)^2 D^2 + \left[L_p\,(R_s + R_{eq}) + L_s\,R_p\right]D + (R_s + R_{eq})\,R_p} \tag{9}$$

The effect of adding 'R' would be equivalent to reducing R_m of Fig. 1.

(b) Introducing a resistance and capacitance as shown in Fig. 4.19 (b)

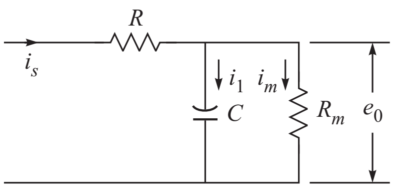


Fig. 3

The new impedance due to 'R' and 'C' is given by

$$Z(D) = R + \frac{R_m}{R_m CD + 1} \tag{10}$$

By replacing R_m by $Z(D)$ in Eq. (5)

$$\det = (R_p + L_p\,D)\left(R_s + R + \frac{R_m}{R_m CD + 1} + L_s D\right) + (M_1 - M_2)^2 D^2 \tag{11}$$

$$i_s = \frac{(M_2 - M_1)D}{\det} \times e_{ex} \tag{12}$$

$$i_m = \frac{i_s \dfrac{1}{CD}}{\dfrac{1}{CD} + R_m} = \frac{i_s}{1 + R_m CD} \tag{13}$$

$$e_0 = i_m\,R_m \tag{14}$$

From Eq. (11) through (14)

$$\frac{e_0}{e_{ex}} = \frac{R_m (M_2 - M_1)D}{(1 + R_m CD)\left[(M_1 - M_2)^2 D^2 + (R_p + L_p\,D)\left(R_s + R + \dfrac{R_m}{R_m CD + 1} + L_s\,D\right)\right]} \tag{15}$$

(c) Introducing a resistance and capacitance as shown in Fig. 4.19 (c).

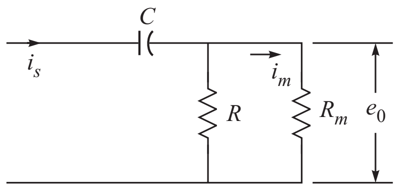


Fig. 4

The new impedance due to 'R' and 'C' shown in Fig. 4 is given by

$$Z(D) = \frac{1}{CD} + \frac{R\,R_m}{R + R_m} \tag{16}$$

By replacing R_m by $Z(D)$ (eq. (16)) in Eq. 5

$$\det = (R_p + L_p\,D)\left(R_s + \frac{1}{CD} + \frac{R\,R_m}{R + R_m} + L_s\,D\right) + (M_1 - M_2)^2 D^2 \tag{17}$$

$$i_s = \frac{(M_2 - M_1)D}{\det} \times e_{ex} \tag{18}$$

$$i_m = \frac{i_s\,R}{R + R_m} \tag{19}$$

$$e_0 = i_m\,R_m \tag{20}$$

From Eq. (17) through (20)

$$\frac{e_0}{e_{ex}} = \left(\frac{R\,R_m}{R + R_m}\right)\frac{(M_2 - M_1)D}{\left[(R_p + L_p\,D)\left(R_s + \frac{1}{CD} + \frac{R\,R_m}{R + R_m} + L_s\,D\right) + (M_1 - M_2)^2\,D^2\right]} \tag{21}$$

(d) By introducing a series resistance R in the primary circuit as shown in Fig. 4.19 (d) The only change is R_p replaced by $R + R_p$ in Eq. (7)

$$\frac{e_0}{e_{ex}} = \frac{R_m\,(M_2 - M_1)D}{(M_1 - M_2)^2\,D^2 + \left[L_p\,(R_s + R_m) + L_s\,(R_p + R)\right]D + (R_s + R_m)(R_p + R)} \tag{22}$$

4.13

f_{ex} = 10000 Hz excitation frequency

$f_{i_{\max}}$ = 500 Hz Maximum input frequency

$f_{\text{ripple}} = 2\,f_{ex} \pm f_{i_{\max}}$ = 20000 ± 500

= 19500 Hz & 20500 Hz

For the galvanometer $\zeta = 0.65$

$$|H| = \frac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}; f_n = 1000 \text{ Hz} \quad (1)$$

$$r = \frac{f_{i_{\max}}}{f_n} = \frac{500}{1000} = 0.5 \quad (2)$$

$$|H|_{r=0.5} = \frac{1}{\sqrt{(1-0.5^2)^2 + (2\times 0.65\times 0.5)^2}} = 1.0076 \quad (3)$$

for

$$r = \frac{f_{\text{ripple}_{\min}}}{f_n} = \frac{19500}{1000} = 19.5 \quad (4)$$

$$|H|_{r=19.5} = \frac{1}{\sqrt{\left[1-(19.5)^2\right]^2 + (2\times 0.65\times 19.5)^2}} = 0.0026 \quad (5)$$

Therefore, the Galvanometer has a flat response upto the maximum input frequency of 500 Hz and does not respond to the ripple frequency at all. The phase is also linear from 0-500 Hz.

4.14

Error = 5% $\quad f_l = 20$ Hz

$\omega_l = 2\pi 20 = 125.66$ rad/s

$A = 3 \text{ cm}^2 = 300 \text{ mm}^2$ (area of the capacitor)

$x_0 = 125\ \mu\text{m} = 0.125$ mm (distance between the plates)

From Eq. (4.33)

$$C = \frac{0.00885A}{x_0} \quad (1)\ (4.33)$$

C : Capacitance in pF

A: Area, mm^2

x_0: Distance between plates mm

$$C = \frac{0.00885\times 300}{0.125} = 21.24 \text{ pF} \quad (2)$$

From Eq. (4.38),

$$\left|\frac{e_0}{x_i K}(D)\right| = \frac{\tau\omega_l}{\sqrt{1+(\tau\omega_l)^2}} = 0.95 \quad (3)$$

$$\tau\,\omega_l = \frac{0.95}{\sqrt{1-0.95^2}} = 3.04;\ \tau = \frac{3.04}{\omega_l} = \frac{3.04}{125.66} = 0.024\text{s} \quad (4)$$

$$\tau = RC$$

$$R = \frac{\tau}{C} = \frac{0.024}{21.24\times 10^{-12}} = 1130 \text{ M}\Omega \quad (5)$$

4.15 This problem is same as the example on page 237

Assumptions

(1) $R_{cr} >> R_{\text{amp}} \quad \therefore \quad R_{eq} \approx R_{\text{amp}}$

(2) $C_b >> C_{cr} + C_{\text{cable}} + C_{\text{amp}}$

$$C_{eq} = C_{cr} + C_{\text{cable}} + C_{\text{amp}} \tag{1}$$

(3) The piezoelectric transducer is considered as a second stage element with relative displacement as the input from the first stage seismic pickup that is assumed to be in the flat response region.

$$C_{cr} = 1000\ pF \quad C_{\text{cable}} = 300\ pF \quad C_{\text{amp}} = 50\ \text{pF}$$
$$R_{\text{amp}} = 1\ \text{M}\,\Omega$$
$$k_q = 0.4\ \mu\text{c/mm}$$

(a) Sensitivity of the transducer alone

$$K_{\text{tran}} = \frac{K_q}{C_{cr}} = \frac{0.4\times10^{-6}}{1000\times10^{-12}} = 400\ \text{V/mm} \tag{2}$$

(b) High frequency sensitivity

$$K_{\text{tot}} = \frac{K_q}{C_{eq}} = \frac{0.4\times10^{-6}}{(1000+300+50)\times10^{-12}} = \frac{0.4}{1350\times10^{-6}} \tag{3}$$
$$= 296\ \text{V/mm}$$

Please note that the movement of the crystal is in the order of nano and micrometer. So we can expect output voltage of the order of mV only.

(c) Lowest frequency for an amplitude error of 5%

$$\left|\frac{e_0}{x_0}\right| = \frac{\tau\omega}{\sqrt{1+(\tau\omega)^2}} = 0.95 \tag{4}$$

$$(\tau\,\omega_l)^2\ \{1 - 0.95^2\} = 0.95^2$$

$$\tau\,\omega_l = \frac{0.95}{\sqrt{1-0.95^2}} = 3.04 \tag{5}$$

$$\tau = R_{eq}\,C_{eq} = 1\times10^6\times1350\times10^{-12} \tag{6}$$
$$= 0.00135\text{s}$$

$$\omega_l = \frac{3.04}{0.00135} = 2252\ \text{rad/s (358 Hz)} \tag{7}$$

(d) To reduce f_l to 10 Hz or ω_l to 62.83 rad/s

$$\tau'\omega_l = 3.04 \text{ From part (e)}$$

$$\tau' = \frac{3.04}{\omega_l} = \frac{3.04}{62.83} = 0.048\text{s} \tag{8}$$

$$C'_{eq} = \frac{\tau}{R_{eq}} = \frac{0.048}{1\times10^6} = 48000 \text{ pF} \tag{9}$$

$$48000 = C_{cr} + C_{\text{cable}} + C_{\text{amp}} + C_{\text{add}} \tag{10}$$

$$48000 = 1000 + 300 + 50 + C_{\text{add}}; \quad C_{\text{add}} = 46650 \text{ pF} \tag{11}$$

Therefore, by adding a very high value of capacitance in parallel, the low frequency limit can be reduced to 10 Hz (62.83 rad/s)

(e) High frequency sensitivity with revised C'_{eq}

$$K_{\text{tot}} = \frac{K_q}{C'_{eq}} = \frac{0.4\times10^{-6}}{48000\times10^{-12}} = 8.33 \text{ V/mm} \tag{12}$$

Therefore, sensitivity has decreased by a factor of 37, for widening the low frequency limit

PS: This example is a practical problem encountered in the design of dynamic transducers wherein frequency range and sensitivity will have to be compromised.

4.16

$$x_i = \begin{cases} At & 0 \le t < T \\ 0 & T < t < \infty \end{cases} \tag{1}$$

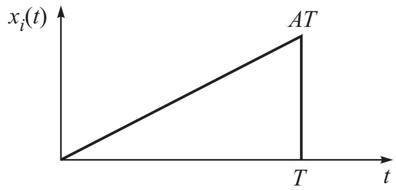


Fig. 1

$$(\tau D + 1)e_0 = (K\tau D)x_i \tag{4.63}$$

τ : Time constant of the piezoelectric transducer

K : Static sensitivity

e_0 : Voltage output

x_i : Displacement input

From Eq. (1)

$$(\tau D + 1)e_0 = K\tau A \qquad (0 \le t < T)$$

P.s. (Particular solution) = $K\tau A$

C.s. (Complementary solution) = $Ce^{-t/\tau}$

$$e_0 = Ce^{-t/\tau} + k\tau A \qquad 0 \le t < T \tag{2}$$

$$e_0(0) = 0 = C + K\tau A \quad \Rightarrow \quad C = -K\tau A$$

$$e_0 = K\tau A\ (1 - e^{-t/\tau}) \qquad 0 \le t < T \tag{3}$$

$$e_{0_{\text{approx}}} = K\tau A \left\{1 - \left[1 - \frac{t}{\tau} + \frac{1}{2}\left(\frac{t}{\tau}\right)^2\right]\right\} \tag{4}$$

Equations (1), (3) and (4) are plotted on the next page

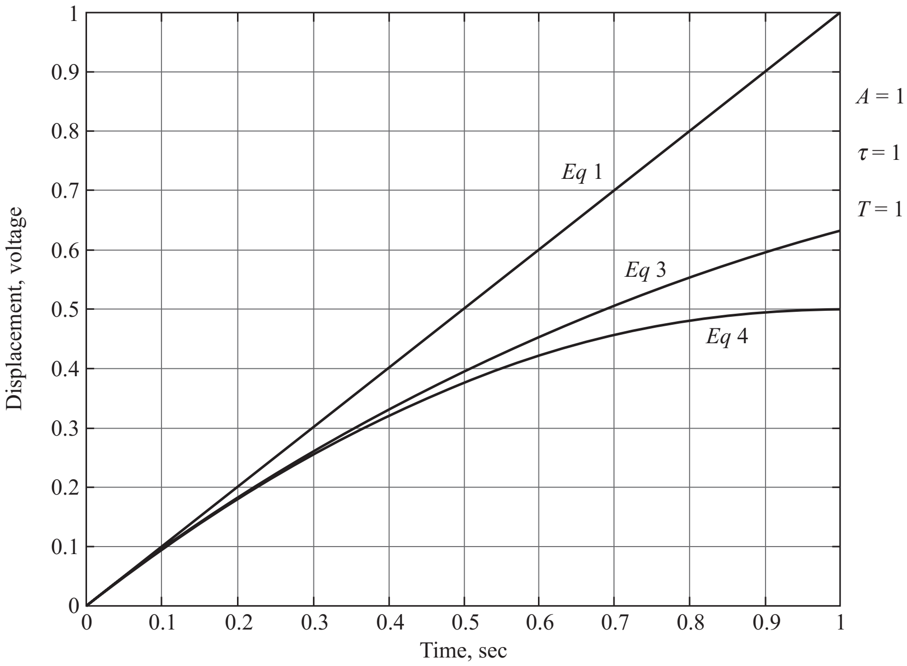

Displacement, voltage
Time, sec
Eq 1
Eq 3
Eq 4
A = 1
τ = 1
T = 1

Fig. 2

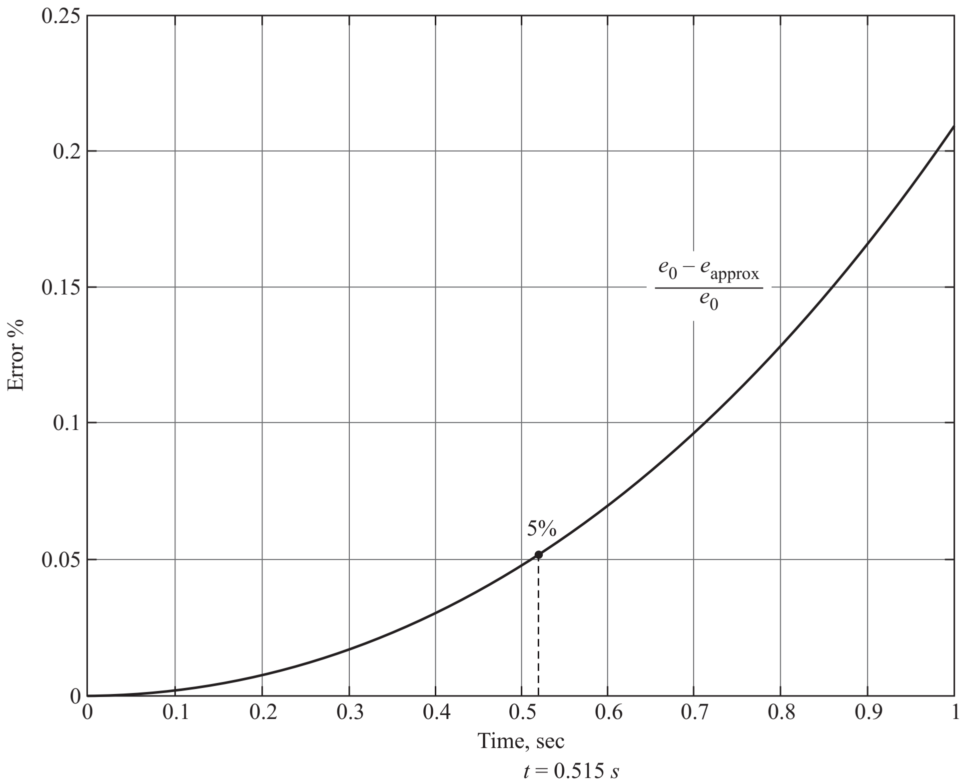

Error %
Time, sec
$\frac{e_0 - e_{approx}}{e_0}$
5%
t = 0.515 s

Fig. 3

From Fig. 3, there is a 5% error in the output voltage calculation due to truncation at

$$t = 0.515\text{s}; \qquad \tau = 1$$

$$e^{-t/\tau}\big|_{t=0.515} = e^{-0.515} = 0.5975 \tag{5}$$

$$1 - \frac{t}{\tau} + \frac{1}{2}\left(\frac{t}{\tau}\right)^2\Bigg|_{t=0.515} = 1 - 0.515 + 0.5 \times (0.515)^2 \tag{6}$$

$$= 0.6176$$

$$\text{Error due to truncation} = \frac{0.5975 - 0.6176}{0.5975} = -\,3.36\% \tag{7}$$

4.17 Nozzle-flapper devices have been removed for this edition. However, we shall discuss a single-flapper device first, followed by a double-flapper device

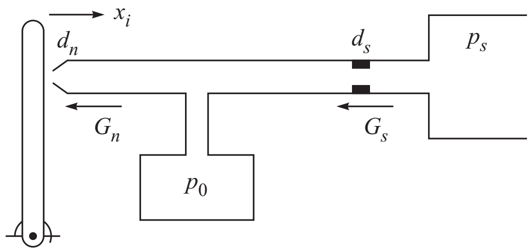


Fig. 1

G_s : Mass flow rate through supply orifice

$$G_s = \frac{C_d\,\pi d_s^2}{4}\sqrt{2\rho\,(p_s - p_0)} \tag{1}$$

G_n : Mass flow rate through discharge orifice

$$G_n = C_d\,\pi\,d_n\,x_i\sqrt{2\rho\,(p_0 - p_{\text{ambient}})} \tag{2}$$

The flow area = $\pi d_n\, x_i$ (peripheral area of the cylinder surrounding the nozzle)

For steady-state incompressible flow

$$G_s = G_n,\ p_{\text{ambient}} = 0 \text{ (gage)}$$

$$p_0 = \frac{p_s}{1 + 16\,\dfrac{d_n^2\,x_i^2}{d_s^4}} \tag{3}$$

The sensitivity $\dfrac{d\,p_0}{d\,x_i}$ varies with x_i

$$\frac{d\,p_0}{d\,x_i} = -\left\{\frac{p_s}{\left[1+\dfrac{16\,d_n^2\,x_i^2}{d_s^4}\right]^2}\right\}\frac{32\,d_n^2\,x_i}{d_s^4} \tag{4}$$

$$\frac{d^2\,p_0}{d\,x_i} = \frac{\dfrac{-32\,d_n^2}{d_s^4}\,p_s\left\{\left(1+\dfrac{16\,d_n^2\,x_i^2}{d_s^4}\right)^2 - 2\,x_i\left[1+\dfrac{16\,d_n^2\,x_i^2}{d_s^4}\right]\times\dfrac{32\,d_n^2\,x_i}{d_s^4}\right\}}{\left[1+\dfrac{16\,d_n^2\,x_i^2}{d_s^4}\right]^4} \tag{5}$$

$$\frac{d^2\,p_0}{d\,x_i} = 0 \text{ at } x_i^m = \sqrt{\frac{1}{48}}\,\frac{d_s^2}{d_n} = \frac{0.14\,d_s^2}{d_n} \tag{6}$$

position of maximum sensitivity

For a differential nozzle flopper shown in Fig. *P* 4.2

$$\Delta p = \frac{p_s}{1+k\,(x_0-x)^2} - \frac{p_s}{1+k\,(x_0+x)^2} \tag{7}$$

Where $$k = \frac{16\,d_n^2}{d_s^4} \tag{8}$$

For a single flapper

$$\frac{d\,p_0}{d\,x_i} = \frac{2\,p_s\,k\,(x_0-x)}{\left\{1+k\,(x_0-x)^2\right\}}2 \qquad \text{Where } x_i = x_0 - x \tag{9}$$

(From Eqs (4) and (8))

For a double–nozzle flapper (from Eq. 7)

$$\frac{d\,\Delta p_0}{d\,x_i} = \frac{2\,p_s\,k\,(x_0-x)}{\left\{1+k\,(x_0-x)^2\right\}^2} + \frac{2\,p_s\,k\,(x_0+x)}{\left\{1+k\,(x_0-x)^2\right\}^2} \tag{10}$$

(where $x_i = x_0 - x$ or $x_0 + x$)

Let $p_s = 200$ kPa

$d_s = 1$ mm $d_n = 1$ mm

Equations for pressure and sensitivity are plotted for single and double flapper systems. The double flapper system has higher sensitivity. They are shown in Figs. 2 and 3.

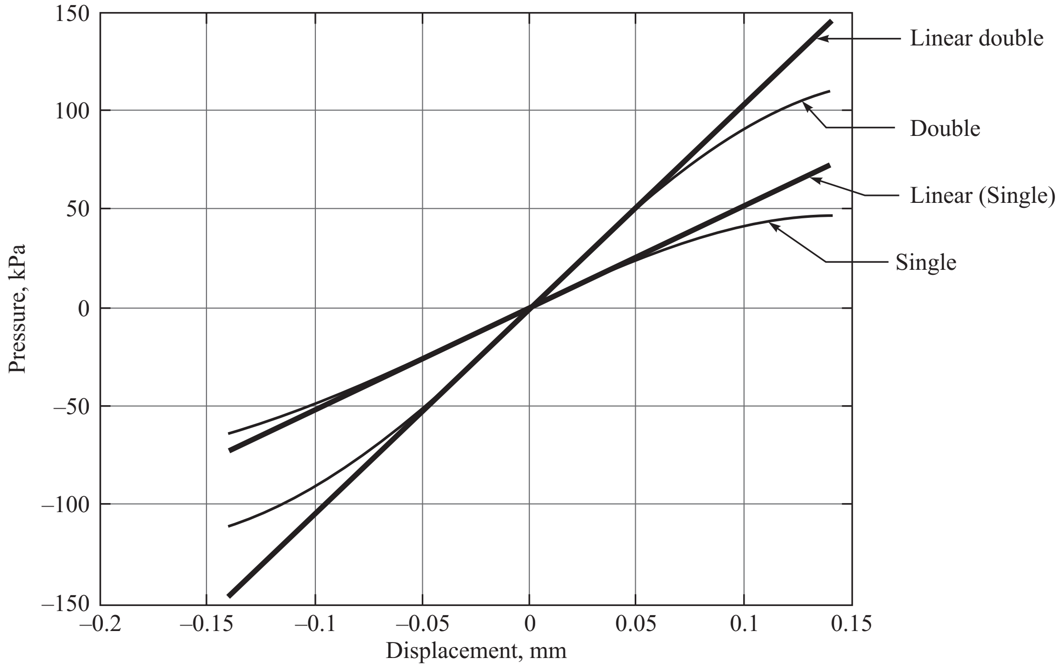

150
100
50
0
–50
–100
–150
Pressure, kPa
–0.2
–0.15
–0.1
–0.05
0
0.05
0.1
0.15
Displacement, mm
Linear double
Double
Linear (Single)
Single

Fig. 2

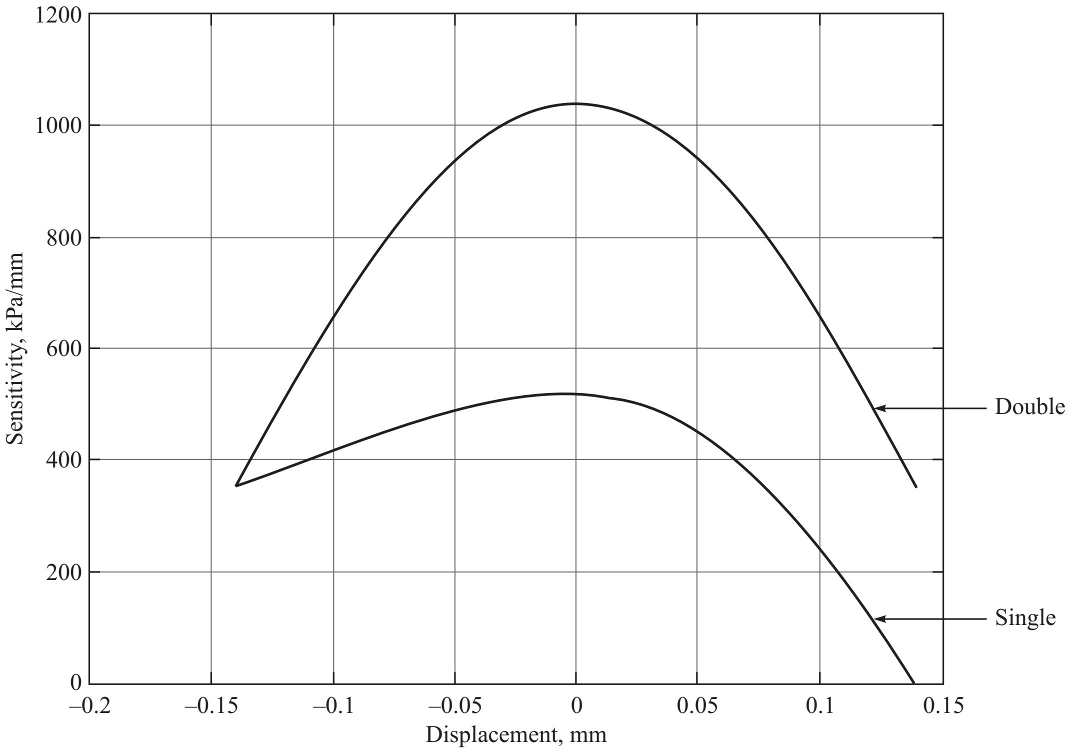

1200
1000
800
600
400
200
0
Sensitivity, kPa/mm
–0.2
–0.15
–0.1
–0.05
0
0.05
0.1
0.15
Displacement, mm
Double
Single

Fig. 3

4.18 To derive Eq. 4.73

$$n = \frac{r_1 r_2}{r_1 - r_2} \tag{4.73}$$

n: Speed of the machine RPM
r_1: No. of flashes per minute FPM at j^{th} submultiple of speed
r_2: No. of flashes per minute, FPM at $(j + 1)$ submultiple of speed
$r_2 < r_1$

$$r_1 = \frac{1}{j} n \tag{1}$$

$$r_2 = \frac{1}{j+1} n \tag{2}$$

(Because r_1, r_2 … are integral submultiples of the speed

$$r_1 - r_2 = n \left\{ \frac{1}{j} - \frac{1}{j+1} \right\} = \frac{n}{j(j+1)} \tag{3}$$

From Eqs. (1), (2) and (3)

$$r_1 - r_2 = \cancel{n} \times \frac{r_1}{\cancel{n}} \frac{r_2}{n}$$

$$n = \frac{r_1 r_2}{r_1 - r_2} \tag{4}$$

Equation (4) is same as Eq. 4.73

To derive Eq. 4.74

$$n = \frac{r_1 r_N (N-1)}{r_1 - r_N} \tag{4.74}$$

At very high speeds, r_1 and r_2 are very close is each other. Therefore the flashes are reduced below r_2

$$r_1 = \frac{n}{j} \tag{1}$$

$$r_2 = \frac{n}{j+1} \tag{2}$$

$$r_3 = \frac{n}{j+2} \tag{3}$$

$$r_N = \frac{n}{j+N-1} \tag{4}$$

From Eq. (1) and (4)

$$r_1 - r_N = n \left\{ \frac{1}{j} - \frac{1}{j+N-1} \right\}$$

$$r_1 - r_N = \frac{n\,(N-1)}{j\,(j+N-1)} \tag{5}$$

Once again using Eqs. (1), (4) and (5)

$$r_1 - r_N = n\ (N - 1) \times \mathrm{r}_1\ \frac{r_N}{n}$$

$$n = \frac{r_1\,r_N}{r_1 - r_N}\ (N - 1) \qquad (6)\ (4.74)$$

4.19

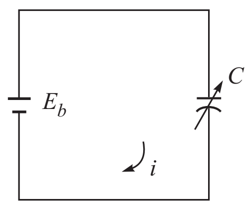


Fig. 1

$$q = Ce \tag{1}$$

Where C is the capacitance, q charge and e voltage

The capacitance is given by

$$C = \frac{0.00885A}{x} \tag{2}$$

C in pF, A = area (mm^2), x = distance (mm)

$$A = K_\theta \theta \tag{3}$$

$$i = \frac{dq}{dt} = c\frac{de}{dt} + e\frac{dc}{dt} \tag{4}$$

For the present problem, E_b is constant $\therefore\ \dfrac{de}{dt} = 0$

$$i = \frac{E_b\, 0.00885 \times 10^{-12}}{x}\, K_\theta\, \frac{d\theta}{dt} \tag{5}$$

In order to measure the above current in terms of voltage, a slight modification is required in the circuit, as shown in Fig. 2

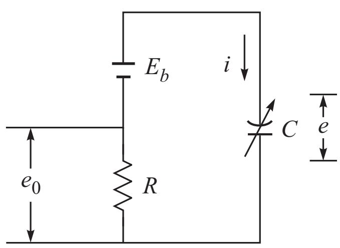


Fig. 2

The voltage across the capacitor is now

$$e = E_b - iR \tag{6}$$

Since i is a variable, e is also variable

From Eqs. (3), (4) and (6)

$$i = \frac{0.00885\, K_\theta\, \theta \times 10^{-12}}{x} \times \left\{\frac{-di}{dt} R\right\} + (E_b - iR)\, \frac{0.00885 \times 10^{-12}}{x}\, K_\theta \,\frac{d\theta}{dt} \tag{7}$$

Let $$C_1 = \frac{0.00885 \times 10^{-12}\, K_\theta}{x}$$

$$i = C_1 \left\{-\theta \frac{di}{dt} R + (E_b - iR)\frac{d\theta}{dt}\right\}$$

$$i\left\{1 + C_1 R \frac{d\theta}{dt}\right\} + C_1\ \theta R\ \frac{di}{dt} = C_1\ E_b\ \frac{d\theta}{dt} \tag{8}$$

Even for a linear time variation of θ, the above equation is a linear differential equation with varying coefficients, that can only be solved through simulation.

4.20

L_T (inductance) = 20 mH

R_T (resistance) = 10 Ω, R_M = 900 Ω

S_v = 1000 mV|m|s (Sensitivity of LVT)

f_n: Natural frequency of the first stage element = 5 Hz

ζ: Damping factor of the first stage = 0.1

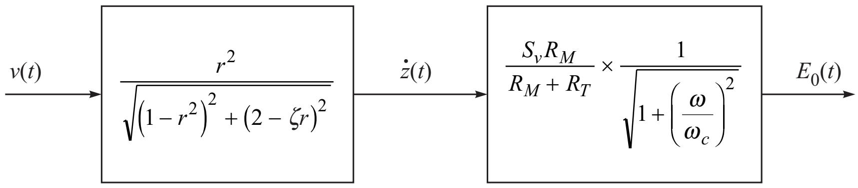


Fig. 1

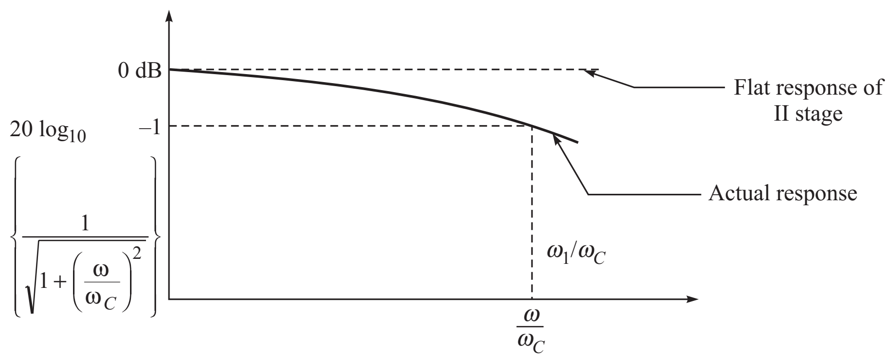


Fig. 2

$$20 \log_{10} \frac{1}{\sqrt{1+\left(\frac{\omega_1}{\omega_c}\right)^2}} = -1 \tag{1}$$

$$\omega_1 = \omega_c \sqrt{10^{0.1}-1} = 0.509\ \omega_c \tag{2}$$

$$\omega_c = \frac{R_M + R_T}{L_T} = \frac{1000}{20\times10^{-3}} = 5 \times 10^4 \text{ rad/s (7958 Hz)}$$

From Eq. (2)

$$\omega_1 = 0.509 \times 5 \times 10^4 = 25450 \text{ rad/s (4051 Hz)} \tag{3}$$

(b) $$E_0 = \frac{r^2}{\sqrt{(1-r^2)^2+(2\zeta r)^2}} \cdot \frac{S_v R_M}{(R_M + R_T)} \times \frac{1}{\sqrt{1+\left(\frac{\omega}{\omega_c}\right)^2}} \tag{4}$$

Static sensitivity of I stage = 1

$$\text{Static sensitivity of II stage} = \frac{S_v R_M}{R_M + R_T}$$

At $f = 600$ Hz i.e. $\omega = 3770$ rad/s

$$r = \frac{600}{5} = 120$$

$$dB_{\text{error}} \text{ (Stage I)} = 20 \log_{10} \left\{\frac{120^2}{\sqrt{(1-120^2)^2+(2\times0.1\times120)^2}}\right\} \tag{5}$$

$$= 5.9 \times 10^{-4}\ dB \tag{6}$$

$$dB_{\text{error}} \text{ (Stage II)} = 20 \log_{10} \frac{1}{\sqrt{1+\left(\frac{3770}{5\times10^4}\right)^2}} = -\ 0.0246\ dB \tag{7}$$

$$\text{Total error} = 5.9 \times 10^{-4} - 0.0246 = -\ 0.0240\ dB$$

4.21 From Eq. (4.79)

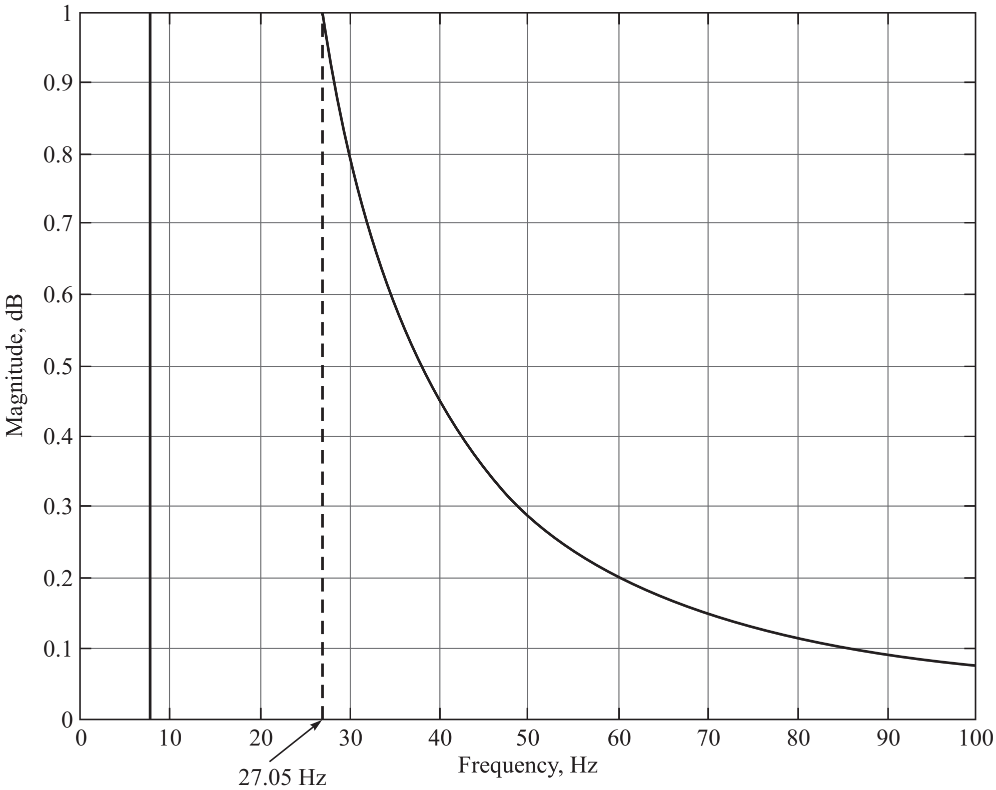
$$\frac{x_0}{x_i}(j\omega) = \frac{(j\omega)^2/\omega_n^2}{\left(\frac{j\omega}{\omega_n}\right)^2 + 2\zeta\, j\omega/\omega_n + 1} \qquad (1)\ (4.79)$$

Given $\zeta = 0.3 \quad f_n = 10$ Hz

For the above values of ζ and f_n, Eq. (4.79) is used to compute the magnitude of the above transfer function in dB, Let $f_{\max} = 100$ Hz

$$dB = 20 \log_{10} \frac{x_0}{x_i} (j\omega) \tag{2}$$

Equation (2) is plotted on the next page. Only the portion of the graph between 0 – 1 dB is displayed. Since we know that only the portion $\sqrt{2}$ times the natural frequency is the useful range of flat response, it can be seen that 27-100 Hz magnitude is within 1 dB of the flat response.



27 – 100 Hz within 1 dB

4.22

	Modifying	Interfering	Effect
a.	e_{ex}		gives different e_0 for same x_i
		temperature	thermal expansion relative motion of wiper and resistance element
		vibration	causes "jitter" in e_0
	meter resistance change		changes the loading effect

b.	e_{ex}		gives different e_0 for same x_i
	temperature	temperature	temperature coefficient of R differential thermal expansion
c.	e_{ex} amplitude		gives different e_0 for same x_i
	e_{ex} frequency		gives different e_0 for same x_i
	radial motion	radial motion	changes the sensitivity to axial motion causes an output voltage with no change in axial motion
	temperature	temperature	changes resistance, changes dimensions, changes iron magnetic properties
d.	temperature	temperature	changes dimensions, changes dielectric properties of air, changes resistance
	humidity		changes dielectric properties of air
	battery voltage		changes sensitivity
e.	temperature	temperature	changes piezoelectric properties, pyroelectric effect gives an output voltage when temperature changes, changes resistance and capacitance
		cable motion	triboelectric effect generates electric charge when cable flexes
		base strain	strain of surface on which sensor is mounted causes output unrelated to desired input
		temperature	random noise in resistances
		vibration	for a force, or pressure sensor
		acoustic noise	for force, pressure, or acceleration sensors
		off-axis motion	motion in axes other than that intended causes output ("cross-talk")
f.	temperature	temperature	changes air properties, changes dimensions
	supply pressure		changes sensitivity
g.	superimpose the effects of parts *c* and *f*		
h.	temperature	temperature	changes the spring constant, changes dimensions, changes damping
		vibration	changes frictional effects, introduces inertial forces
i.	temperature	temperature	changes dimensions, changes coil resistance, changes magnetic properties
	time ("ageing")		changes magnetic properties
j.	temperature	temperature	changes dimensions, changes magnetic properties, changes cup resistance, changes spring stiffness
		angular vibration	introduces inertial torques
k.	temperature	temperature	changes dimensions, changes magnetic properties, changes resistance, changes damping, changes spring stiffness, various effects on the motion pickup, amplifier and filter
	amplifier supply voltage		changes amplifier gain
	motion pickup supply voltage		changes motion pickup sensitivity
		off-axis motion	causes "cross-talk"
l.	temperature	temperature	changes spring stiffness, damping, changes dimensions
		wheel speed	changes gyro sensitivity
		off-axis motion	various subtle effects (discussed in gyro sources referenced in the text)

4.23 Derive $\frac{\theta_0}{\theta_i}(D)$ for Fig. 4.54b

Assume that the spring and dampers are attached at a radius R.

Since the ends of the springs and dampers undergo the same amount of deflection, they are equivalent to be connected in parallel. So, they add up.

$$\Sigma \text{ moments} = J(\ddot{\theta}_i - \ddot{\theta}_0) \tag{1}$$

J is the moment of inertia of the inner mass.

$$(2K_s R)\, R\theta_0 + (2BR)\, R\theta_0 = J(\ddot{\theta}_i - \ddot{\theta}_0) \tag{2}$$

θ_0: Relative angular motion between inner and outer rings. $(\ddot{\theta}_i - \ddot{\theta}_0)$ is the absolute acceleration of the inner ring

$$(J\,D^2 + 2BR^2\,D + 2\,K_s\,R^2)\,\theta_0 = JD^2\,\theta_i$$

$$\frac{\theta_0}{\theta_i}(D) = \frac{JD^2}{(JD^2 + 2BR^2 D + 2K_s R^2)} \tag{3}$$

$$= \frac{\frac{J}{2K_s R^2} D^2}{\frac{J}{2K_s R^2} D^2 + \frac{B}{K_s} D + 1} \tag{4}$$

Equation (4) has characteristics similar to a linear displacement pickup

4.24

$$\frac{K_q}{C\omega_n^2} = 0.04 \text{ V/m/s}^2$$

$$\left| \frac{E}{\frac{K_q}{C\omega_n^2}} \right| = \frac{\tau\omega}{\sqrt{(1 + (\tau\omega)^2)}\,(1 - r^2)} \quad \text{for } \zeta = 0 \tag{1}$$

$$\tau = 0.1$$
$$f_n = 10000 \text{ Hz}$$

Equation (1) is plotted for various values of $\omega(0 - 0.4\,\omega_n)$. The static sensitivity 0.04 V/m/s^2 does not play any role in this problem. The plot is shown in Fig. 1. Depending on the desired accuracy, the flat response region can be figured out.

Regarding shock pulse measurements, τ should be greater than $7T$. For the present problem $T = 0.05$ for the half-sine pulse.

$$\therefore \quad \tau = 7T,\ 0.05 = 0.35\text{s (from Fig. 4.61)} \tag{2}$$

This will help measure the peak of the shock pulse accurately, and in addition, can extend the lower frequency limit of flat response for steady excitation. A satisfactory value of τ for measuring half-sine pulse can also be verified through direct simulation.

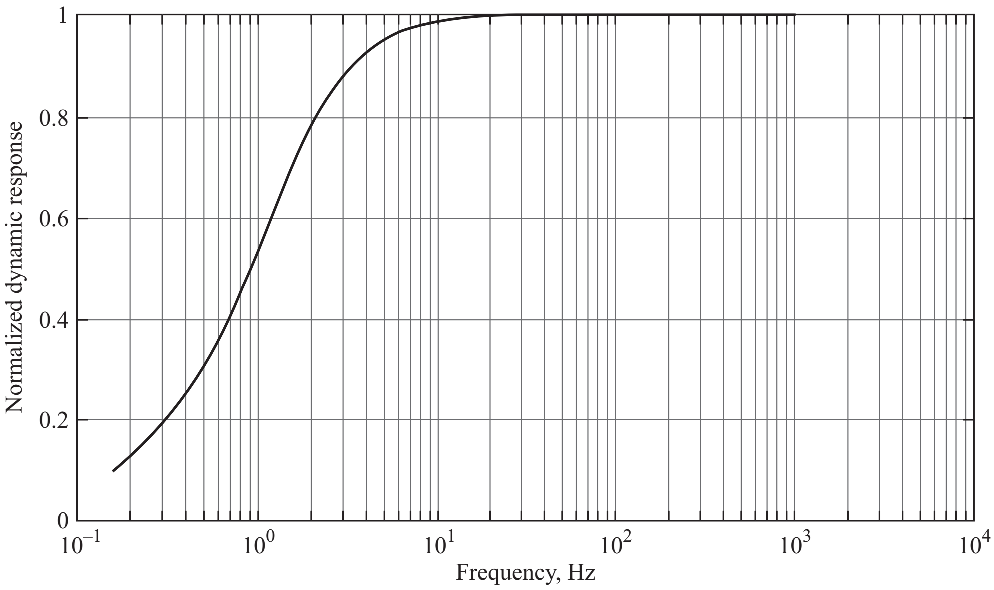


Fig. 1

4.25 The system shown in Fig. 4.62 can be adapted to measure angular acceleration by adding a torque coil and amplifier.

4.26 In the Fig. of 4.62, damping and stiffness are made zero.

The first block after the summing point has stiffness and damping. It can be rewritten as:

$$\frac{1}{K_s\left\{\dfrac{D^2}{\omega_{n,1}^2}+\dfrac{2\zeta_1 D}{\omega_{n,1}}+1\right\}} \tag{1}$$

$$\left.\begin{aligned} K_s\frac{D^2}{\omega_{n,1}^2} &= \frac{\cancel{K_s} D^2 J}{\cancel{K_s}} = JD^2 \\ \frac{K_s 2\zeta_1 D}{\omega_{n,1}} &= \frac{K_s}{\sqrt{\dfrac{K_s}{J}}}\times 2\zeta_1 D = \frac{K_s\sqrt{J}}{\sqrt{K_s}}\times\frac{2B}{2\sqrt{K_s J}}D = BD = 0 \\ K_s &= 0 \end{aligned}\right\} \tag{2}$$

From Eqs. (1) and (2), the block becomes

$$\frac{1}{JD^2} \tag{3}$$

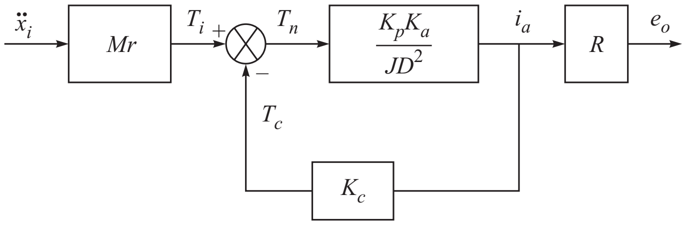


Fig. 1

$$\frac{i_a}{T_i}(D) = \frac{\dfrac{K_p\, K_a}{JD^2}}{1+\dfrac{K_c\, K_p\, K_a}{JD^2}} = \frac{1}{\dfrac{JD^2}{K_p\, K_a}+K_c} \tag{4}$$

$$T_i = Mr\,\frac{d^2 x_i}{dt^2} \tag{5}$$

$$e_0 = i_a\, R \tag{6}$$

From Eqs. (4), (5) and (6)

$$e_0 = \frac{R\times Mr\dfrac{d^2 x_i}{dt^2}}{\dfrac{JD^2}{K_p\, K_a}+K_c} = \frac{Mr\dfrac{d^2 x_i}{dt^2}}{\dfrac{JD^2}{R\, K_p\, K_a}+\dfrac{K_c}{R}} \tag{7}$$

$$\left(\frac{JD^2}{R\, K_p\, K_a}+\frac{K_c}{R}\right)e_0 = Mr\,\frac{d^2 x_i}{dt^2} \tag{8}$$

The defect in the above system is that since it has no damping, the roots of the characteristic equation are purely imaginary. The system is therefore marginally stable and the closed loop response is purely oscillatory.

The above defect can be fixed by adding a lead-lag system of Fig. *P* 4.4. This is shown in Fig. 2.

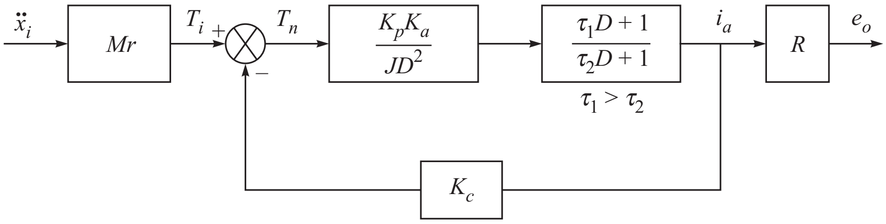


Fig. 2

$$\frac{i_a}{T_i}(D) = \frac{\dfrac{K_p K_a}{JD^2}\cdot\dfrac{\tau_1 D+1}{\tau_2 D+1}}{1+\dfrac{K_c K_p K_a}{JD^2}\cdot\dfrac{\tau_1 D+1}{\tau_2 D+1}} = \frac{K_p K_a(\tau_1 D+1)}{JD^2(\tau_2 D+1)+K_c K_p K_a(\tau_1 D+1)} \tag{9}$$

$$T_i = Mr\,D^2\,x_i \tag{10}$$

$$e_0 = i_a\,R \tag{11}$$

From Eqs. (9), (10) and (11)

$$e_0 = \frac{(MrD^2 x_i)K_p K_a(\tau_1 D+1)R}{JD^2(\tau_2 D+1)+K_c K_p K_a(\tau_1 D+1)}$$

$$\left(\frac{J\tau_2}{R}D^3+\frac{J}{R}D^2+\frac{K_c K_p K_a \tau_1}{R}D+\frac{K_c K_p K_a}{R}\right)e_0 = Mr\,K_P\,K_a\,(\tau_1\,D+1)D^2\,x_i \tag{12}$$

Since the characteristic equation is now cubic its terms can be adjusted to give satisfactory response. However, τ_1 must be greater than τ_2.

4.27 For a seismic pickup, in the accurate frequency range of flat response, the seismic mass is stationary. Thus the relative displacement measures the desired displacement. Under such conditions, the loading effect on the object whose displacement is being measured is the instruments housing mass, together with the spring and damper connected between the moving object and the seismic mass.

4.28 Jerkmeter (Fig. 4.64)

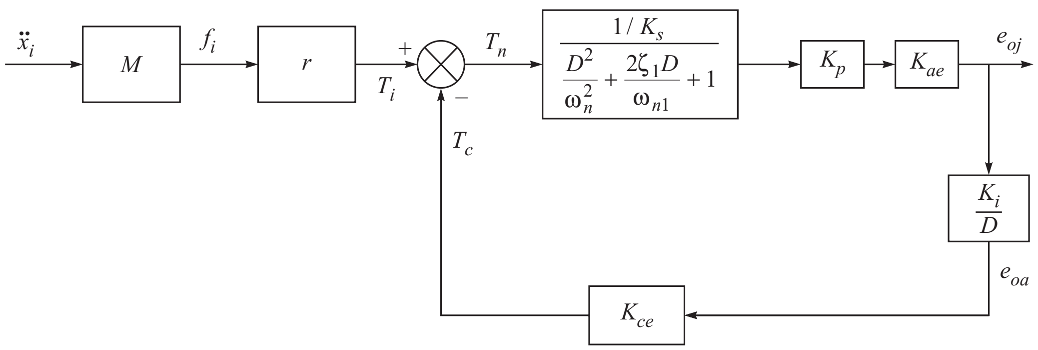


Fig. 1 [Fig. 4.64]

Let

$$G(D) = \frac{K_p K_{ae}}{K_s\left(\dfrac{D^2}{\omega_{n1}^2}+\dfrac{2\zeta_1 D}{\omega_{n1}}+1\right)} = \frac{K_p K_{ae}}{JD^2+BD+K_s} \tag{1}$$

$$H(D) = \frac{K_{ce} K_i}{D} \tag{2}$$

$$\frac{e_{0j}}{T_i}(D) = \frac{G(D)}{1+G(D)\,H(D)} = \frac{\dfrac{K_p\,K_{ae}}{D^2+BD+K_s}}{1+\dfrac{K_p\,K_{ae}\,K_{ce}\,K_i}{JD^3+BD^2+K_s\,D}} \tag{3}$$

$$\frac{e_{0j}}{T_i}(D) = \frac{D\,K_p\,K_{ae}}{(JD^3+BD^2+K_s\,D+K_p\,K_{ae}\,K_{ce}\,K_i)} \tag{4}$$

$$\frac{e_{0j}}{T_i}(D) = \frac{D}{K_{ce}\,K_i}\,\frac{1}{\left(\dfrac{JD^3+BD^2+K_s\,D}{K_p\,K_{ae}\,K_{ce}\,K_i}+1\right)} \tag{5}$$

$$D^2\,x_i\,Mr = T_i \tag{6}$$

From Eqs. (5) and (6)

$$\left(\frac{JD^3}{K_p\,K_{ae}\,K_{ce}\,K_i}+\frac{BD^2}{K_p\,K_{ae}\,K_{ce}\,K_i}+\frac{K_s\,D}{K_p\,K_{ae}\,K_{ce}\,K_i}+1\right)e_{0j} = \frac{Mr}{K_{ce}\,K_i}\,D^3\,x_i \tag{7}$$

(a) B and $k_s = 0 \quad J = Mr^2$

$$\left(\frac{Mr^2\,D^3}{K_p\,K_{ae}\,K_{ce}\,K_i}+1\right)e_{0j} = \frac{Mr}{K_{ce}\,K_i}\,D^3\,x_i \tag{8}$$

Let $D = j\omega$ for LHS

$$\frac{e_{0j}}{D^3\,x_i} = \frac{\dfrac{Mr}{K_{ce}\,K_i}}{1-\dfrac{j\,Mr^2\,\omega^3}{K_p\,K_{ae}\,K_{ce}\,K_i}} \tag{9}$$

$$\left|\frac{e_{0j}}{D^3\,x_i}\right| = \frac{\dfrac{Mr}{K_{ce}\,K_i}}{\sqrt{1+\left(\dfrac{Mr^2\,\omega^3}{K_p\,K_{ae}\,K_{ce}\,K_i}\right)^2}} \tag{10}$$

Given $\dfrac{Mr}{K_i\,K_{ce}} = 0.16\ \text{V/m/s}^2$

$$\left|\frac{e_{0j}}{D^3\,x_i}\right| = \frac{0.16}{\sqrt{1+\left(\dfrac{r\,\omega^3\times 0.16}{K_p\,K_{ae}}\right)^2}} \tag{11}$$

For 5% error at 5 Hz

$$\left|\frac{e_{0j}}{D^3 x_i}\right| \times \frac{1}{0.16} = \frac{1}{\sqrt{1+\left(\dfrac{r\omega^3 \times 0.16}{K_p K_{ae}}\right)^2}} = 1 \pm 0.05 \tag{12}$$

$$\text{since } \left(\frac{r\omega^3}{K_p K_{ae}}\right)^2 > 1,\ 1 - 0.05 \text{ is relevant} \tag{13}$$

Let

$$Y = \left(\frac{r \times 0.16}{K_p K_{ae}}\right) \tag{14}$$

$$0.95 = \frac{1}{\sqrt{1+Y^2\omega^6}} \tag{15}$$

$$0.95^2(1 + Y^2\omega^6) = 1 \tag{16}$$

$$1 + Y^2\omega^6 = \frac{1}{0.95^2} \tag{17}$$

$$\omega^6 Y^2 = \frac{1}{0.95^2} - 11 \tag{18}$$

$$Y = \sqrt{\frac{1}{\omega^6}\left\{\frac{1}{0.95^2} - 1\right\}} = \sqrt{\frac{1}{(2\pi 5)^6}\left\{\frac{1}{0.95^2} - 1\right\}} = 1.06 \times 10^{-5} \tag{19}$$

$$\frac{1}{Y} = \frac{K_p K_{ae}}{0.16r} \Rightarrow \frac{1}{1.06 \times 10^{-5}} = \frac{K_p K_{ae}}{0.16r} \tag{20}$$

$$\frac{K_p K_{ae}}{r} = 1.5094 \times 10^4 \tag{21}$$

(b) If r = 30 mm K_p = 57.3 V/rad, K_{ae} = ?

$$\frac{K_p K_{ae}}{r} = 1.5094 \times 10^4 \tag{22}$$

$$\frac{57.3}{30 \times 10^{-3}} \times K_{ae} = 1.5094 \times 10^4 \tag{23}$$

$$K_{ae} = 7.902 \text{ m/V/rad}$$

(c) M = 0.005 kg

Given

$$\frac{Mr}{K_i K_{ce}} = 0.16 \tag{24}$$

$$\frac{0.005 \times 30 \times 10^{-3}}{K_i K_{ce}} = 0.16 \tag{25}$$

$$K_i\,K_{ce} = 9.375 \times 10^{-4}$$

(d) **If B and K_S are not zero**

From (a) through (c), $K_p\,K_{ae}\,K_{ce}\,K_i = 57.3 \times 7.902 \times 9.375 \times 10^{-4} = 0.424$

Equation (7) now becomes,

$$2.355\,JD^3 + 2.355\,BD^2 + 2.355\,K_sD + 1 = 0$$

From here on, it needs knowledge of controls.

So, we shall stop here.

4.29

$$(I_x\,I_y\,D^2 + B\,I_x\,D + H_s^2 + I_x\,K_s)\phi = -\frac{H_s}{D}\,T_y \qquad (1)\ (4.101)$$

If $B = K_s = 0$, Eq. (1) becomes

$$(I_x\,I_y\,D^2 + H_s^2)\phi = -\frac{H_s}{D}\,T_y \qquad (2)$$

Taking Laplace transform on both sides

$$(I_x\,I_y\,s^2 + H_s^2)\Phi(s) = -\frac{H_s}{s}\,\hat{T}_y\,(s) \qquad (3)$$

If $T_y\,(t) = \delta(t),\ \hat{T}_y\,(s) = 1$

$$\Phi(s) = \frac{-H_s}{s(I_x\,I_y\,s^2 + H_s^2)} \qquad (4)$$

Let $C = \dfrac{-H_s}{I_x I_y},\ \omega^2 = \dfrac{H_s^2}{I_x I_y}$ (5)

From Eq. (5), Eq. (4) becomes

$$\Phi(s) = \frac{C}{s(s^2+\omega^2)} = \frac{D}{s} + \frac{Es+F}{s^2+\omega^2} = \frac{D(s^2+\omega^2)+Es^2+Fs}{s(s^2+\omega^2)} \qquad (6)$$

$$Ds^2 + D\omega^2 + Es^2 + Fs = C \qquad (7)$$

Comparing terms on both sides

$$D = \frac{C}{\omega^2} \quad E = \frac{-C}{\omega^2} \quad F = 0$$

$$\Phi(s) = \frac{C}{\omega^2}\left\{\frac{1}{s} - \frac{s}{s^2+\omega^2}\right\} \qquad (8)$$

$$\zeta^{-1}\{\Phi(s)\} = \phi(t) = \frac{C}{\omega^2}\{1 - \cos\,\omega t\}$$

$$\phi(t) = \frac{1}{H_s}\left\{1-\cos\frac{H_s}{\sqrt{I_x I_y}}\,t\right\} \tag{9}$$

$$\frac{\theta}{T_y}(D) = \frac{I_x}{I_x I_y D^2 + BI_x D + H_s^2 + I_x K_s} \qquad (10)\ (4.103)$$

$$B = 0 = K_s,\ T_y\ (t) = \delta(t)$$

$$\Theta(s) = \frac{I_x}{I_x I_y s^2 + H_s^2}$$

$$\Theta(s) = \frac{1}{\omega I_y}\,\frac{1}{(s^2+\omega^2)} \tag{11}$$

where $$\omega^2 = \frac{H_s^2}{I_x I_y}$$

$$\theta(t) = \zeta^{-1}\{\Theta(\mathrm{s})\} = \frac{1}{\omega I_y}\ \sin\ wt$$

$$\theta(t) = \sqrt{\frac{I_x}{I_y}}\,\frac{1}{H_s}\ \sin\frac{H_s}{\sqrt{I_x I_y}} \tag{12}$$

The composite motion of Eqs. (9) and (12) describe a conical motion in space known as nutation.

4.30 0.1° oscillations at 50 Hz (slope of the gyro location)
Rigid body
rotation 10 rad/s

$$\theta = 0.1° = 0.1 \times \frac{\pi}{180} = 1.745 \times 10^{-3}\ \text{rad} \tag{1}$$

$$\omega = 2\pi \times 50\ \text{rad/s} \tag{2}$$

Equation for angular motion

$$\Theta - \theta = (\sin\ 314\ t)\ 1.745 \times 10^{-3} \quad \{\text{Rotation of the gyro}\}$$

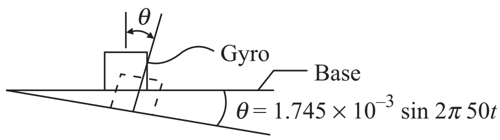
$$\omega = \frac{d\theta}{dt} = 1.745 \times 10^{-3} \times 314\ \cos\ 314\ t \tag{3}$$

$|\omega| = 1.745 \times 10^{-3} \times 314 = 0.548$ rad/s is the peak angular velocity of the missile

Rigid body rotation = 10 rad/s

$$\%\ \text{of gyro signal due to rotation} = \frac{0.548}{10} \tag{4}$$

$$= 5.48\%$$



gyro and the base rotate with a peak amplitude of 0.1°.

To solve the bending vibration problem the gyro should be located at a node point. If the rigid body motions are of very low frequency, a low pass filter can be used that would filter the bending vibrations.

4.31 For linearizing Eq. 4.96 it was assumed that $\cos\theta = 1$ If $\cos\theta = 0.99$, non-linearity will be 1%

(a) $\cos\theta = 0.99 \quad \theta = \cos^{-1} 0.99 = 8.1096° = 0.142$ rad (1)

(b) $\omega_n = 100$ rad/s

$$I_y = 0.00015 \text{ kg-m}^2$$

$$\omega_n^2 = \frac{K_s}{I_y} \tag{2}$$

$$K_s = 100^2 \times 0.00015 = 1.5 \text{ N-m/rad}$$

(c) $\dot{\phi} = 10$ rad/s

Sensitivity
$$K = \frac{0.142 \text{ rad}}{10 \text{ rad/s}} = 0.0142 \text{ rad/rad/s} \tag{3}$$

$$\left\{\because \frac{\theta}{D\phi} = K \text{ (from 4.104)}\right\}$$

$$K = \frac{H_s}{K_s} \qquad (4)\ (4.106)$$

$$H_S = K\,K_s = 0.0142 \times 1.5 = 0.0213 \text{ kg-m}^2 - \text{rad/s}$$

(d) $\omega = \dfrac{24000}{60} \times 2\pi = 2513$ rad/s

$$H_s = I_s\,\omega,\ I_s = \frac{H_s}{\omega} = \frac{0.0213}{2513} = 8.475 \times 10^{-6} \text{ kg} - \text{m}^2 - \text{rad/s} \tag{5}$$

$$I_s = \pi\rho R^5 \Rightarrow R = \left(\frac{I_s}{\pi\rho}\right)^{1/5} = (8.475 \times 10^{-6}/\pi \times 2800)^{1/5} = 16 \text{ mm} \tag{6}$$

4.32

x_0 : Position of the centre of gravity of the float at equilibrium, it coincides with the water surface. x_0 is measured by a potentiometer

h_i : Height of the wave which pulls up the float. It acts like an input to the system

p_w : Density of water
A : Area of cross section of the float
B : Damping coefficient of the external damper
B_w : Damping provided by water
M : Mass of the float

$$\rho_w\ g\ A\ (h_i - x_0) - B\,\dot{x}_0 - B_w\ (\dot{x}_0 - \dot{h}_i) = M\,\ddot{x}_0 \tag{1}$$

$$[MD^2 + (B + B_w)\ D + \rho_w\ g\ A]\ x_0 = B_w\ Dh_i + \rho_w\ g\ A\ h_i$$

$$\frac{x_0}{h_i}(D) = \frac{(B_w D + \rho_w g A)}{MD^2 + (B + B_w)D + \rho_w g A} \tag{2}$$

$$\frac{x_0}{h_i}(D) = \frac{\left(\dfrac{B_w}{\rho_w\, g A}D + 1\right)}{\dfrac{M}{\rho_w g A}D^2 + \left(\dfrac{B + B_w}{\rho_w\, g A}\right)D + 1} \tag{3}$$

$$\omega_n = \sqrt{\frac{\rho_w g A}{M}} \tag{4}$$

$$\frac{2\zeta}{\omega_n} = \frac{B + B_w}{\rho_w g A} \qquad \zeta = \frac{(B + B_w)\omega_n}{2\rho_w g A}$$

$$\zeta = \frac{B + B_w}{2\rho_w g A}\sqrt{\frac{\rho_w g A}{M}} = \frac{B + B_w}{2\sqrt{\rho_w g A M}} \tag{5}$$

$$\tau = \frac{B_w}{\rho_w g A} \tag{6}$$

$$\frac{x_0}{h_i}(D) = \frac{(\tau D + 1)}{\dfrac{D^2}{\omega_n^2} + \dfrac{2\zeta D}{\omega_n} + 1} \tag{7}$$

Assumptions $\quad B_w << B \quad \tau \approx 0$

$$\frac{x_0}{h_i}(D) = \frac{1}{\dfrac{D^2}{\omega_n^2} + \dfrac{2\zeta D}{\omega_n} + 1} \tag{8}$$

Let $\quad \zeta = 0.65 \quad \omega_{max} = 0.2\ \omega_n$ {Flat response region for 95% accuracy}

$\omega_{max} = 0.2 \times 5 \times 2\pi$

$$\omega_n = \frac{\omega_{max}}{0.7} = \frac{0.2 \times 5 \times 2\pi}{0.7} = 8.975 \text{ rad/s} = \sqrt{\frac{\rho_w g A}{M_{max}}}$$

$$M_{\max} = \frac{\rho_w g A}{80.6}$$

4.33 If the vehicle rotates in both yaw and roll, and we use gyros to measure the roll motion, the output angle pick-off will incur an error due to the yaw motion. Equation 4.98 needs to be modified as follows

$$T_y - B(D\,\theta_g - D\theta_f) - K_s\,(\theta_g - \theta_f) = -\,H_s\,D\,\phi + I_y\,D^2\,\theta_g$$

$$= -\,H_s\,D\,\phi + I_y\,D^2(\theta_0 + \theta_f) \quad (1)$$

θ_0 is the pick-off output angle $= (\theta_g - \theta_f)$ (2)

$$(I_y\,D^2 + BD + K_s)\,\theta_0 = H_s\,D\,\phi - I_y\,D^2\,\theta_f \quad (3)$$

$$\frac{\theta_0}{\theta_f}(D) = \frac{-I_y D^2}{I_y D^2 + BD + K_s} \quad (4)$$

Because of the presence of D^2 in the numerator, the above equation acts like a "high-pass" filter. Therefore, very slow variation of yaw angle will not affect the gyro.

4.34

$$M_g - F_C = M\ddot{y}$$

M : Mass of the stylus
F_C : Contact force
If V is very large, the stylus will bounce off at which $F_C = 0$

$$M_g - 0 = M\ddot{y}$$

$$M(\ddot{y} - g) = 0, \;\; \ddot{y} = g \quad (1)$$

$$y = A\,\sin\frac{2\pi V t}{L} \quad (2)$$

L: Surface wavelength 2.5×10^{-2} cm

$$A = 2.5 \times 10^{-4}\ \text{cm}$$

$$\dot{y} = A\,\frac{2\pi V}{L}\,\cos\frac{2\pi V t}{L} \quad (3)$$

$$\ddot{y} = -A\left(\frac{2\pi V}{L}\right)^2 \sin\frac{2\pi V t}{L} \quad (4)$$

$$|\ddot{y}| = A\left(\frac{2\pi V}{L}\right)^2 = g \quad (5)$$

$$\frac{A\,4\pi^2 V^2}{L^2} = g \quad (6)$$

$$V^2 = \frac{gL^2}{4A\pi^2} \qquad V = \frac{L\sqrt{g}}{\pi\sqrt{4A}} = \frac{L}{\pi}\sqrt{\frac{g}{4A}} \quad (7)$$

$$V = \frac{2.5 \times 10^{-2}}{\pi} \sqrt{\frac{981}{4 \times 2.5 \times 10^{-4}}} = 7.88 \text{ cm/s} \tag{8}$$

4.35

(a) There would not be any error, because the line of action of the y-axis inertia force would be exactly through the axis of rotation and thus not produce any torque.

(b) Now there will be an error because the system as shown does have a small deflection when experiencing a steady x acceleration. A simultaneous y acceleration then causes an inertia torque and an error. We need to calculate the deflection for a given x acceleration and then multiply this by $r \sin \theta$.

Refer to Fig. 4.62(a)

Let us assume that the rotary spring, mass and dashpot system is replaced by a unity amplifier

$$D^2 x_i Mr - \theta K_p K_a K_c = 0$$

$$\theta = \frac{D^2 x_i Mr}{K_p K_a K_c} \tag{1}$$

The spurious inertia torque is given by

$$T_d = Mr\ \theta D^2 y_i \tag{2}$$

$$e_d = \frac{R}{K_c} Mr\ \theta D^2 y_i \tag{3}$$

Equation (3) represents the error

(c) $\ddot{x}_i = \ddot{y}_i = 0\ \ddot{\theta}_f$

θ_J = Absolute angle of the pivoted arm

θ_f = Absolute angle of the instrument frame

$\theta = \theta_J - \theta_f$: Angle measured by the sensor

$$-T_c - K_s\theta - BD\theta = JD^2\ \theta_f = JD^2\theta + JD^2\ \theta_f \tag{4}$$

$$T_c = +\ e_0 \frac{K_c}{R} \tag{5}$$

$$\theta = \frac{e_0}{K_p K_a R} \tag{6}$$

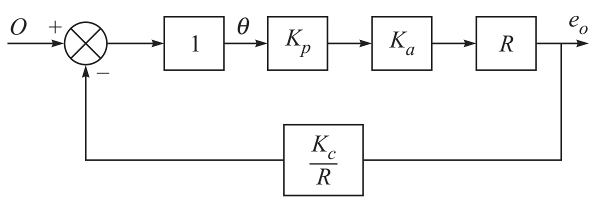


Fig. 1 (Modified 4.62 with $\ddot{x}_i = 0$ and pushing R inside).

From Eqs. (4), (5) and (6)

$$-\frac{e_0 K_c}{R} - K_S \frac{e_0}{K_p K_a R} - BD \frac{e_0}{K_p K_a R} = JD^2 \frac{e_0}{K_p K_a R} + JD^2 \theta_f$$

$$\frac{e_0}{\theta_f}(D) = \frac{JD^2}{-\left(\dfrac{JD^2}{K_p K_a R} + \dfrac{BD}{K_p K_a R} + \dfrac{K_S}{K_p K_a R} + \dfrac{K_c}{R}\right)} \tag{7}$$

4.36 In Fig. 4.15, the two horizontal members are treated as rigid and the vertical strain-gaged member is very flexible. When the specimen is loaded and the two knife edges move apart by the total full stroke X_{fs}, each horizontal member will move an amount $X_{fs}/_2$. This causes the flexible member to undergo a rotation θ, given by $\dfrac{X_f}{2L_m}$ (for small angles). By symmetry, we treat the strain gaged member as two cantilever beams and calculate for one of them.

The force exerted by the specimen, on the beam, is called F and this force creates a pure moment

$$M = \frac{F L_M}{2} \quad \text{(Because we have two beams)} \tag{1}$$

For a cantilever with pure moment

$$\theta = \frac{M L_B}{2EI} \tag{2}$$

$$\sigma = \frac{M_C}{I} = \frac{F L_M t}{4I} \qquad \because c = t/_2 \tag{3}$$

$$\varepsilon_{\text{des}} = \frac{\sigma}{E} = \frac{F L_m t}{4EI} \tag{4}$$

$$t = \frac{4EI\,\varepsilon_{\text{des}}}{F L_m} = \frac{2 L_B L_M \varepsilon_{\text{des}}}{X_{fs}} \left\{\because \frac{2EI X_{fs}}{L_B L_M^2} = F\right\} \tag{5) (4.15}$$

Let $2L_B = 25$ mm $\varepsilon_{\text{des}} = 1500$ μs (given)

$L_M = 50$ mm $X_{fs} = 0.5$ mm (given)

$$t = \frac{25 \times 50}{0.5} \times 1500 \times 10^{-6} = 3.75 \text{ mm} \tag{6}$$

Let $b = 8$ mm

$$I = \frac{bt^3}{12} = \frac{(8\times10^{-3})(3.75\times10^{-3})^3}{12}, \quad E = 200 \text{ GPa for steel} \tag{7}$$

$$= 3.5 \times 10^{-11} \text{ m}^4$$

$$F = \frac{2\times 200\times 10^{9}\times 3.5\times 10^{-11}\times 0.5\times 10^{-3}}{0.025\times 0.050^{2}} = 112.0 \text{ N} \tag{8}$$

4.37

$$q = C_{ec} \tag{1}$$

$$i = \frac{dq}{dt} = C\,\frac{de_c}{dt} + e_c\,\frac{dc}{dt} \tag{2}$$

$$C = \frac{0.00885A}{x_0 - x_i} \tag{3}$$

$$\frac{dc}{dt} = \frac{0.00885A}{(x_0 - x_i)^2}\,\frac{dx_i}{dt} \tag{4}$$

$$i = 0.00885\ A\left\{\frac{1}{x_0 - x_i}\,\frac{de_c}{dt} + \frac{e_c}{(x_0 - x_i)^2}\,\frac{dx_i}{dt}\right\} \tag{5}$$

$$e_c = E_b - iR \tag{6}$$

$$\frac{de_c}{dt} = -\,\mathrm{R}\,\frac{di}{dt} \tag{7}$$

$$i = 0.00885\ \mathrm{A}\left\{\frac{1}{x_0 - x_i}\,\frac{de_c}{dt} + \frac{(E_b - iR)}{(x_0 - x_i)^2}\,\frac{dx_i}{dt}\right\} \tag{8}$$

$$i = 0.00885\ \mathrm{A}\left\{\frac{1}{x_0 - x_i}\left(-R\,\frac{di}{dt}\right) + \frac{E_b - iR}{(x_0 - x_i)^2}\,\frac{dx_i}{dt}\right\} \tag{9}$$

Equation (9) can only be solved through simulation