Chapter 3

Solutions

3.1

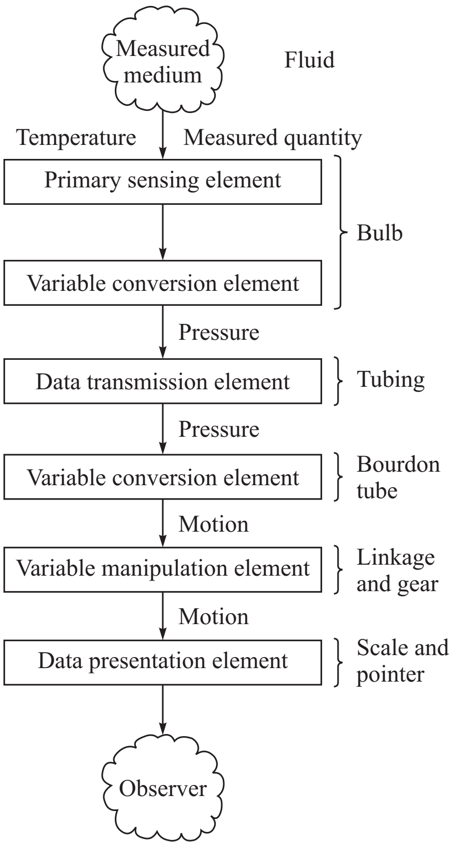


Fig. 1 Pressure Thermometer

(a) Commercial calibrators are available that have a liquid bath for immersing the bulb and an accurate temperature sensor that gives the bath temperature. One can also fabricate a liquid bath,

which has a heater and stirrer arrangement and use a liquid-in glass standard thermometer. For any of the above setups, set the bath temperature to 10 different values in the range of interest (both increasing and decreasing). Allow sufficient time for the system to achieve equilibrium at each temperature. Compare the temperature indicated by the instrument with a standard temperature immersed in the bath.

(b) Temperature of the air surrounding the capillary tube as an interfering input: The temperature of the bulb should be maintained constant at the room temperature and subject the tubing to a range of temperatures. The same experiment could be repeated for several different bulb temperatures, Although it is difficult to maintain the same tubing temperature over its length, the above calibration would give conservative predictions corresponding to the highest tubing temperature.

(c) Elevation difference between the bourdon tube and bulb:
First the bulb and tubing temperatures are kept fixed and experiments are conducted for different elevations. The same is repeated for other temperatures ensuring that there is no variation between the bulb and tubing temperature.

3.2 Whether the digital revolution counter needs calibration
Like any system, the system can malfunction. However, this system does not need any calibration-one revolution gives exactly one count. Higher shaft speeds can cause vibration in the sensing arm and/or micro switch mechanism that could result in loss of counts. So one has to be really sure about the highest speed at which this counter can be used. This is definitely not calibration in the usual sense.

3.3 Difficulties in defining true temperature of a body

(a) When a body has some physical size and shape, the temperature varies from point to point
(b) No temperature is absolutely constant with time
(c) If the body is a gas at very low pressures, the temperature variation will be more pronounced.

3.4 This is a very simple exercise that can be done at home. Results will vary depending on the individual, surface and the coin. If several students do this independently, more interesting variations are possible.

3.5 $e_0 = \dfrac{(GF)\,R_g\,\varepsilon\,E_b R_a}{(R_g + R_a)^2}$ Eq. (2.6) (see also Fig. 2.11)

ε: Strain, e_0: output voltage, GF: gage factor
R_g: Gage resistance, E_b: Bridge voltage
R_a: Resistance of the adjacent arm
Equation (2.6) can be written as:

$$\varepsilon = \frac{e_0 (R_g + R_a)^2}{GF\, R_g\, E_b\, R_a} \tag{1}$$

Taking natural log on both sides

$$\ln\varepsilon = \ln e_0 + 2\ln(R_g + R_a) - \ln(GF) - \ln R_g - \ln E_b - \ln R_a \tag{2}$$

Differentiating Eq. (2)

$$\frac{d\varepsilon}{\varepsilon} = \frac{de_0}{e_0} + \frac{2d(R_g + R_a)}{R_g + R_a} - \frac{dGF}{GF} - \frac{dR_g}{R_g} - \frac{dE_b}{E_b} - \frac{dR_a}{R_a} \tag{3}$$

using Eq. (3.20) and letting $R = R_g + R_a$

$$\left|\frac{d\varepsilon}{\varepsilon}\right|_{\max} = \left|\frac{de_0}{e_0}\right|_{\max} + 2\left|\frac{dR}{R}\right|_{\max} + \left|\frac{dGF}{GF}\right|_{\max} + \left|\frac{dR_g}{R_g}\right|_{\max} + \left|\frac{dE_b}{E_b}\right|_{\max} + \left|\frac{dR_a}{R_a}\right| \tag{4}$$

If each term of Eq. (4) contributed the same error of 1%

$$\left|\frac{d\varepsilon}{\varepsilon}\right|_{\max} = 1 + 2 + 1 + 1 + 1 + 1 = \mathbf{7\%}$$

Applying the root mean square formula of Eq. (3.21) to Eq. (3)

$$\left|\frac{d\varepsilon}{\varepsilon}\right|_{\max} = \sqrt{1^2 + 2^2 + 1^2 + 1^2 + 1^2 + 1^2} = \mathbf{3\%}$$

3.6 No.

The logarithmic differentiation method of Prob. 3.5 is only applicable to functions that are products and ratios of powers or roots of independent variables.

Therefore, for a function of the form $w = \sin x + 5y^3 - 6e^z$, the above method cannot be applied.

3.7

$$c_q = \frac{w}{t\rho A\sqrt{2gh}} \tag{1}$$

The nominal value of the discharge coefficient is given by

$$c_q = \frac{390}{600 \times 1050 \times \dfrac{\pi \times (12 \times 10^{-3})^2}{4} \times \sqrt{2 \times 9.81 \times 3.6}} \tag{2}$$

$$c_q = 0.6513$$

The partial derivative of the discharge coefficient w.r.t. all the parameters of Eq. (1) at the corresponding nominal values can be computed as follows:

$$\frac{\partial c_q}{\partial w} = \frac{1}{t\rho A\sqrt{2gh}} = \frac{1}{600 \times 1050 \times \dfrac{\pi \times (12 \times 10^{-3})^2}{4} \times \sqrt{2 \times 9.81 \times 3.6}} \tag{3}$$

$$\frac{\partial c_q}{\partial w} = 0.0017$$

$$\frac{\partial c_q}{\partial t} = -\frac{W}{t^2 \rho A \sqrt{2gh}} = -\frac{390}{600^2 \times 1050 \times \dfrac{\pi \times (12 \times 10^{-3})^2}{4} \times \sqrt{2 \times 9.81 \times 3.6}} \tag{4}$$

$$\frac{\partial c_q}{\partial t} = -0.0011$$

$$\frac{\partial c_q}{\partial \rho} = -\frac{W}{t\rho^2 A \sqrt{2gh}} = -\frac{390}{600 \times 1050^2 \times \dfrac{\pi \times (12 \times 10^{-3})^2}{4} \times \sqrt{2 \times 9.81 \times 3.6}} \tag{5}$$

$$\frac{\partial c_q}{\partial \rho} = -6.20 \times 10^{-4}$$

$$\frac{\partial c_q}{\partial d} = -\frac{8W}{t\rho d^3 \pi \sqrt{2gh}} = -\frac{8 \times 390}{600 \times 1050 \times (12 \times 10^{-3})^3 \times \pi \times \sqrt{2 \times 9.81 \times 3.6}} \tag{6}$$

$$\frac{\partial c_q}{\partial d} = -108.55$$

$$\frac{\partial c_q}{\partial g} = -\frac{1}{2} \frac{W}{t\rho A \sqrt{2h}\,(g)^{3/2}} = -\frac{1}{2} \frac{390}{600 \times 1050 \times \dfrac{\pi \times (0.012)^2}{4} \times \sqrt{2 \times 3.6} \times 9.81^{3/2}} \tag{7}$$

$$\frac{\partial c_q}{\partial g} = -0.0332$$

$$\frac{\partial c_q}{\partial h} = -\frac{1}{2} \frac{W}{t\rho A \sqrt{2g}\, h^{3/2}} = -\frac{1}{2} \frac{390}{600 \times 1050 \times \dfrac{\pi \times (0.012)^2}{4} \times \sqrt{2 \times 9.81} \times 3.6^{3/2}} \tag{8}$$

$$\frac{\partial c_q}{\partial h} = -0.0905$$

from Eq. (3.21), uncertainty of the flow coefficient can be expressed as

$$U_{c_q\,(\mathrm{rss})} = \sqrt{\left(\frac{\partial c_q}{\partial w} u_w\right)^2 + \left(\frac{\partial c_q}{\partial t} u_t\right)^2 + \left(\frac{\partial c_q}{\partial \rho} u_\rho\right)^2 + \left(\frac{\partial c_q}{\partial d} u_d\right)^2 + \left(\frac{\partial c_q}{\partial g} u_g\right)^2 + \left(\frac{\partial c_q}{\partial h} u_h\right)^2} \tag{9}$$

Before using Eq. (9), all uncertainties are expressed in terms of absolute values (instead of percentages as given for g and p)

$$w = \pm 0.25 \text{ kg} \qquad u_d = \pm 0.03 \text{ mm}$$
$$u_t = \pm 2s \qquad u_g = \pm 9.81 \times 0.001 \text{ m/s}^2$$
$$u_\rho = \pm 1050 \times 0.001 \text{ kg/m}^3 \qquad u_h = \pm 0.03 \text{ m}$$

$$U_{c_q(\text{rss})} = \sqrt{\begin{aligned}&(0.0017\times 0.25)^2 + (0.0011\times 2)^2 + (6.20\times 10^{-4}\times 1050\times 0.001)^2\\ &+(-108.55\times 0.03\times 10^{-3})^2 + (0.0332\times 9.81\times 0.001)^2 + (0.0905\times 0.03)^2\end{aligned}}$$

$$= 0.0048 \qquad (\text{rss} = \text{root mean square})$$

$$U_{c_q}(abs) = \left|\frac{\partial c_q}{\partial w}u_w\right| + \left|\frac{\partial c_q}{\partial t}u_t\right| + \left|\frac{\partial c_q}{\partial \rho}u_\rho\right| + \left|\frac{\partial c_q}{\partial d}u_d\right| + \left|\frac{\partial c_q}{\partial g}u_g\right| + \left|\frac{\partial c_q}{\partial h}u_h\right| \qquad (10)$$

$$\begin{aligned}U_{c_q}(abs) &= 0.0017 \times 0.25 + 0.0011 \times 2 + 6.20 \times 10^{-4} \times 1050 \times 0.001\\ &\quad + 108.55 \times 0.03 \times 10^{-3} + 0.0332 \times 9.81 \times 0.001 + 0.0905 \times 0.03\\ &= 0.0095\end{aligned}$$

3.8 If $\dfrac{U_{c_q}}{c_q} = 0.5\%$, uncertainties of all the parameters of the previous example are to be determined.

From the principle of equal effects

$$u_{x_i} = \frac{U_y}{\sqrt{n}\,\dfrac{\partial f}{\partial x_i}} \qquad (3.22)$$

For this problem, $U_y = U_{c_q}$ $n = 6$, $i = 1 \ldots 6$

$\dfrac{\partial f}{\partial x_i}$ can be obtained from Eq. (2) through (7) of Prob. 3.7

Since U_{c_q} is given as a percentage, Eq. (3.22) must be divided by c_q on both sides

$$\frac{u_{x_i}}{c_q} = \frac{U_{c_q}}{c_q\sqrt{6}\,\dfrac{\partial f}{\partial x_i}} \qquad (2)$$

$$u_{x_i}(\text{rss}) = \frac{c_q\left(\dfrac{U_{c_q}}{c_q}\right)}{\sqrt{6}\,\dfrac{\partial f}{\partial x_i}} \qquad (3)$$

$$u_w(rss) = \frac{0.65\times 0.005}{\sqrt{6}\times 0.0017} = 0.7961$$

$$u_t(rss) = \frac{0.65 \times 0.005}{\sqrt{6} \times 0.0011} = 1.2247$$

$$u_\rho(rss) = \frac{0.65 \times 0.005}{\sqrt{6} \times 6.20 \times 10^{-4}} = 2.14 \text{ kg/m}^3$$

$$u_d(rss) = \frac{0.65 \times 0.005}{\sqrt{6} \times 108.55} = 12.24 \ \mu\text{m}$$

$$u_g(rss) = \frac{0.65 \times 0.005}{\sqrt{6} \times 0.0332} = 0.04 \text{ m/s}^2$$

$$u_h(rss) = \frac{0.65 \times 0.005}{\sqrt{6} \times 0.0905} = 0.0147 \text{ m}$$

Similarly, uncertainty values for absolute errors can be expressed as

$$u_{x_i}(abs) = \frac{c_q\left(\dfrac{U_{c_q}}{c_q}\right)}{6\left|\dfrac{\partial f}{\partial x_i}\right|} \tag{4}$$

$$u_w(abs) = \frac{0.65 \times 0.005}{6 \times 0.0017} = 0.3250 \text{ kg}$$

$$u_t(abs) = \frac{0.65 \times 0.005}{6 \times 0.0011} = 0.5 \text{ s}$$

$$u_\rho(abs) = \frac{0.65 \times 0.005}{6 \times 6.20 \times 10^{-4}} = 0.8750 \text{ kg/m}^3$$

$$u_d(abs) = \frac{0.65 \times 0.005}{6 \times 108.55} = 5 \ \mu\text{m}$$

$$u_g(abs) = \frac{0.65 \times 0.005}{6 \times 0.0332} = 0.0164 \text{ m/s}^2$$

$$u_h(abs) = \frac{0.65 \times 0.005}{6 \times 0.0905} = 0.006 \text{ m}$$

Results of Prob. 3.7 and 3.8 are summarized as follows

	problem 3.7		problem 3.8	
Parameter	uncertainty		uncertainty	
	rss	abs	rss	abs
u_{C_q}	0.74%, 0.0048	1.45%, 0.0095	0.5%, 0.0032	0.5%, 0.0032
u_W kg	± 0.25	± 0.25	± 0.7961	± 0.325
u_t s	± 2	± 2	± 1.2247	± 0.5
u_p kg/m^3	± 1.05	± 1.05	± 2.14	± 0.8750
u_d, µm	± 30	± 30	± 12.24	± 5
u_g, m/s^2	± 0.00981	± 0.00981	± 0.04	± 0.0164
u_h, mm	± 30	± 30	± 14.7	± 6

3.9

$$\Sigma q_i = 360 \qquad \Sigma q_0 = 77.94$$
$$\Sigma q_i\, q_0 = 2211.9 \qquad \Sigma q_i^2 = 10200$$
$$N = 18 \text{ (Total number of data points)}$$

From Eq. 3.10

$$m = \frac{N\,\Sigma q_i q_0 - (\Sigma q_i)(\Sigma q_0)}{N\,\Sigma q_i^{\,2} - (\Sigma q_i)^2} \tag{3.10}$$

$$m = \frac{18 \times 2211.9 - 360 \times 77.94}{18 \times 10200 - 360^2} = 0.2177 \text{ (SLOPE)}$$

$$b = \frac{(\Sigma q_0)(\Sigma q_i^{\,2}) - (\Sigma q_i q_0)(\Sigma q_i)}{N\,\Sigma q_i^{\,2} - (\Sigma q_i)^2} \tag{3.11}$$

$$b = \frac{77.94 \times 10200 - 2211.9 \times 360}{18 \times 10200 - 360^2}$$

$$b = -0.024 \text{ (INTERCEPT)}$$

$$S_{q_0}^2 = \frac{1}{N-2} \Sigma\, (m\, q_i + b - q_0)^2 \tag{3.14}$$

S_{q_0}: Standard deviation of the output

$$\Sigma\, (mq_i + b - q_0)^2 = 0.0459$$

$$S_{q_0}^2 = \frac{0.0459}{18-2} = 0.0029;\ S_{q_0} = 0.054$$

$$Sq_i^2 = \frac{s_{q_0}^2}{m^2} \tag{3.16}$$

S_{q_i}: Standard deviation of the input

$$S_{q_i}^{\ 2} = \frac{0.054^2}{0.2177^2} = 0.0605$$

$$S_{q_i} = 0.2460$$

$$q_i^m = \frac{q_0 - b}{m} \tag{1}$$

For a known output q_0 of the instrument, the mean value of the input can be determined from Eq. (1)

However, due to uncertainties of the measurement system, the true value can only be determined in a band. Assuming 95% uncertainties ($\pm 2S_{q_i}$)

$$q_i = q_i^m \pm 2 \times S_{q_i} \tag{2}$$

Equations (1) and (2) are plotted in Fig. 1.

A magnified view around point a is shown in Fig. 2. Point 'a' is the intersection of q_0 = 5.72 and the best fit line of Eq. (1).

For $\quad q_0 = 5.72,\ q_i^m = \dfrac{(5.72-(-0.024))}{0.2177} = 26.38$

From $\quad$ Eq. (2) $q_i^{\text{lower}} = 26.38 - 2 \times 0.2460 = 25.89$

$q_i^{\text{upper}} = 26.38 + 2 \times 0.2460 = 26.87$

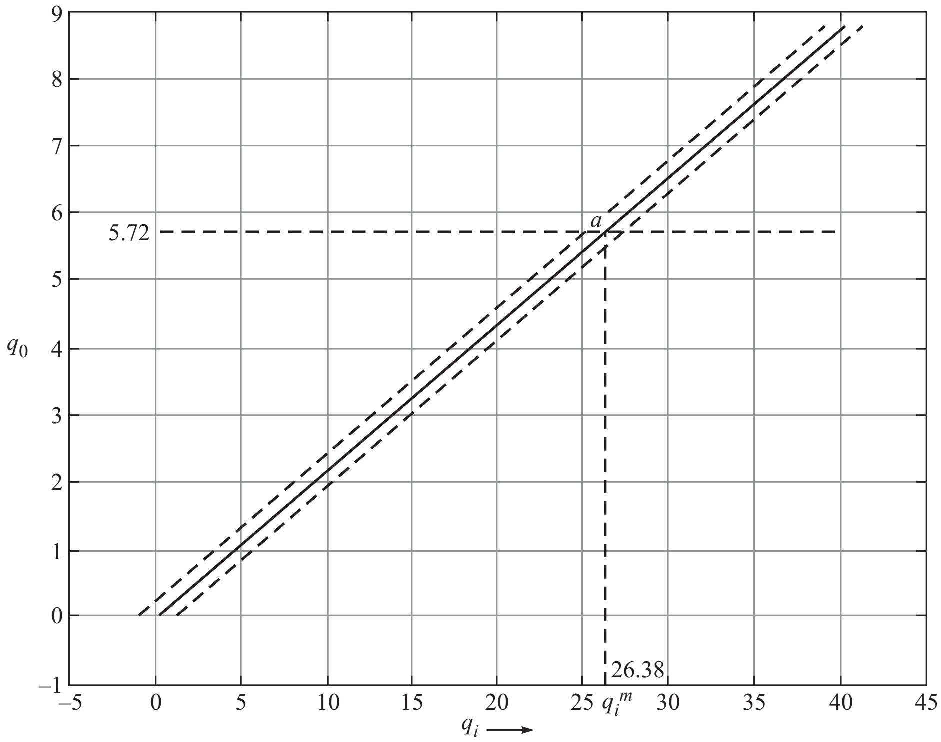


Fig. 1

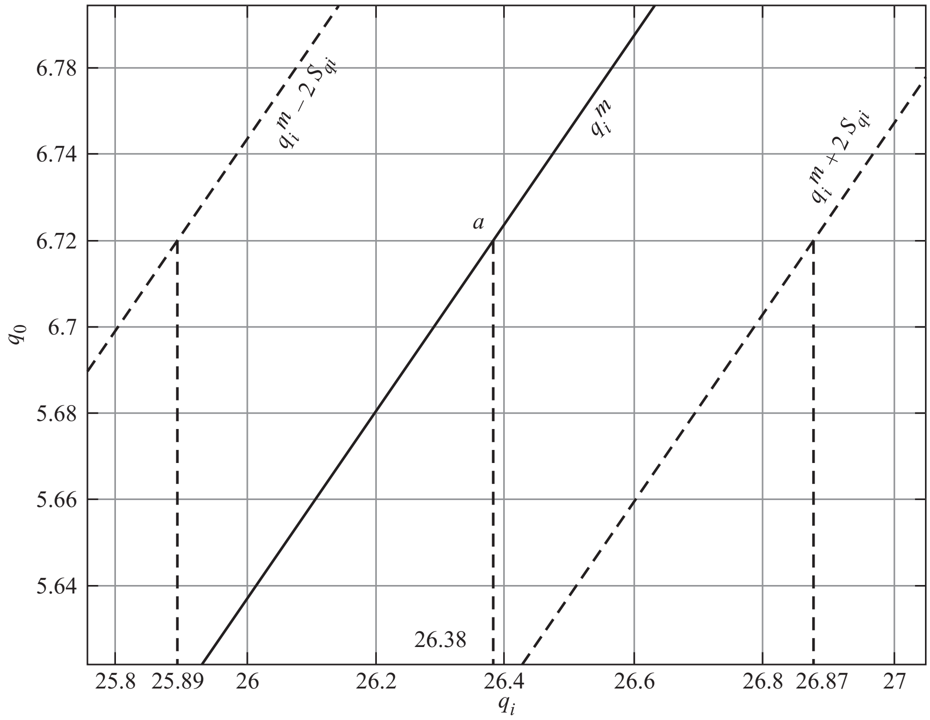


Fig. 2

3.10 Refer to Fig. 3.21.

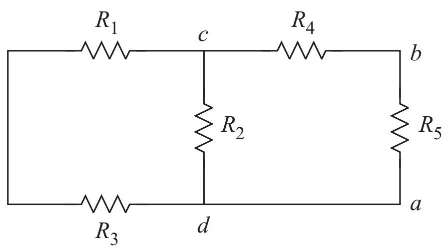


Fig. 1

The first step would be to obtain the equivalent resistance as seen through '*ab*', by short circuiting all the power sources.

Figure (1) can be simplified as

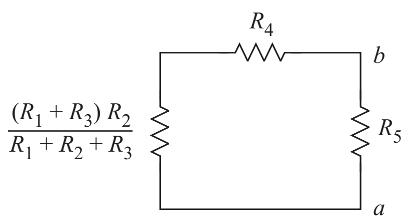


Fig. 2

$$R_{ab} = \frac{\left[\dfrac{(R_1+R_3)\,R_2}{R_1+R_2+R_3}+R_4\right]\times R_5}{\dfrac{(R_1+R_3)\,R_2}{R_1+R_2+R_3}+R_4+R_5} \tag{1}$$

Given $R_1 = R_2 = R_3 = R_4 = R_5 = 100\ \Omega$

R_{ab} from Eq. (1) $= 62.5\ \Omega$

From Eq. (3.41)

$$\frac{E_m}{E_0} = \frac{R_m}{R_m + R_{ab}} \tag{3.41}$$

Error due to loading resistance R_m

$$\varepsilon = 1 - \frac{E_m}{E_0} = 1 - \frac{R_m}{R_m + R_{ab}} \tag{2}$$

for $R_m = 1000\ \Omega$

$$\varepsilon = 1 - \frac{1000}{1000 + 62.5} = 5.88\%$$

for $R_m = 1000\ \Omega$

$$\varepsilon = 1 - \frac{10000}{10000 + 62.5} = 0.62\%$$

3.11 Objective: Error due to loading resistance R_m, if voltage across R_3 has to be measured.

Short circuit all the power sources and determine the equivalent resistance across R_3.

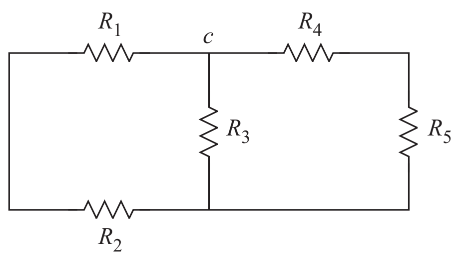


Fig. 1

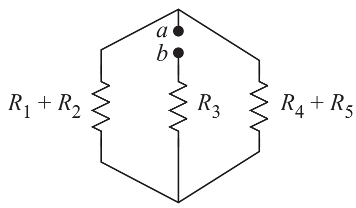


Fig. 2

$$\frac{1}{R_{ab}} = \frac{1}{R_1 + R_2} + \frac{1}{R_3} + \frac{1}{R_4 + R_5} \tag{1}$$

$$R_{ab} = \frac{R_3\,(R_1 + R_2)(R_4 + R_5)}{(R_1 + R_2)\,R_3 + R_3\,(R_4 + R_5) + (R_1 + R_2)(R_4 + R_5)} \tag{2}$$

$$R_1 = R_2 = R_3 = R_4 = R_5 = 100\ \Omega$$

$$R_{ab} = 50\ \Omega \text{ (from Eq. (2))}$$

Error due to loading resistance R_m

$$\varepsilon = 1 - \frac{R_m}{R_m + R_{ab}} \tag{3}$$

for $R_m = 1000\ \Omega$

$$\varepsilon = 1 - \frac{1000}{1000 + 50} = 4.76\%$$

for $R_m = 10000\ \Omega$

$$\varepsilon = 1 - \frac{10000}{10000 + 50} = 0.5\%$$

3.12 As a first step, short circuit all the power sources of Fig. 3.25 (a) (page 81)

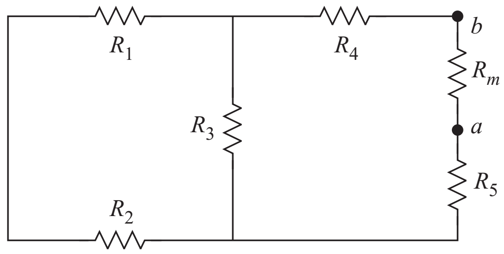


Fig. 1

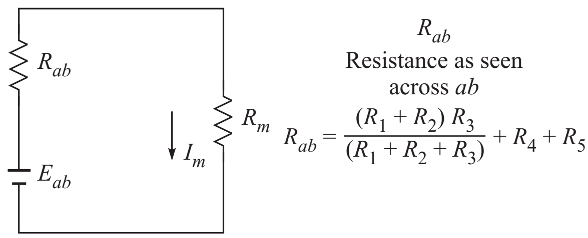


Fig. 2

$$I_u = \frac{E_{ab}}{R_{ab}} \text{ (3.47) (without meter)} \tag{1}$$

$$I_m = \frac{E_{ab}}{R_{ab} + R_m} \text{ (with meter)} \tag{2}$$

Error due to meter loading

$$\varepsilon = \frac{I_u - I_m}{I_u} = \frac{\dfrac{E_{ab}}{R_{ab}} - \dfrac{E_{ab}}{R_{ab} + R_m}}{\dfrac{E_{ab}}{R_{ab}}} \tag{3}$$

$$\varepsilon = \frac{R_m}{R_{ab} + R_m} \tag{4}$$

If $R_1 = R_2 = R_3 = R_4 = R_5 = 100\,\Omega$

$R_{ab} = 267\ \Omega$ (from Fig. 2)

for $R_m = 10\ \Omega$ $\quad \varepsilon = \dfrac{10}{267 + 10} = 3.6\%$

for $R_m = 1\ \Omega$ $\quad \varepsilon = \dfrac{1}{267 + 1} = 0.37\%$

3.13

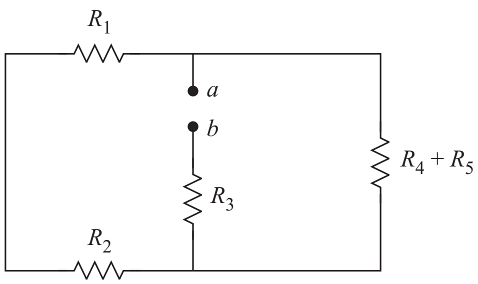


Fig. 1

$$R_{ab} = \frac{(R_1 + R_2)(R_4 + R_5)}{R_1 + R_2 + R_4 + R_5} + R_3 \tag{1}$$

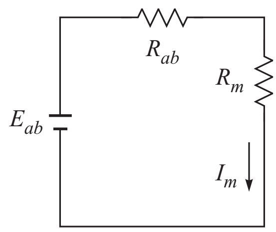


Fig. 2

If $R_1 = R_2 = R_3 = R_4 = R_5 = 100\ \Omega$

$R_{ab} = 200\ \Omega$ {from Eq. (1)}

From Prob. 3.12, $\quad \varepsilon = \dfrac{R_m}{R_{ab} + R_m}$ (2)

Error due to loading

If $$R_m = 10\ \Omega,$$

$$\varepsilon = \frac{10}{200+10} = 4.76\%$$

If $$R_m = 1\Omega$$

$$\varepsilon = \frac{1}{200+1} = 0.5\%$$

3.14 Objective: Percentage error in measuring force in k_2

Given $$k_1 = k_2 = k_3 = k_4 = 100 \text{ N/cm}$$
$$k_m = 1000 \text{ N/cm, } 10000 \text{ N/cm}$$

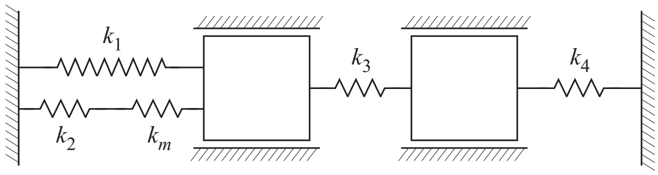


Fig. 1

From Eq. 3.62 (page 84), the output stiffness S_{g0} is given by

$$S_{g0} = \frac{1}{\dfrac{1}{k_2} + \dfrac{(k_3+k_4)}{(k_3+k_4)(k_1+k_3)-k_3^2}} \tag{3.62}$$

$$= \frac{1}{\dfrac{1}{100} + \dfrac{(100+100)}{(100+100)(100+100)-100^2}}$$

$$S_{g0} = 60 \text{ N/cm}$$

Case (i) $k_m = 1000$ N/cm $\quad S_{g_i} = k_m = 1000$ N/cm (Input stiffness)
Error = $S_{g0}/(S_{g0} + S_{g_i}) = 60/(60 + 1000) = 5.66\%$

Case (ii) $k_m = 10000$ N/cm $\quad S_{g_i} = k_m = 10000$ N/cm
Error = $60/(60 + 10000) = 0.596\%$

3.15 Objective: Same as 3.14, except that the force in k_3 is to be measured.

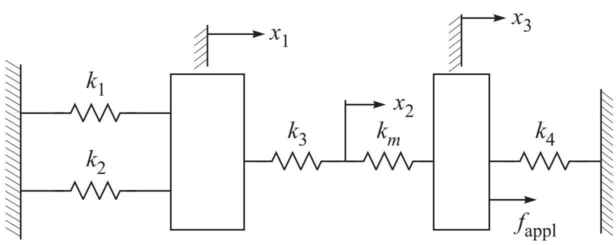


Fig. 1

The stiffness matrix $[K]$ for the above system is given by

$$[K] = \begin{bmatrix} k_1 + k_2 + k_3 & -k_3 & 0 \\ -k_3 & k_3 + k_m & -k_m \\ 0 & -k_m & k_4 + k_m \end{bmatrix} \tag{1}$$

It is easier to write down the above stiffness matrix since it follows a certain pattern-the diagonal elements are formed by adding stiffness of all the elements connected to a degree of freedom; off-diagonal elements are formed by subtracting stiffness of elements connecting to the other degrees of freedom.

Stiffness matrix is related to applied forces by

$$[K]\boldsymbol{x} = \boldsymbol{f} \text{ (2) Where } \boldsymbol{x} = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} \boldsymbol{f} = \begin{Bmatrix} 0 \\ 0 \\ f_{\text{appl}} \end{Bmatrix}$$

$$\begin{bmatrix} k_1 + k_2 + k_3 & -k_3 & 0 \\ -k_3 & k_3 + k_m & -k_m \\ 0 & -k_m & k_4 + k_m \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ f_{\text{appl}} \end{Bmatrix} \tag{3}$$

$$(k_1 + k_2 + k_3)x_1 - k_3 x_2 = 0 \tag{4}$$

$$-k_3 x_1 + (k_3 + k_m)x_2 - k_m x_3 = 0 \tag{5}$$

$$-k_m x_2 + (k_4 + k_m)x_3 = f_{\text{appl}} \tag{6}$$

The measured force f_m is

$$f_m = k_3(x_3 - x_2) \tag{7}$$

In order that we do not have to solve explicitly for x_1, x_2, x_3 to determine the error due to the measuring system, Eqs. (4), (5), (6) are rearranged so they can be expressed in terms of f_m.
From Eqs. (4)

$$(k_1 + k_2)x_1 = k_3(x_2 - x_1) \tag{8}$$

From Eq. (5)

$$k_3(x_2 - x_1) = k_m(x_3 - x_2) = f_m \tag{9}$$

From Eq. (6)

$$k_m(x_3 - x_2) + k_4 x_3 = f_{\text{appl}}$$

i.e.

$$f_m + k_4 x_3 = f_{\text{appl}} \tag{10}$$

From Eq. (8)

$$(k_1 + k_2)x_1 = f_m \tag{11}$$

which gives

$$x_1 = \frac{f_m}{k_1 + k_2} \tag{12}$$

From Eqs. (9) and (12)

$$k_3\left(x_2 - \frac{f_m}{k_1 + k_2}\right) = f_m \tag{13}$$

Which gives

$$x_2 = f_m\left\{\frac{1}{k_3} + \frac{1}{k_1 + k_2}\right\} \tag{14}$$

From Eqs. (9) and (14)

$$k_m\left\{x_3 - f_m\left[\frac{1}{k_3} + \frac{1}{k_1 + k_2}\right]\right\} = f_m \tag{15}$$

Which gives

$$x_3 = f_m \left\{\frac{1}{k_m} + \frac{1}{k_3} + \frac{1}{k_1 + k_2}\right\} \tag{16}$$

From Eqs. (10) and (16)

$$f_m + k_4\, f_m \left\{\frac{1}{k_m} + \frac{1}{k_3} + \frac{1}{k_1 + k_2}\right\} = f_{\text{appl}} \tag{17}$$

$$\frac{f_m}{f_{\text{appl}}} = \frac{1}{1 + \dfrac{k_4}{k_m} + \dfrac{k_4}{k_3} + \dfrac{k_4}{k_1 + k_2}} \tag{18}$$

Equation (18) relates the force measured by the instrument in response to the applied force, which accounts for the spring stiffness of the instrument. If one were to be interested in directly obtaining the applied force on the measuring instrument, the calibration constant defined by Eq. (18) has to be accounted for. However, the objective of this problem is to determine the percentage error in measurement due to k_m. If k_m were to be extremely large, it represents the situation of least error in measurement.

Let $f_m^{\,u}$ (undisturbed value of force due to k_m) be given by

$$\frac{f_m^{\,u}}{f_{appl}} = \frac{1}{1 + \dfrac{k_4}{k_3} + \dfrac{k_4}{k_1 + k_2}} \tag{19}$$

(Eq. (18) for $k_m = \infty$ gives Eq. (19))
From Eqs. (18) and (19)

$$\frac{f_m}{f_m^u} = \frac{1+\dfrac{k_4}{k_3}+\dfrac{k_4}{k_1+k_2}}{1+\dfrac{k_4}{k_m}+\dfrac{k_4}{k_3}+\dfrac{k_4}{k_1+k_2}} \tag{20}$$

$$\text{Error} = 1 - \frac{f_m}{f_m^u} \tag{21}$$

$$k_1 = k_2 = k_3 = k_4 = 100 \text{ N/cm}$$

From Eq. (20), for k_m = 1000 N/cm

$$\frac{f_m}{f_m^u} = \frac{1+1+\dfrac{1}{2}}{1+\dfrac{100}{1000}+1+\dfrac{1}{2}} = 0.9615$$

$$\text{Error} = 1 - 0.9615 = 3.85\% \qquad \text{(from Eq. 21)}$$

From Eq. (20), for k_m = 10000 N/cm

$$\frac{f^m}{f_m^u} = \frac{1+1+\dfrac{1}{2}}{1+\dfrac{100}{10000}+1+\dfrac{1}{2}} = 0.9960$$

$$\text{Error} = 1 - 0.9960 = 0.4\% \qquad \text{(from Eq. 21)}$$

Therefore, increasing the stiffness of the force transducer will decrease error, but will have other disadvantages related to sensitivity, resolution, etc.

3.16

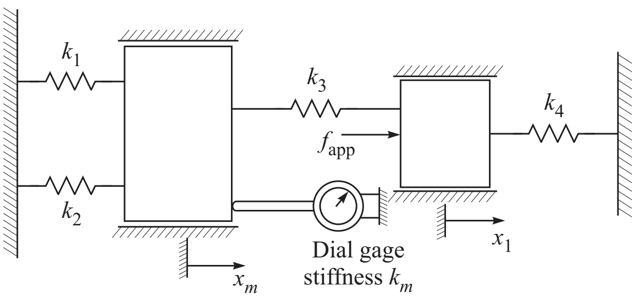


Fig. 1

$$[K]\boldsymbol{x} = \boldsymbol{f} \tag{1}$$

$$\begin{bmatrix} k_1+k_2+k_3+k_m & -k_3 \\ -k_3 & k_3+k_4 \end{bmatrix} \begin{Bmatrix} x_m \\ x_1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ f_{\text{app}} \end{Bmatrix} \tag{2}$$

Where x is measured by the dial gage.
from Eq. (2)

$$(k_1 + k_2 + k_3 + k_m)x_m = k_3\, x_1 \tag{3}$$

Which gives

$$x_1 = \frac{1}{k_3}\,\{k_1 + k_2 + k_3 + k_m\}x_m \tag{4}$$

From Eqs. (2) and (4)

$$x^m = \frac{f_{\text{appl}}}{(k_1+k_2+k_4+k_m) + \dfrac{k_4}{k_3}(k_1+k_2+k_m)} \tag{5}$$

The undisturbed value of x^m can be obtained by putting $k_m = 0$ in Eq. (5) as follows. (PS: Eq. (5) is also part of Prob. 3.17)

$$x_u^m = \frac{f_{\text{appl}}}{(k_1+k_2+k_4) + \dfrac{k_4}{k_3}(k_1+k_2)} \tag{6}$$

From Eqs. (5) and (6)

$$\frac{x^m}{x_u^m} = \frac{(k_1+k_2+k_4) + \dfrac{k_4}{k_3}(k_1+k_2)}{(k_1+k_2+k_4+k_m) + \dfrac{k_4}{k_3}(k_1+k_2+k_m)} \tag{7}$$

$$\text{Error} = 1 - \frac{x^m}{x_u^m} \tag{8}$$

Given $\quad k_1 = k_2 = k_3 = k_4 = 1$ N/cm
for $\quad k_m = 0.1$ N/cm

From Eq. (7)

$$\frac{x^m}{x_u^m} = \frac{1+1+1+(1+1)}{(1+1+1+0.1)+(1+1+0.1)} = 0.9615$$

Error = 1 – 0.9615 = 3.85% (from Eq. (8))

for $\quad k_m = 0.01$ N/cm

$$\frac{x^m}{x_u^m} = \frac{1+1+1+(1+1)}{(1+1+1+0.01)+(1+1+0.01)} = 0.9960$$

$$\text{Error} = 1 - 0.9960 = 0.4\%$$

3.18

(a) Step input

$$t = 0.05\text{s accuracy } 95\%$$

From Eq. 3.98,

$$\frac{q_0}{K_{q_i}} = 0.95 = 1 - e^{-0.05/\tau_{\text{step}}}$$

$$\tau_{\text{step}} = \frac{-0.05}{\ln 0.05} = 0.0166\text{s}$$

(b) Ramp input

$$e_{m,\text{ss}} = 14 \text{ kPa} \qquad q_{is} = 700 \text{ kPa/s}$$

From Eq. 3.104,

$$e_{m,\text{ss}} = \tau_{\text{ramp}}\, q_{is}$$

$$\tau_{\text{ramp}} = \frac{14}{700} = 0.02\text{s}$$

(c) Sine input

$$\frac{q_0}{K_{q_i}} = 0.9; \qquad \omega = 2\pi \times 25 = 157.08 \text{ rad/s}$$

From Eq. 3.106,

$$\frac{q_0}{K_{q_i}} = 0.9 = \frac{1}{\sqrt{(\omega\tau_{\text{sine}})^2 + 1}}$$

$$\tau_{\text{sine}} = \sqrt{\frac{(1/0.9)^2 - 1}{157.08}} = 0.0386\text{s}$$

The maximum value of the time constant of the first order system is for sine input. Therefore $\tau = \tau_{\text{sine}} = 0.0386$s must be used. This satisfies condition for sine input and results in more accuracy than required for ramp and step inputs.

3.19 Diameter of the capillary tube $d_c = 0.25$ mm

K: Sensitivity 4 mm/°C
Operating temperature 20°C
Bulb is spherical and immersed in stationary air

$$K = \frac{K_{ex} V_b}{A_c} \tag{3.91}$$

K_{ex} : Differential volume expansion coefficient of thermometer fluid and glass
V_b : Bulb volume
A_c : Area of capillary tube
α_{mercury} : Linear expansion coefficient of mercury

$$= 60 \times 10^{-6} \text{ m/m} - °\text{C}$$

$$K_{ex} = 3 \times \alpha_{\text{mercury}} = 180 \times 10^{-6} \text{ m}^3/\text{m}^3 - °\text{C}$$

$$A_c = \frac{\pi d_c^2}{4} = \frac{\pi \times (0.25 \times 10^{-3})^2}{4} = 4.91 \times 10^{-8} \text{ m}^2$$

$$V_b = \frac{K A_c}{K_{ex}} = \frac{4 \times 10^{-3} \times 4.91 \times 10^{-8}}{180 \times 10^{-6}} = 1.09 \times 10^{-6} \text{ m}^3$$

For a spherical bulb, $V_b = \dfrac{\pi d_b^3}{6}$

$$d_b = \left(\frac{6V_b}{\pi}\right)^{1/3} = \left(\frac{6 \times 1.09 \times 10^{-6}}{\pi}\right)^{1/3} = 12.77 \text{ mm}$$

Surface area of the bulb, $A_b = \pi d_b^2 = \pi \times (12.77 \times 10^{-3})^2$

$$A_b = 5.12 \times 10^{-4} \text{ m}^2$$

$$\tau = \frac{\rho c V_b}{U A_b} \tag{3.92}$$

τ : Time constant's; C: specific heat of thermometer fluid J/kg – °C
ρ : Density of thermometer fluid kg/m^3
C : Specific heat kJ/kg – °C
V_b : Volume of the bulb, m^3
U : Heat transfer coefficient J/s – m^2 – °C
A_b : Area of bulb, m^2

$$\rho(\text{mercury}) = 13600 \text{ kg/m}^3$$
$$\text{C(mercury)} = 0.15 \text{ kJ/kg} - °\text{C}$$
$$\text{U(Air)} = 5.36 \text{ J/s} - \text{m}^2 - °\text{C}$$

$$\tau = \frac{13600 \times 0.15 \times 10^3 \times 1.09 \times 10^{-6}}{5.36 \times 5.12 \times 10^{-4}} = 810\text{s}$$

3.20

τ : Time constant = 15s
ΔT : Variation of temperature with height

$$= 0.15°\text{C}/30\text{m}$$

Ramp response of a first-order system

$$q_0 = K \dot{q}_{is} (\tau e^{-t/\tau} + t - \tau) \text{ (Eq. 3.102)}$$

K : Static sensitivity

For steady-state response, $\tau e^{-t/\tau} \approx 0$

$$q_0 = K\dot{q}_{is}(t - \tau) \tag{2}$$

$\dot{q}_{is}$: Ramp input = $\Delta T \times v$

v : Velocity (given) = 6 m/s

$$\dot{q}_{is} = \frac{0.15}{30} \times 6 = 0.03°\text{C/s}$$

(a) From Eq. (2), there is a time lag of τ seconds and the balloon moves up by $\tau \times v$ meters during this period

If $h = 3000$ m is the current altitude,

$h - \tau v = 3000 - 15 \times 6 = 2910$ m is the altitude at which 0°C occurs

(b) Steady-state error = $\dot{q}_{is}\,\tau$ [Eq. 3.104]

$= 0.03 \times 15 = 0.45°\text{C}$

Correct temperature at 3000 m = 0 – 0.45°C

= –0.45°C

P.S: (1)

(1) part (b) has not been asked in the problem.

3.21

τ : Time constant = 0.48

K : Static sensitivity = 0.01 mV/°C

q_{i_i} : Initial input temperature = 15°C

q_{i_f} : Final input temperature = 80°C

q_0 : Output voltage, mV

$$q_0 = K(q_i - q_{i_i})(1 - e^{-t/\tau})$$

Given
$$\frac{q_0}{K(q_i - q_{i_i})} = 0.7 = 1 - e^{-t/\tau} \tag{1}$$

(When q_0 records 70% of the reading)

Taking natural logarithm of Eq. (1)

$$\ln(1 - 0.7) = -t/\tau$$

$$t = -\tau \ln(1 - 0.7) = 0.48\text{s}$$

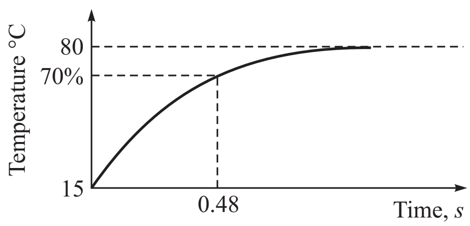


Fig. 1

$$q_0 = 0.01(80 - 15)\left(1 - e^{\frac{-0.48}{0.4}}\right)$$

$$= 0.45 \text{ mV}$$

3.22

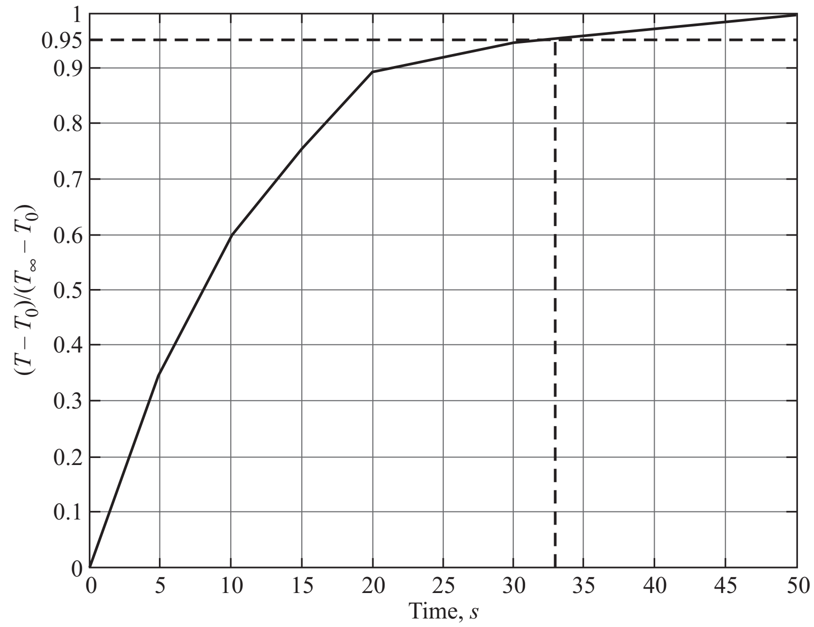


Fig. 1

(a) The given data are first normalized as follows so they can be compared against standard values. The following Eq. is used

$$\frac{T - T_0}{T_\infty - T_0} \tag{1}$$

Table 1

t, Sec	*Equation (1)*
0	0
5	0.35
10	0.6
15	0.76
20	0.9
25	0.92
30	0.946
35	0.9568

Data of Table 1 are plotted in Fig. 1.

(b) The time instant at which the temperature reaches 95% of its final value is 32.5s = 3τ

Time constant $\tau = \dfrac{32.5}{3} = 10.83\text{s}$

(c) The first order Eq. is given by

$$\frac{T - T_0}{T_\infty - T_0} = e^{-t/\tau} \tag{2}$$

τ is obtained from part (b)

Equation (2) is used to obtain the normalized values of temperature at the same instants corresponding to the experiment.

(d)

Table 2

t, Sec	*Equation (2)*
0	0
5	0.37
10	0.60
15	0.75
20	0.84
25	0.90
30	0.937
35	0.96
40	0.97

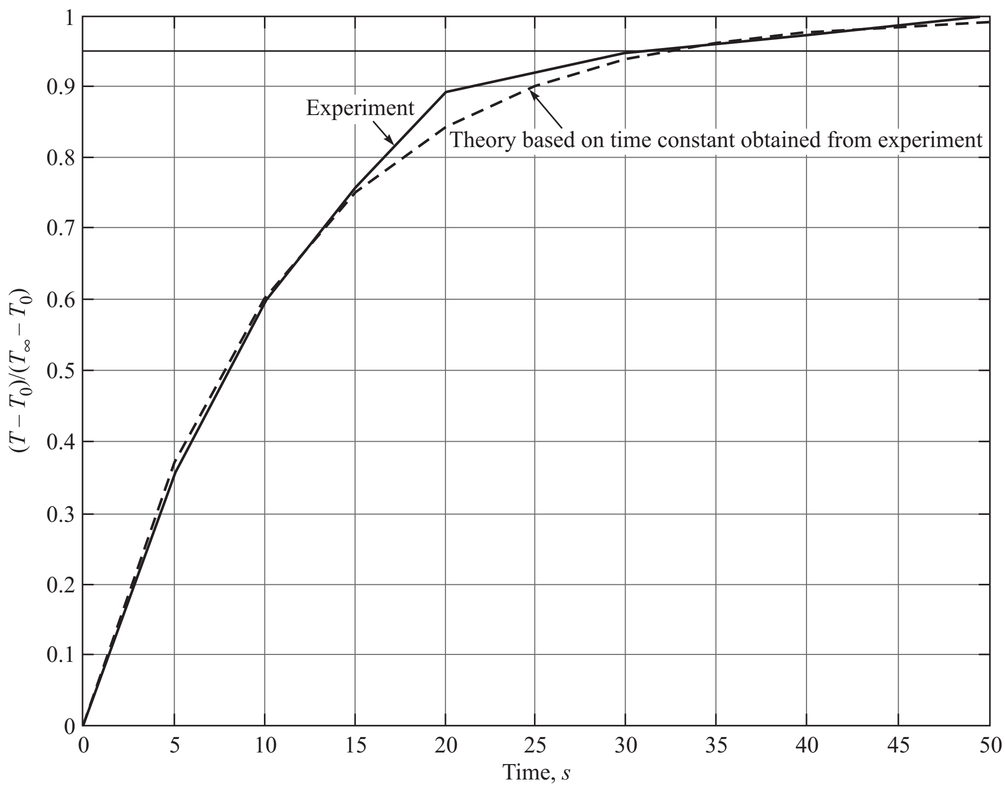


Fig. 2

(e) Data of Table 2 are plotted in Fig. 2. It can be seen from Fig. 2 that the instrument behaves like a typical first-order system at most of the points (except at t = 15s)

PS: The accuracy of τ determined from Fig. 1 depends mainly on the value of temperature at 3τ (95% of the value) the experimental data at other points do not account for determining τ.

This can be avoided by expressing Eq. (2)

as
$$\ln \frac{T - T_0}{T_\infty - T_0} = -\frac{t}{\tau} \tag{3}$$

or
$$\frac{t}{\tau} = -\ln \frac{T - T_0}{T_\infty - T_0} \tag{4}$$

(see Eq. 3.196)

Eq. (4) is plotted against time in Fig. 3 and a best fit curve of the above points is also drawn on the same figure. The negative reciprocal of the slope of this best fit curve gives the time constant τ = 10.95s

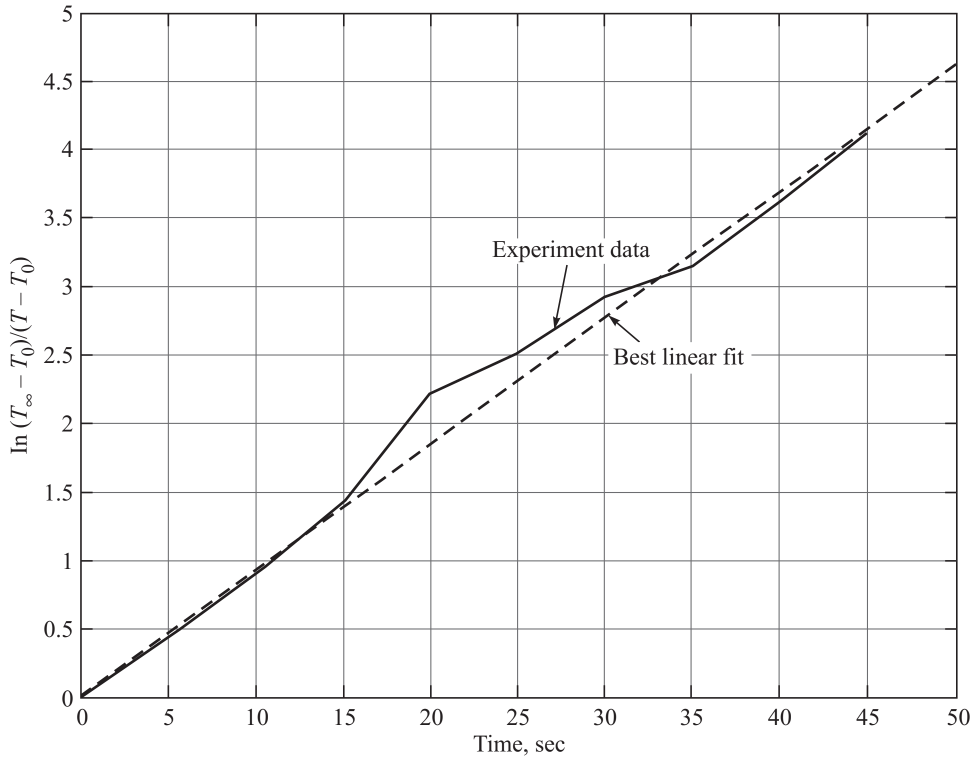


Fig. 3

3.23 The differential Eq. for a two-time constant transducer can be written as

$$(1 + \tau_s D)(1 + \tau_p D)q_0 = Kq_i \tag{1}$$

By defining $\zeta = \dfrac{\tau_s}{\tau_p}$ and redefining the input and output variables in terms of temperature, we get

$$\frac{\Delta T}{\Delta T_{\max}} = \left(\frac{\zeta}{\zeta - 1}\right) e^{\frac{-t}{\zeta \tau_p}} - \left(\frac{1}{\zeta - 1}\right) e^{-t/\tau_p} \tag{2}$$

ΔT = Difference between the temperature at time t and final temperature

$\Delta T_{\max}$: Difference between final temperature and initial temperature

Given T_i : Initial temperature = 100°C

T_f : Final temperature = 400°C

T_t : Temperature at t = 5s needs to be determined

$$\Delta T_{\max} = T_f - T_i = 400 - 100 = 300$$

$$\zeta = 5, \qquad \tau_p - 2\text{s (Given)}$$

$$\frac{T_f - T_t}{T_f - T_i} = \left(\frac{5}{5-1}\right) e^{-5/5\times 2} - \left(\frac{1}{5-1}\right) e^{-5/2} = 0.7376$$

$$T_t = T_f - 0.7376\ (T_f - T_i) = 400 - 0.7376\ (400 - 100)$$
$$= 179°\text{C (temperature at } t = 5\text{s)}$$

3.24

$$f_{\max} = 100 \text{ Hz}$$

$$\frac{q_0}{q_i}(j\omega) = \frac{K}{\sqrt{\omega^2 \tau^2 + 1}} < \tan^{-1}(-\omega\tau)\ (3.105)$$

Given
$$\left|\frac{q_0}{K q_i}\right| = 0.95 = \frac{1}{\sqrt{(2\pi f_{\max})^2 \tau^2 + 1}}$$

$$\tau = \sqrt{\frac{\left(\frac{1}{0.95}\right)^2 - 1}{(2\pi f_{\max})^2}} = \sqrt{\frac{\frac{1}{0.95^2} - 1}{(2\pi \times 100)^2}}$$

$$\tau = 5.23 \times 10^{-4}\text{s}$$

$$\phi(50 \text{ Hz}) = \tan^{-1}(-2\pi \times 50 \times 5.23 \times 10^{-4}) = -9.33°$$

$$\phi(100 \text{ Hz}) = \tan^{-1}(-2\pi \times 100 \times 5.23 \times 10^{-4}) = -18.19°$$

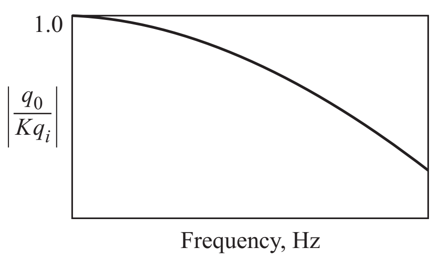


Fig. 1

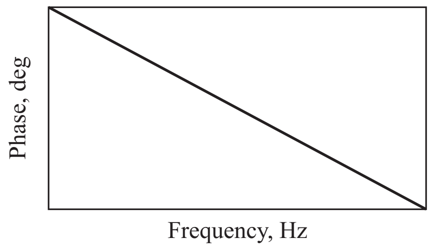


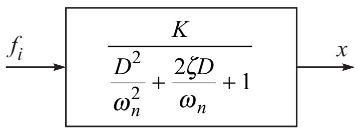
Fig. 2

Figure 1 shows amplitude accuracy versus frequency and Fig. 2 shows phase between input and output of a first order system subjected to sine excitation. Amplitude accuracy decreases with frequency and phase linearly decreases with frequency. Therefore, it is only the amplitude accuracy that decides the maximum frequency. In the present case, the system has 95% accuracy at 100 Hz and a higher amplitude accuracy at 50 Hz.

3.25 From Eq. 3.145, increase in stiffness reduces static sensitivity

From Eq. 3.146, increase in stiffness increases speed of response

3.26 *Second order system* of Fig. 3.43



From Fig. 3.45, time-lag τ for the second order system for ramp input

$$= \frac{2\zeta}{\omega_n} = \frac{2B}{2\sqrt{k_s m}\sqrt{\dfrac{k_s}{m}}} = \frac{B}{k_s} \tag{1}$$

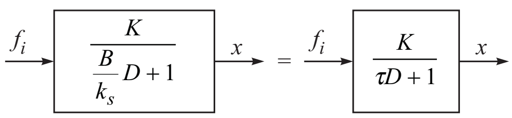
B : Damping coefficient $\dfrac{N-s}{m}$

k_s : Stiffness, N/m

First order system of Fig. 3.43

$$\left.\begin{array}{r}\text{if } m \text{ is negligible}\\ \omega_n \text{ is very large}\end{array}\right\} \frac{D^2}{\omega_n^2} \approx 0$$

Since $\dfrac{2\delta}{\omega_n} = \dfrac{B}{k_s}$ from Eq. (1), the transfer function of the first order system becomes



Where $$\tau = \frac{B}{k_s}$$

From Fig. 3.37, steady state time lag for ramp input to a first order system = $\tau = B/k_s$

Therefore, irrespective of whether mass is neglected or not, time lag remains the same because of constant velocity.

3.27

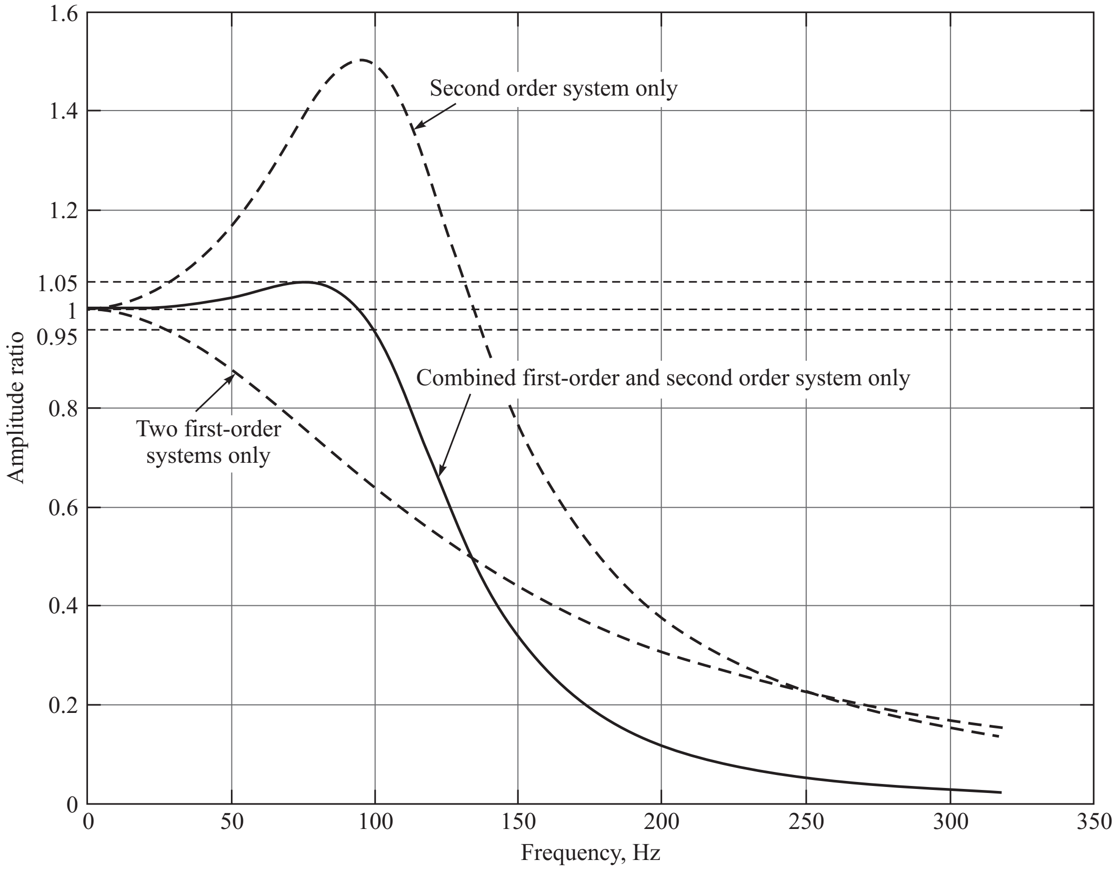


Fig. 1

This problem does'nt have any analytical solution. Therefore, computer has to be used. Fig. 1 has been drawn after conducting an exhaustive search for all possible values. Following values were used

$$\tau_1 = \tau_2 = 0.0012\text{s (Both first-order systems were assumed to have the same time constant)}$$
$$\zeta = 0.356$$
$$\omega_n = 690 \text{ rad/s (109.8 Hz)}$$

In Fig. 1, first-order systems and second-order systems were drawn separately. Due to the block diagram representation, the amplitudes of each block multiplies with the other. A stand-alone second

order system has 50% more response than the flat response region. On the other hand, the two first-order systems together have a dropping response after 10 Hz. A combination of all the systems gives the desired amplitude flat response of ± 5% from 0-100 Hz.

3.28 Any periodic function of period $2L$ can be expressed in terms of Fourier series as

$$q_i(t) = q_{i,av} + \frac{1}{L}\sum_{n=1}^{\infty} a_n \cos\frac{n\pi t}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi t}{L} \tag{1}$$

where,

$$q_{i,av} = \frac{1}{2L}\int_{-L}^{L} q_i(t)\,dt \tag{2}$$

$$a_n = \int_{-L}^{L} q_i(t)\cos\frac{n\pi t}{L}\,dt \tag{3}$$

$$b_n = \int_{-L}^{L} q_i(t)\sin\frac{n\pi t}{L}\,dt \tag{4}$$

For the given problem $q_i(t) = -1$ for $-L < t < 0$ (5)

$$q_i(t) = 1 \text{ for } 0 < t < L$$

$$L = 0.01\text{s}$$

From Eq. (2) and (5)

$$q_{i,av} = 0$$

From Eq. (3) and (5)

$$a_n = -\int_{-L}^{0} \cos\frac{n\pi t}{L}\,dt + \int_{0}^{L} \cos\frac{n\pi t}{L}\,dt$$

$$= \frac{L}{n\pi}\left\{-\sin\frac{n\pi t}{L}\bigg|_{-L}^{0} + \sin\frac{n\pi t}{L}\bigg|_{0}^{L}\right\}$$

$$a_n = \frac{L}{n\pi}\{\sin n\pi + \sin n\pi\} = \frac{2L}{n\pi}\sin n\pi = 0$$

From Eq. (4) and (5)

$$b_n = -\int_{-L}^{0} \sin\frac{n\pi t}{L}\,dt + \int_{0}^{L} \sin\frac{n\pi t}{L}\,dt$$

$$= \frac{L}{n\pi}\left\{\cos\frac{n\pi t}{L}\bigg|_{-L}^{0} - \cos\frac{n\pi t}{L}\bigg|_{0}^{L}\right\}$$

$$= \frac{L}{n\pi}\{1 - \cos n\pi - \cos n\pi + 1\}$$

$$b_n = \frac{2L}{n\pi}\{1 - \cos n\pi\} \tag{6}$$

$$b_1 = \frac{4L}{\pi}; \quad b_2 = 0 \quad b_3 = \frac{4L}{3\pi}\ldots \tag{7}$$

Substituting $q_{i,av}$, a_n and b_n in Eq. (1)

$$q_i(t) = \frac{4L}{\pi}\sin\frac{\pi t}{L} + \frac{4L}{3\pi}\sin\frac{3\pi t}{L} + \frac{4L}{5\pi}\sin\frac{5\pi t}{L} + \ldots \tag{8}$$

Equation (8) can be written as

$$q_i(t) = \frac{4L}{\pi}\sin 2\pi f t + \frac{4L}{3\pi}\sin 6\pi f t + \frac{4L}{5\pi}\sin 10\pi f t + \ldots$$

$f = \dfrac{1}{2L} = \dfrac{1}{2 \times 0.01}$ = 50 Hz (fundamental frequency) 150, 250, 350 …Hz higher harmonics

3.29

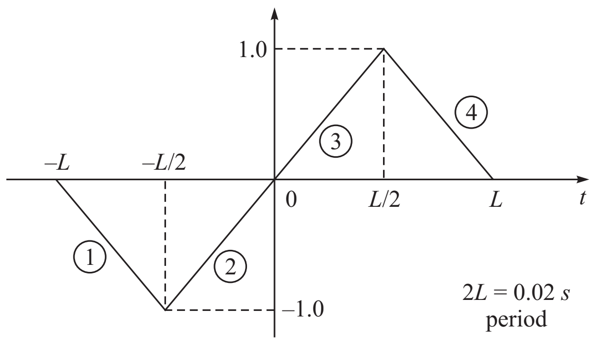


Fig. 1

One period of the signal is shown in Fig. 1 the four parts of the above signal can be written as

$$\left.\begin{aligned} q_i(t) &= -2\left(1+\frac{t}{L}\right) && -L < t < -L/2 \\ &= \frac{2t}{L} && \frac{-L}{2} < t < 0 \\ &= \frac{2t}{L} && 0 < t < L/2 \\ &2\left(1-\frac{t}{L}\right) && L/2 < t < L \end{aligned}\right\} \tag{1}$$

$$q_{i,av} = \frac{1}{2L}\left\{\int_{-L}^{-L/2} -2\left(1+\frac{t}{L}\right)dt + \int_{-L/2}^{0} \frac{2t}{L}\,dt + \int_{0}^{L/2} \frac{2t}{L}\,dt + \int_{L/2}^{L} 2\left(1-\frac{t}{L}\right)dt\right. = 0 \quad (2)$$

$$a_n = \int_{-L}^{L} q_i(t)\cos\frac{n\pi t}{L}\,dt \quad (3)$$

From Eq. (1) and (3)

$$a_n = \int_{-L}^{-L/2} 2\left(1+\frac{t}{L}\right)\cos\frac{n\pi t}{L}\,dt + \int_{-L/2}^{0}\frac{2t}{L}\cos\frac{n\pi t}{L}\,dt +$$

$$\int_{0}^{L/2}\frac{2t}{L}\cos\frac{n\pi t}{L}\,dt + \int_{L/2}^{L} 2\left(1-\frac{t}{L}\right)\cos\frac{n\pi t}{L}\,dt = 0 \quad (4)$$

$$b_n = \int_{-L}^{L} q_i(t)\sin\frac{n\pi t}{L}\,dt \quad (5)$$

From Eq. (1) and (5)

$$b_n = \int_{-L}^{-L/2} 2\left(1+\frac{t}{L}\right)\sin\frac{n\pi t}{L}\,dt + \int_{-L/2}^{0}\frac{2t}{L}\sin\frac{n\pi t}{L}\,dt$$

$$+ \int_{0}^{L/2}\frac{2t}{L}\sin\frac{n\pi t}{L}\,dt + \int_{L/2}^{L} 2\left(1-\frac{t}{L}\right)\sin\frac{n\pi t}{L}\,dt \quad (6)$$

$n = 1, 3, 5, 7$

$$b_1 = \frac{1}{50\pi^2},\ b_2 = -\frac{1}{450\pi^2}\quad b_3 = \frac{1}{1250\pi^2} \quad (7)$$

$$q_i(t) = \frac{8}{\pi^2}\sin(100\,\pi t) - \frac{8}{9\pi^2}\sin(300\,\pi t) + \frac{8}{25\pi^2}\sin(500\,\pi t)$$

$$f = \frac{1}{2L} = 50 \text{ Hz (fundamental frequency)} \quad (8)$$

$f = 150, 250, 350$ Hz, etc Harmonics

3.30

$$\frac{q_0}{q_i}(j\omega) = \frac{K}{\sqrt{\omega^2\tau^2+1}} < \tan^{-1}(-\omega\tau) \quad (1)$$

Given $K = 1,\ \tau = 0.001\text{s}\ \&\ \omega = 2\pi f$

For the problem of 3.28, the square wave has the following frequencies: 50, 150, 250, … Hz

Frequency, Hz *f*	*Amplitude ratio*	*Phase angle degrees*
50	0.954	–17°
150	0.728	–43°
250	0.537	–57°

The implication of the above table is that for 50 Hz, only 0.954 of the input amplitude at 50 Hz delayed by a phase angle of 17° will be obtained. From Prob. 3.28, q_i is given by

$$q_i(t) = \frac{4L}{\pi}\sin(2\pi \times 50\,t) + \frac{4L}{3\pi}\sin(2\pi \times 150\,t) + \frac{4L}{5\pi}\sin(2\pi \times 250\,t) \tag{2}$$

Based on the above table

$$q_0(t) = \frac{4L}{\pi}(0.954)\sin(2\pi\,50\,t - 17°) + \frac{4L}{3\pi}(0.728)\sin(2\pi\,150\,t - 43°)$$

$$+\ \frac{4L}{5\pi}(0.537)\sin(2\pi\,250\,t - 57°) \tag{3}$$

Therefore, the output signal of Eq. (3) is not a true representation of the input signal $q_i(t)$ of Eq. (2).

3.31 Since the input signal of Prob. 3.29 has the same frequencies of 50, 150, 250 … Hz of Prob. 3.30, the same amplitude and phase information of the first order system in Prob. 3.30 can be used here also.

From Prob. 3.29, the input signal is given by

$$q_i(t) = \frac{8}{\pi^2}\sin(2\pi\,50\,t) - \frac{8}{9\pi^2}\sin(2\pi\,150\,t) + \frac{8}{25\pi^2}\sin(2\pi\,250\,t) \tag{1}$$

Using the amplitude and phase information of the first-order system in Prob. 3.30, the output signal $q_0(t)$ for the input signal q_i of Eq. (1) is given by

$$q_0(t) = \frac{8}{\pi^2}(0.954)\sin(2\pi\,50\,t - 17°) - \frac{8}{9\pi^2}(0.728)\sin(2\pi\,150\,t - 43°)$$

$$+\ \frac{8}{25\pi^2}(0.537)\sin(2\pi\,250\,t - 57°) \tag{2}$$

The output signal is therefore not a true representation of the input signal.

3.32 The carrier wave is given by

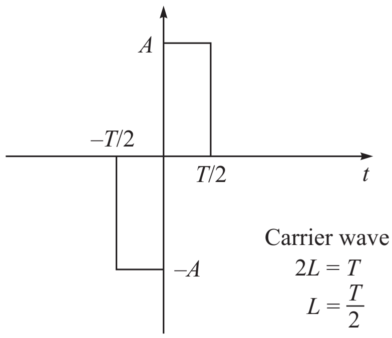


Fig. 1

From Prob. 3.28, the carrier signal is given by

$$f_c(t) = \frac{4A}{\pi}\frac{T}{2} \sin 2\pi\, 50\, t + \frac{4AT}{6\pi} \sin 2\pi\, 150\, t$$

$$+ \frac{4AT}{10\pi} \sin 2\pi\, 250\, t + \ldots \tag{1}$$

The signal which is being modulated is given by

$$f_s(t) = A_s \sin \omega_s t \tag{2}$$

$$\text{output} = f_c(t)\, f_s(t) \tag{3}$$

using Eq. 3.177 (page 123) and Eqs. (1), (2) and (3)

$$\text{output} = \frac{AA_sT}{\pi} \sin ((2\pi\, 50 - \omega_s)t + 90°)$$

$$+ \frac{AA_sT}{\pi} \sin ((2\pi\, 50 + \omega_s)t - 90°)$$

$$+ \frac{AA_sT}{3\pi} \sin ((2\pi\, 150 - \omega_s)t + 90°)$$

$$+ \frac{AA_sT}{3\pi} \sin ((2\pi\, 150 + \omega_s)t - 90°)$$

$$+ \frac{AA_sT}{5\pi} \sin ((2\pi\, 250 - \omega_s)t + 90°)$$

$$+ \frac{AA_sT}{5\pi} \sin ((2\pi\, 250 + \omega_s)t - 90°) + \ldots$$

3.33

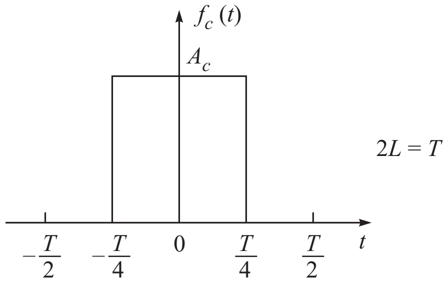


Fig. 1

$$f_{c,av} = \frac{1}{2L}\int_{-L}^{L} f_c(t)\,dt$$

$$= \frac{1}{T}\int_{\frac{-T}{2}}^{-T4} 0.dt + \int_{\frac{-T}{4}}^{T/4} A_c\,dt + \int_{T/4}^{T/2} 0\,dt$$

$$= \frac{1}{T}\left\{A_c\left(\frac{T}{4}+\frac{T}{4}\right)\right\} = \frac{A_c}{2} \qquad (1)$$

$$a_n = \int_{-T/4}^{T/4} A_c \cos\frac{n\pi t}{L}\,dt = \int_{\frac{-L}{2}}^{L/2} A_c \cos\frac{n\pi t}{L}\,dt$$

$$a_n = A_c \frac{L}{n\pi}\sin\frac{n\pi t}{L}\Big|_{-L/2}^{L/2} = \frac{A_c L}{n\pi}\left\{\sin\frac{n\pi}{2} - \sin\frac{-n\pi}{2}\right\}$$

$$a_n = \frac{2A_c L}{n\pi}\sin\frac{n\pi}{2} \qquad (2)$$

$$a_1 = \frac{2A_c L}{\pi};\ a_2 = 0;\ a_3 = -\frac{2A_c L}{3\pi}\quad a_4 = 0$$

$$b_n = \int_{-L}^{L} f_c(t)\sin\frac{\pi t}{L}\,dt$$

$$b_n = \int_{-T/4}^{T/4} A_c \sin\frac{n\pi t}{L}\,dt = \int_{-L/2}^{L/2} A_c \sin\frac{n\pi t}{L}\,dt$$

$$b_n = -\frac{A_c L}{n\pi}\cos\frac{n\pi t}{L}\Bigg|_{-L/2}^{L/2} = -\frac{A_c L}{n\pi}\left(\cos\frac{n\pi}{2} - \cos-\frac{n\pi}{2}\right) = 0 \tag{3}$$

$$f_c = \frac{A_c}{2} + \frac{1}{L}\sum_{n=1}^{\infty} a_n \cos\frac{n\pi t}{L}$$

$$\frac{A_c}{2} + \frac{1}{L}\sum_{n=1}^{\infty}\frac{2A_c L}{n\pi}\sin\frac{\pi n}{2}\cos\frac{n\pi t}{L} \tag{4}$$

The signal which is being modulated is given by

$$f_s(t) = A_s \sin \omega_s t \tag{5}$$

$$\text{output} = f_c(t)\, f_s(t)$$

$$= \left(\frac{A_c}{2} + \frac{1}{L}\sum_{n=1}^{\infty}\frac{2A_c L}{n\pi}\sin\frac{n\pi}{2}\cos\frac{n\pi t}{L}\right) A_s \sin \omega_s t$$

$$= \sin \omega_s t\,\frac{A_c A_s}{2} + \frac{2A_c A_s}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi}{2}\cos\frac{n\pi t}{L}\sin \omega_s t$$

$$\text{output} = \frac{A_c A_s}{2}\sin \omega_s t + \frac{2A_c A_s}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi}{2}\left\{\sin\left(\omega_s + \frac{n\pi}{L}\right)t + \sin\left(\omega_s - \frac{n\pi}{L}\right)t\right\}$$

3.34

$$\text{Sinusoidal transfer function} = \frac{Q_0(j\omega)}{Q_i(j\omega)} \tag{1}$$

Where $Q_0\,(j\omega)$ is the frequency-domain function of the output and $Q_i\,(j\omega)$ is the frequency-domain function of the input. $q_0(t)$ is the time record of the output and $q_i(t)$ is that of the input. Irrespective of the shape of these time records, it is possible to obtain the corresponding frequency-domain functions that can be used to get the system transfer function defined by Eq. (1)

Measuring instruments known as FFT (Fast-Fourier Transform) analyzers, which are based on an extended theory of Fourier series can easily compute the frequency domain function of any time record. Many FFT analyzers have more than one input channel, using which transfer function between the input-out of any system can be directly obtained.

3.35

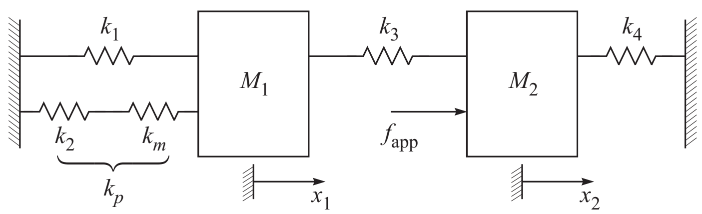


Fig. 1

$$k_p = \frac{k_2\ k_m}{k_2 + k_m}\ (1) \qquad f_m = k_p\ x_1\ (2)$$

Equation of motion

$$\begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_1 + k_p & -k_3 \\ -k_3 & k_3 + k_4 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ f_{\text{app}} \end{Bmatrix} \tag{3}$$

Taking Laplace transform of Eq. (3) and assuming zero initial conditions

$$(k_1 + k_p + s^2 M_1)\ X_1(s) - k_3\ X_2(s) = 0 \tag{4}$$

$$-k_3\ X_1(s) + (k_3 + k_4 + s^2 M_2)\ X_2(s) = F_{\text{app}}(s) \tag{5}$$

From Eq. (4)

$$X_2(s) = \left\{\frac{k_r + k_p + s^2\ M_1}{k_3}\right\} X_1(s) \tag{6}$$

Substituting Eq. (6) in Eq. (5)

$$\left\{-k_3 + \frac{(k_3 + k_4 + s^2\ M_2)\ (k_1 + k_p + s^2\ M_1)}{k_3}\right\} X_1(s) = F_{\text{app}}(s)$$

$$X_1(s) = \frac{k_3\ F_{\text{app}}\ (s)}{\begin{array}{l} s^4\ M_1\ M_2 + s^2 \left[(k_1 + k_p)\ M_2 + (k_3 + k_4)\ M_1\right] \\ +k_1\ k_3 + k_1\ k_4 + k_p\ k_3 + k_p\ k_4 - k_3^2 \end{array}} \tag{7}$$

Taking Laplace transform of Eq. (2)

$$F_m(s) = k_p\ X_1(s)$$

$$F_m(s) = \frac{k_p\ k_3\ F_{\text{app}}\ (s)}{\begin{array}{l} s^4\ M_1\ M_2 + s^2 \left[(k_1 + k_p)\ M_2 + (k_3 + k_4)\ M_1\right] \\ +k_1\ k_3 + k_1\ k_4 + k_p\ k_3 + k_p\ k_4 - k_3^2 \end{array}} \tag{8}$$

$$\frac{F_m\ (s)}{F_{\text{app}}\ (s)} = \frac{k_p\ k_3}{\begin{array}{l} M_1\ M_2\ s^4 + \left[(k_1 + k_p)\ M_2 + (k_3 + k_4)\ M_1\right] s^2 \\ +k_1\ k_3 + k_1\ k_4 + k_p\ k_3 + k_p\ k_4 - k_3^2 \end{array}} \tag{9}$$

Let $k_1 = k_2 = k_3 = k_4 = 1$ N/cm $\qquad k_m = 100$ N/cm

$M_1 = M_2 = 0.01$ kg

A bode diagram of the function of Eq. (9) is shown on next page as Fig. 2. It is clear that, upto 9 rad/s, the measured and applied loads are almost same. Beyond 9 rad/s, they are not acceptable.

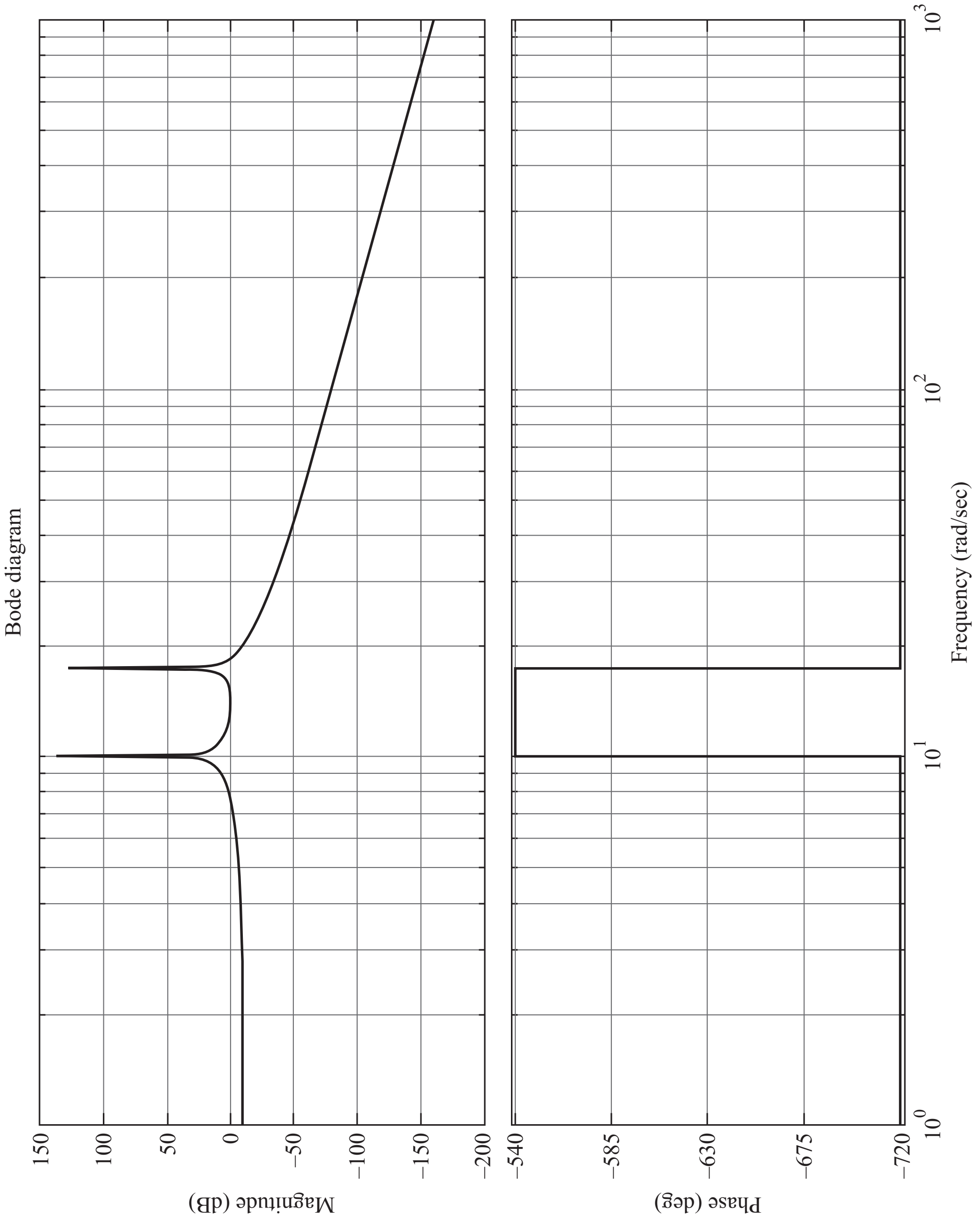

Bode diagram
Magnitude (dB)
150
100
50
0
−50
−100
−150
−200
Phase (deg)
−540
−585
−630
−675
−720
10^0
10^1
10^2
10^3
Frequency (rad/sec)

Fig. 2

3.36 For an oscilloscope with resistance R_m shunted with a capacitor c_m

$$z_m = \frac{R_m}{R_m c_m D + 1} \tag{1}$$

Equation (3.41) can be modified as

$$\frac{E_m}{E_0} = \frac{z_m}{R_{ab} + Z_m} \tag{2}$$

Substituting Eq. (1) in (2)

$$\frac{E_m}{E_0} = \frac{\dfrac{R_m}{R_m c_m D + 1}}{\dfrac{R_m}{R_m c_m D + 1} + R_{ab}} = \frac{R_m}{R_m + R_{ab} + R_{ab} R_m C_m D} \tag{3}$$

By factoring out (R_m + R_{ab}) in Eq. (3)

$$\frac{E_m}{E_0} = \frac{R_m / R_m + R_{ab}}{1 + \dfrac{R_{ab} R_m c_m D}{R_m + R_{ab}}} \tag{4}$$

3.37

$$\frac{I_m}{I_u} = \frac{R_{ab}}{R_{ab} + Z_m} \tag{1}$$

$$z_m = R_m + L_m D \tag{2}$$

From Eq. (1) and (2)

$$\frac{I_m}{I_u} = \frac{R_{ab}}{R_{ab} + R_m + L_m D} \tag{3}$$

By factoring out R_{ab} + R_m in Eq. (3)

$$\frac{I_m}{I_u} = \frac{\dfrac{R_{ab}}{R_{ab} + R_m}}{1 + \dfrac{L_m}{R_{ab} + R_m} D} \tag{4}$$