Technical Chapter Five
Linear Programming with Solver

Linear programming (LP) is a general technique for solving many different types of problems. It is useful for finding "best" outcomes, such as highest profit or lowest cost, for problems based on linear relationships and constrained resources. This technical chapter begins with a formulation of the product-mix problem and then provides graphical and Excel Solver methods for solution of linear programming problems. After describing sensitivity analysis, the transportation problem and mixed integer programming are described as extensions of the use of Solver.

The general setup for LP consists of three elements. First, the problem setup has an **objective function**. This is a mathematical expression of what the user wants maximized or minimized. For example, we may want to maximize sales or minimize costs for a particular product mix. Second, an LP model includes multiple **decision variables.** LP determines what values of the decision variables will maximize or minimize the objective function. These decision variables could be, for example, the amount of each of several products to produce. Third, **constraints** are limitations on the resources needed to accomplish production or use of the decision variables. Constraints could be, for example, the amount of machine time available, the amount of materials available or labor constrains. Constraints can be greater than or equal ($\geq$), less than or equal ($\leq$), or equal ($=$). You will see the use of these three elements below in the product-mix problem.

PRODUCT-MIX PROBLEM

For the sake of simplicity, suppose a furniture company can make only two types of products: tables and chairs. The amounts produced of tables and chairs are the decision variables. The company has limited resources of wood (lumber), labor, and finishing capacity with which to produce these products. The management of the company would like to determine the best mix of products to make: all chairs, all tables, or some mix of chairs and tables. Management defines the best mix of products as the one which maximizes the total contribution to profit, subject to the limited availability of resources already mentioned.

The product-mix problem can be formulated in mathematical terms as follows:

Let: X_1 = the number of tables produced
X_2 = the number of chairs produced

Both X_1 and X_2 are unknown **decision variables** to be determined by the solution of the linear programming problem. Also assume that the following unit production technology matrix is given. This matrix describes the transformation process used to convert the scarce resources (lumber, labor, and finishing capacity) into tables or chairs. For example, it takes 30 board feet to make one table and 20 board feet to make one chair. Likewise, it takes 2 labor hours to make one table and also 2 hours to make one chair. Finally, the resource of finishing capacity is used by 4 hours each for tables and 6 hours each for chairs.

Resources used per unit produced	Table	Chair
Lumber, board feet	30	20
Labor, hours	2	2
Finishing capacity, hours	4	6

We also assume that the amount of resources available is given: 120 board feet of lumber, 9 hours of labor, and 24 hours of finishing capacity. If we multiply the values in the unit production technology matrix by the number of units produced and add over both products, we will obtain the total resources of each type required. For example, $30X_1 + 20X_2$ is the total amount of lumber in board feet needed to make X_1 tables and X_2 chairs. The total must be less than the 120 board feet available. A similar logic can be used to derive the second and third constraints related to labor and finishing capacity needed.

Thus the **constraints** we have are:

$$30X_1 + 20X_2 \le 120$$
$$2X_1 + 2X_2 \le 9$$
$$4X_1 + 6X_2 \le 24$$

Management wishes to find the values of X_1 and X_2 which will maximize the contribution to profit. To formulate this **objective function**, we assume that each table contributes \$10 and each chair \$8 to profit. Then the total contribution for X_1 tables and X_2 chairs is

$$10X_1 + 8X_2$$

Finally, we require that X_1 and X_2 be nonnegative values, since we cannot produce a negative number of tables or chairs. Gathering together the constraints and objective function that we have specified, the product-mix problem can be summarized as follows:

Objective: Max $10X_1 + 8X_2$

Subject to:

$$\begin{aligned} 30X_1 + 20X_2 &\le 120 \\ 2X_1 + 2X_2 &\le 9 \\ 4X_1 + 6X_2 &\le 24 \\ X_1 \ge 0,\quad X_2 &\ge 0 \end{aligned}$$

When this problem is solved by the methods described below, optimal values of X_1 and X_2 will be found. These optimal values, however, are a solution to the mathematical problem as stated and not necessarily a solution to the manager's original decision problem. The mathematical problem is always an abstraction of the manager's real problem; therefore the optimal solution must be carefully evaluated before a decision can be made.

In practice, a series of product-mix problems may be formulated and solved before the

manager is satisfied. For example, suppose the optimal solution to the problem is to produce all tables and no chairs. This solution may conflict with the marketing strategy in the company. Therefore, either the marketing strategy must be modified or the production constraints altered or other changes made in the formulation to make the mathematical solution consistent with the real decision problem. Even though the optimal solution to the product-mix problem is not implemented, it may provide important insights into possible coordination problems between marketing and production, possibly leading to further analysis and ultimate solution of the decision problem.

GENERAL LP PROBLEM

The problem that we have just formulated is a member of a class of general problems called linear programming, or LP, problems - the notation for which follows:

Objective: $\max C_1X_1 + C_2X_2 + \ldots + C_nX_n$

Subject to:

$$
\begin{aligned}
a_{11}X_1 + a_{12}X_2 + \ldots + a_{1n}X_n &\le b_1 \\
a_{21}X_1 + a_{22}X_2 + \ldots + a_{2n}X_n &\le b_2 \\
\vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \quad &\quad \vdots \\
a_{m1}X_1 + a_{m2}X_2 + \ldots + a_{mn}X_n &\le b_m
\end{aligned}
$$

$$X_1 \ge 0, \quad X_2 \ge 0, \ldots, \quad X_n \ge 0$$

In this formulation the C_j, a_{ij}, and b_i values are given constants and the X_j are called decision variables. The problem is to find the values of the n decision variables ($X_1, X_2, \ldots, X_n$) which maximize the objective function subject to the m constraints and the non-negativity conditions on the X_j variables. The resulting set of decision variables which maximizes the objective function is called an optimal solution. In this problem "max" may be replaced by "min" and $\le$ may be replaced by $\ge$ or = signs to obtain the general set of LP problems.

The general LP problem as formulated above is based on the following four assumptions:

1. ***Linearity.*** Both the objective function and the constraints must be linear functions of the X_j. This implies that no cross products, powers of X_j, or other nonlinearities are permitted in the problem. It also implies that resource utilization is proportional and additive, e.g., if it takes 2 hours of labor to produce one table, it will take 4 hours to produce two tables.

2. ***Divisibility.*** The X_j variables are permitted to take on continuous values. Thus we can obtain a solution to the product-mix problem, for example, of 20.5 chairs and 30.2 tables. Sometimes the continuous solution can be rounded off or interpreted as the average production per day. In other cases, special integer programming problems must be formulated to provide integer solutions.

3. ***Nonnegativity.*** The decision variables must have nonnegative values. In many problems this assumption is natural and presents no difficulties. If negative values

are needed, however, the LP formulation can be modified to handle these cases.

4. ***Limited Resources.*** There must be limited resources such as limited workers, finances, machines or materials.

Although LP problems have some rather restrictive assumptions, a large class of real decision problems can still be solved by LP methods. In addition, LP methods sometimes form the nucleus for more advanced nonlinear, stochastic, or integer programming methods. We treat integer programming problems later in the chapter.

LP can be used to solve a general class of resource allocation problems including product-mix, transportation, blending, and scheduling problems. All these decision problems are concerned with finding the best allocation of scarce resources. In the product- mix problem, the scarce resources are allocated to the products; in the blending problem, the best mix of resource inputs to meet a prescribed product blend is determined; and in scheduling problems, available machine times or staff times are allocated to meet demand. For a realistic staff scheduling problem click on the link below[1]. LP may be applied to shipping decisions, to minimize shipping costs. From finance to farming, and from marketing to mixing chemicals or food ingredients, decisions in a wide variety of settings can be addressed using LP. For example, the company Groupon used LP to design the layout of its data centers to maximize both space utilization and power utilization[2]. Airlines use LP extensively to schedule aircraft, pilots and flight attendants to routes. There are many constraints such as pilots are qualified to fly only certain aircraft, crews are restricted to a certain number of hours per month, and only certain aircraft types can be used on particular routes. All these constraints are considered while the LP seeks to minimize the cost of operations. In all of the above cases, scarce resource inputs are allocated among several economic activities. This is a general characteristic of most, if not all, LP problems.

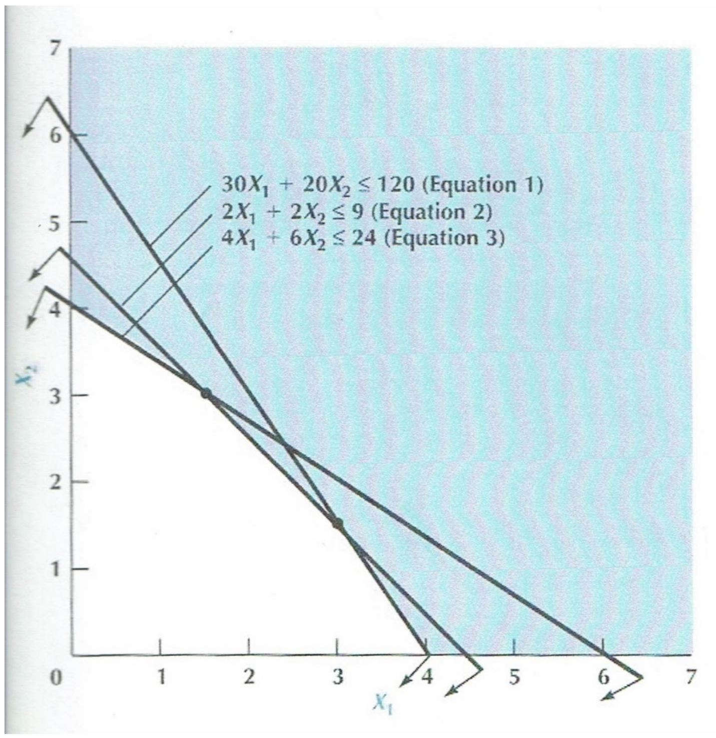
GRAPHICAL SOLUTION METHOD

The graphical solution method may be used to solve LP problems with two unknown variables. Although two variables are not enough to describe real problems, the graphical method provides important insights into LP solution procedures. For illustration purposes, we will solve the product-mix problem formulated above.

The first step in the graphical procedure is to plot the constraint equations. The three constraints from the product-mix example are shown in Figure T5-1. Each constraint is plotted on the figure by first considering the $\leq$ sign as an = sign. The first constraint is thus plotted as the equation $30X_1 + 20X_2 = 120$. By setting $X_1 = 0$ in this equation, we have $X_2 = 6$; and by setting $X_2 = 0$, we have $X_1 = 4$. These two coordinates are placed on the graph and connected with a straight line. The original constraint equation, however, includes all points smaller than or equal to the right-hand side. Thus all points to the left of the lines in Figure T5-1 are permitted by the constraints.

[1] https://youtu.be/OjUmQRRfsEY
[2] https://www.datacenterknowledge.com/archives/2016/01/13/linear-programming-helps-groupon-optimize-data-center-design

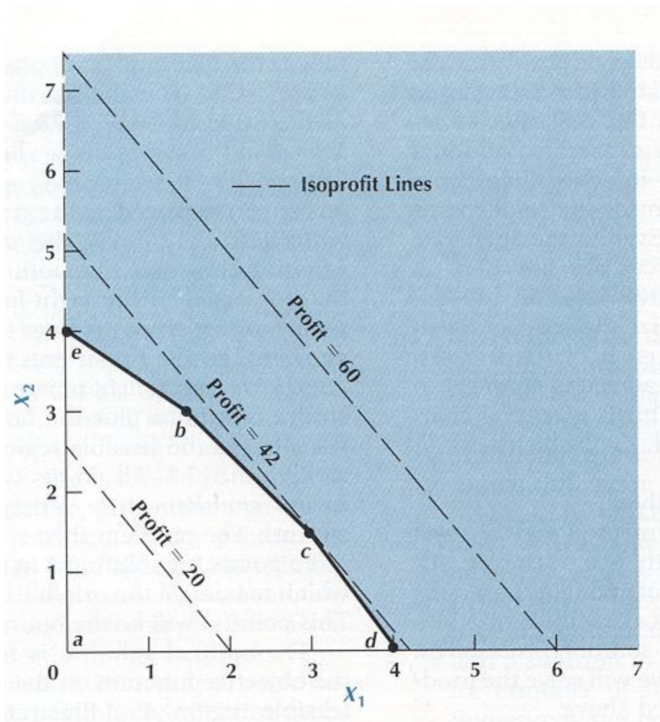
FIGURE T5-1
Product- mix
problem



Likewise, the other two constraints are plotted on the graph in a similar manner. In the second constraint ($X_1 = 0$, $X_2 = 4.5$ and $X_1 = 4.5$, $X_2 = 0$). The third constraint has ($X_1 = 0$, $X_2 = 4$ and $X_1 = 6$, $X_2 = 0$). The intersection of points from all three constraint equations plus the non-negativity conditions forms the feasible region shown in white in Figure T5-1. All points within this feasible region *simultaneously* satisfy all the LP constraints. The problem then is to find the one or more points (or solutions) in the feasible region which maximizes the original objective function. This point(s) will be the optimal solution(s).

The optimal solution is found by graphing the objective function on the same graph as the feasible region. For illustration purposes, we have drawn the feasible region again in Figure T5-2 and plotted a series of objective functions or isoprofit lines on the graph. Each isoprofit line is obtained by setting the objective function equal to an arbitrary value. For example, suppose we arbitrarily set the objective function equal to 60. Then the line $10X_1 + 8X_2 = 60$ is plotted just as we plotted the constraints by setting $X_1 = 0$ and then $X_2 = 0$. Since the value of the objective function is unknown, a series of arbitrary values is selected to generate a series of parallel isoprofit lines, as shown in Figure T5-2.

FIGURE T5-2
Product-mix



The problem now is to find the isoprofit line which has the largest profit and is still in the feasible region. This can be done by starting at the origin or any other point inside the feasible region and moving the isoprofit line out away from the origin parallel to itself until the last point in the feasible region is reached. In this example the last feasible point is the corner labeled c ($X_1 = 3$, $X_2 = 1.5$), which maximizes the value of the isoprofit line and is therefore the optimal solution to the LP problem with a profit of 42 (10*3+8*1.5 = 42).

The exact coordinates of the optimal solutions may be found by solving for the intersection of equations 1 and 2, which define the optimal corner c. These two equations in two unknowns are solved simultaneously for the values of X_1 and X_2. As a result, we find $X_1 = 3$ and $X_2 = 1.5$. Although X_2 is a fractional number of chairs, this solution might still be perfectly realistic. For example, suppose we produce 3 tables and 1.5 chairs a day over a period of several weeks. The 1.5 chairs can then be interpreted as an average production rate per day.

The optimal solution will always occur on at least one corner point (or extreme point) of the feasible region. In the example, there was only one optimal solution because a single corner point was reached. If the isoprofit line had been parallel to a side of the feasible region, then two corner points and all the points in between would have been optimal solutions. In this case, we would have alternative optimal solutions, different values of X_1 and X_2 which yield the same value of the objective function.

The fact that the optimal solution occurs at one or more extreme points is exploited in LP methods. The simplex method used in Excel Solver is an adjacent-extreme-point method. It

moves from one corner point to the next until an optimal solution is found. Since there are only a finite number of corner points but an infinite number of feasible solutions inside the feasible region, the general solution to linear programming problems is greatly simplified when only adjacent extreme points are examined. For example, there are only four extreme points in the feasible region in this example (not including the origin).

SOLVER SOLUTION FOR L.P.

LP problems can be quite easily solved for optimal solutions by using Excel with the add-in Solver. Solver searches adjacent extreme points to find the optimal solution. This is done sequentially in the following way. It is typical to start Solver with all of the decision variables set to zero. In the graphical example from above, Solver starts with $X_1 = 0$ and $X_2 = 0$, at the origin of Figure T5-2. There are only two *adjacent* extreme points from the origin, one along the X_1 axis until $X_1 = 4$, $X_2 = 0$ to still have a feasible solution, or alternatively moving along the X_2 axis to $X_2 = 4$, $X_1 = 0$ and still be feasible. Solver will move to $X_1 = 4$, $X_2 = 0$ since profit increases the most per unit ($10 per unit vs 8 per unit) by moving to $X_1 = 4$ along the X_1 axis. Solver always moves in the direction of the greatest unit increase in profit, until the maximum is reached.

Next, Solver moves from its current position, $X_1 = 4$, $X_2 = 0$, to the next adjacent extreme point of $X_1 = 3$, $X_2 = 1.5$. Notice, profit is increased by moving to this point (c on the graph), from 40 to 42. At this point, the profit will decrease by moving in either direction since the optimum solution has been achieved. In summary, Solver moves algebraically (not graphically) from one extreme point to another adjacent extreme point by increasing profit at each point until it reaches the maximum. Therefore, many more than two unknown variables can be used.

Solver will now be demonstrated on the product mix problem that we solved graphically.

Installation of Solver Add-in

The first task is to be sure that the Solver add-in is installed on your version of Excel[3]. The method of installation is as follows:

1. Click on the <File> tab on the top left side of Excel.
2. Click on <Options> at the bottom of the list
3. Click on <Add-ins> on the left-hand side menu
4. Find the Manage: <Excel Add–ins> box at the bottom of the page and then click on <Go>
5. Click <Solver Add-in> and then click <OK>

Your device should now install the Solver add-in.[4]

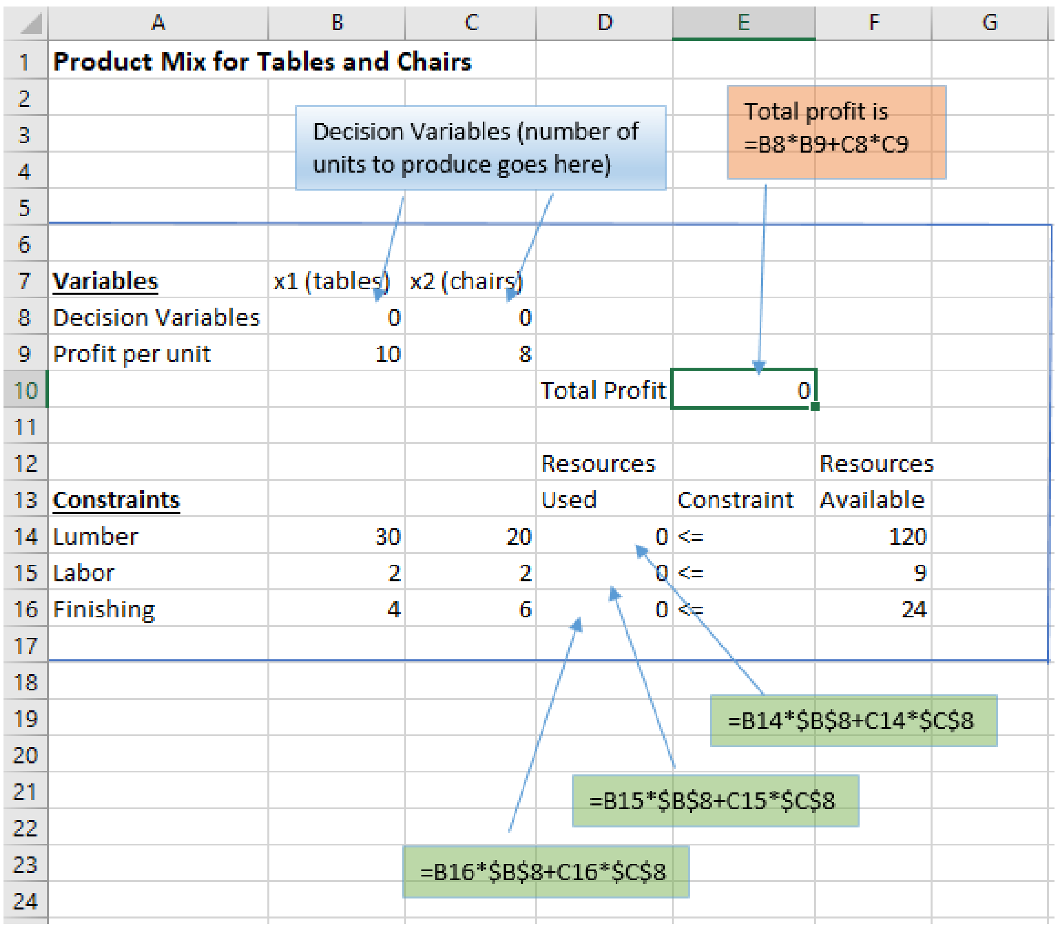
Once the installation of Solver is completed, click on <Data> on the top menu and then find <Solver> on the far right side. You will click on Solver once you have the problem set up in the proper form on Excel.

[3] We are using the Microsoft Office Professional Plus 2016 version in this chapter.
[4] If you are having trouble installing Solver go to the YouTube video https://youtu.be/HdYZkEuNtSc

Setting up the LP problem in Excel

We start with setting up the Product mix problem for tables and chairs that we have already solved graphically. The following discussion refers to Figure T5-3.

Figure T5-3
Initial Excel spreadsheet for Product mix problem



	A	B	C	D	E	F	G
1	**Product Mix for Tables and Chairs**						
2							
3							
4							
5							
6							
7	**Variables**	x1 (tables)	x2 (chairs)				
8	Decision Variables	0	0				
9	Profit per unit	10	8				
10				Total Profit	0		
11							
12				Resources		Resources	
13	**Constraints**			Used	Constraint	Available	
14	Lumber	30	20	0	<=	120	
15	Labor	2	2	0	<=	9	
16	Finishing	4	6	0	<=	24	
17							
18							
19							
20							
21							
22							
23							
24							

Variables
To construct this spreadsheet, first make all the labels shown in column A for Variables and Constraints. Next, enter the labels x1 in column B and x2 in column C. These columns refer to the decision variables x1 and x2 for the entire column.

Now, enter the value of zero (0) for both the Decision Variables in cells B8 and C8. Referring to the graphical example, we are starting at the origin where both decision variables are zero. Solver will put the optimal values of the variables x1 and x2 in these cells after it is run.

In cell B9, enter 10 and in C9, enter 8 as the unit profit coefficients of x1 and x2. These values

are constants and will not change in the Solver solution. Enter the label Total Profit in cell D10. Cell E10 will contain the maximum profit of the objective function found by Solver, but it is initially set to zero.

<u>Constraints</u>

The constraints are taken directly from the unit production technology matrix already shown.

Resources used per unit produced

	Table	**Chair**
Lumber, board feet	30	20
Labor, hours	2	2
Finishing capacity, hours	4	6

The values from this matrix are entered directly into rows 14, 15 and 16 of the spreadsheet in columns B and C, under the x1 and x2 columns. These are sometimes called detached coefficients, because they will be multiplied by x1 and x2 in formulas described below to develop the constraints.

Next, the resources used by Solver are shown in D14, D15 and D16. The values for these cells will be calculated by Solver and are initially set to zero. Since x1 =0 and x2 =0, no resources are initially used.

The direction of the constraints are shown in E14, E15 and E16. They are all <= constraints in this example.

Next, the resources available are entered in F14, F15 and F16 as the maximum resources available to be used, 120 board feet, 9 hours of labor and 24 hours of finishing capacity.

<u>Formulas</u>

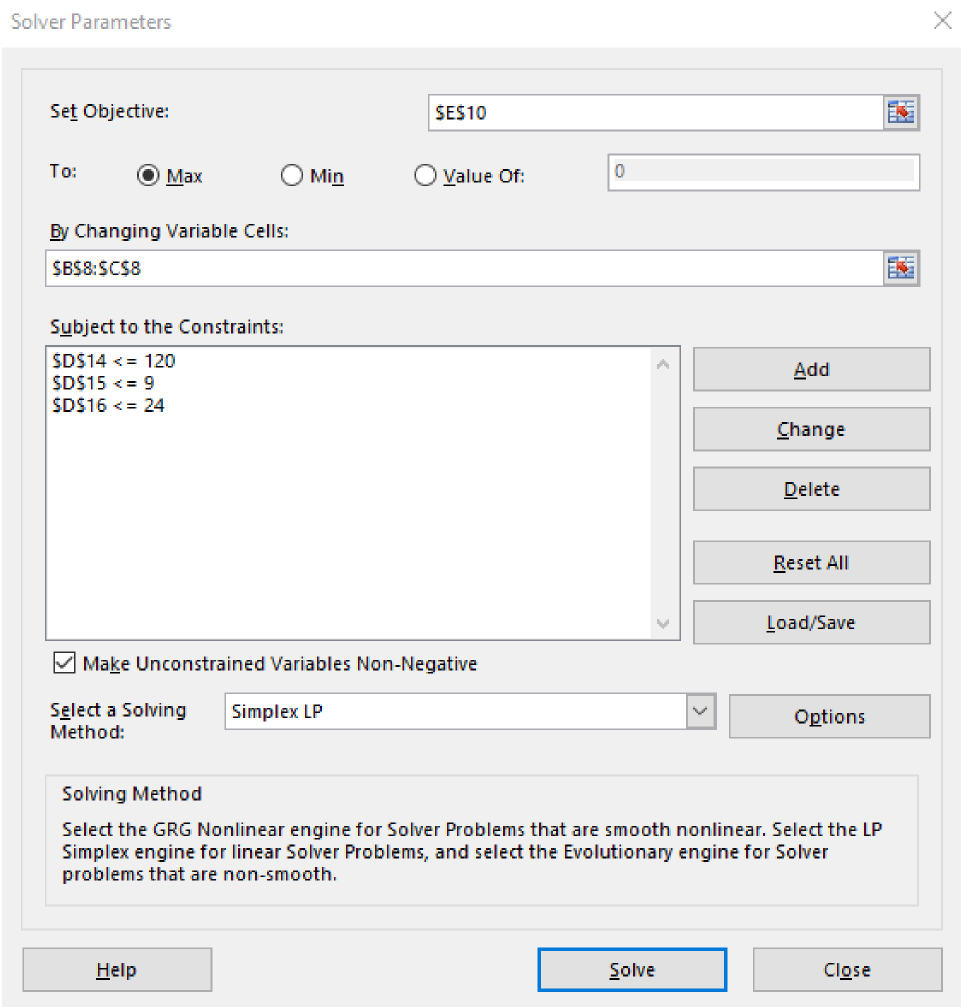
Formulas are entered next. The formula for total profit in cell E10 is Total Profit =B8*B9+C8*C9. We are just multiplying the unit profit for x1 and x2 by the number of units of x1 and x2 and summing to arrive at the total profit.

Formulas for resources used by the constraints in cells D14, D15 and D16 are also shown in Figure T5-3. For example, the resources used for lumber in D14 are simply the board feet per unit of x1 and x2 multiplied by the number of units of x1 and x2 produced (=B14*$B$8+C14*$C$8). The detached coefficients are multiplied by the decision variables to arrive at the formula for each constraint. The formula for cell D14 is entered first and then copied to D15 and D16. The cells $B$8 and $C$8 are absolute references since they represent the variables x1 and x2 and will not change when copied.

Using Solver

Now that the initial spreadsheet has been established, we are ready to use Solver. Click on the Data tab on the top of Excel and then click on Solver on the far right side. The Solver pop-up menu is shown in Figure T5-4.

Figure T5-4
Solver menu for Product mix problem.

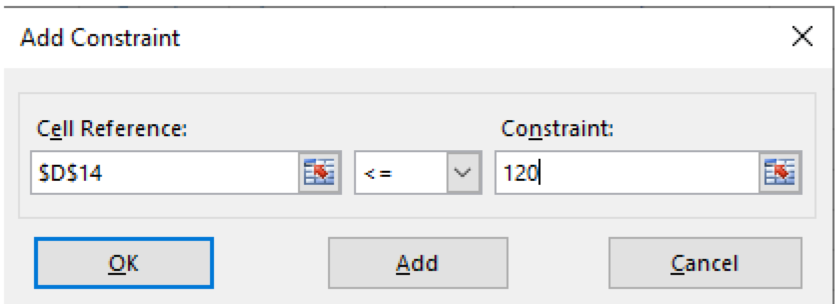


At the top of the menu <Set Objective> cell to read $E$10, if it is not already entered, since this is the total profit to maximize. Also be sure the <Max> button is set for this problem. Next, input to <By Changing Variables Cells> $B$8:$C$8 to indicate that the decision variables are in cells B8 and C8. These are the variables that are changed by Solver to arrive at the optimal solution.

Next, the <Subject to the Constraints> box, when completed, will read as shown to indicate we

are using the formulas entered in D14, D15 and D16 to be <= to the resources available. These constraints are entered one at a time by clicking on the <Add> button on the side of the menu. For example the lumber constraint is entered as follows in Figure T5-5. When finished with the entry, click OK and then add the next constraint.

Figure T5-5
Adding Constraints one at a time

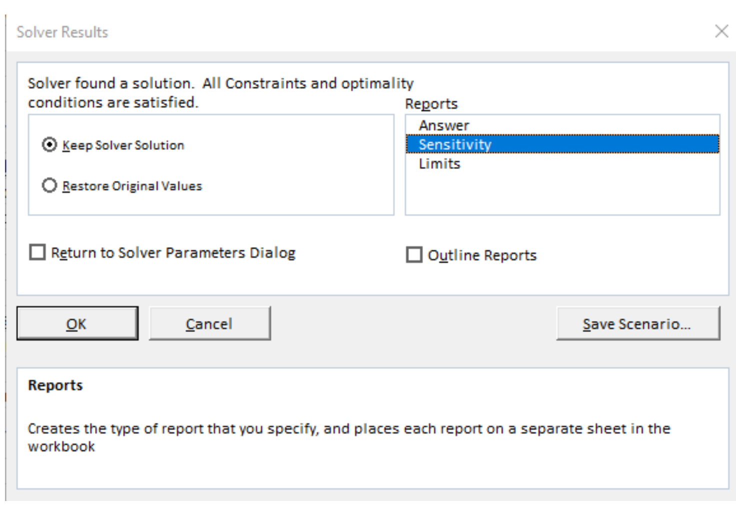


Check to see that the <Make Unconstrained Variables Non-Negative> box is checked since the solution must consist of nonnegative numbers of tables and chairs.

Also check to see that the <Select a Solving Method> box reads <Simplex LP>. You can then press the <Solve> button at the bottom of the menu page to get the Solver Results Screen as shown in Figure T5-6. When first learning LP, instead of immediately pressing <Solve>, students find it helpful to save the formulation they have at this point. If an error is encountered by Solver it can freeze (or crash) your computer and you will have to start over from the beginning by formulating the problem over again.

If the formulation is correct, and no errors are found, the final screen will read that a solution has been found. We want to get the Sensitivity report, so check the box on the right to highlight that option and press <OK>.

Figure T5-6
Solver Results Screen

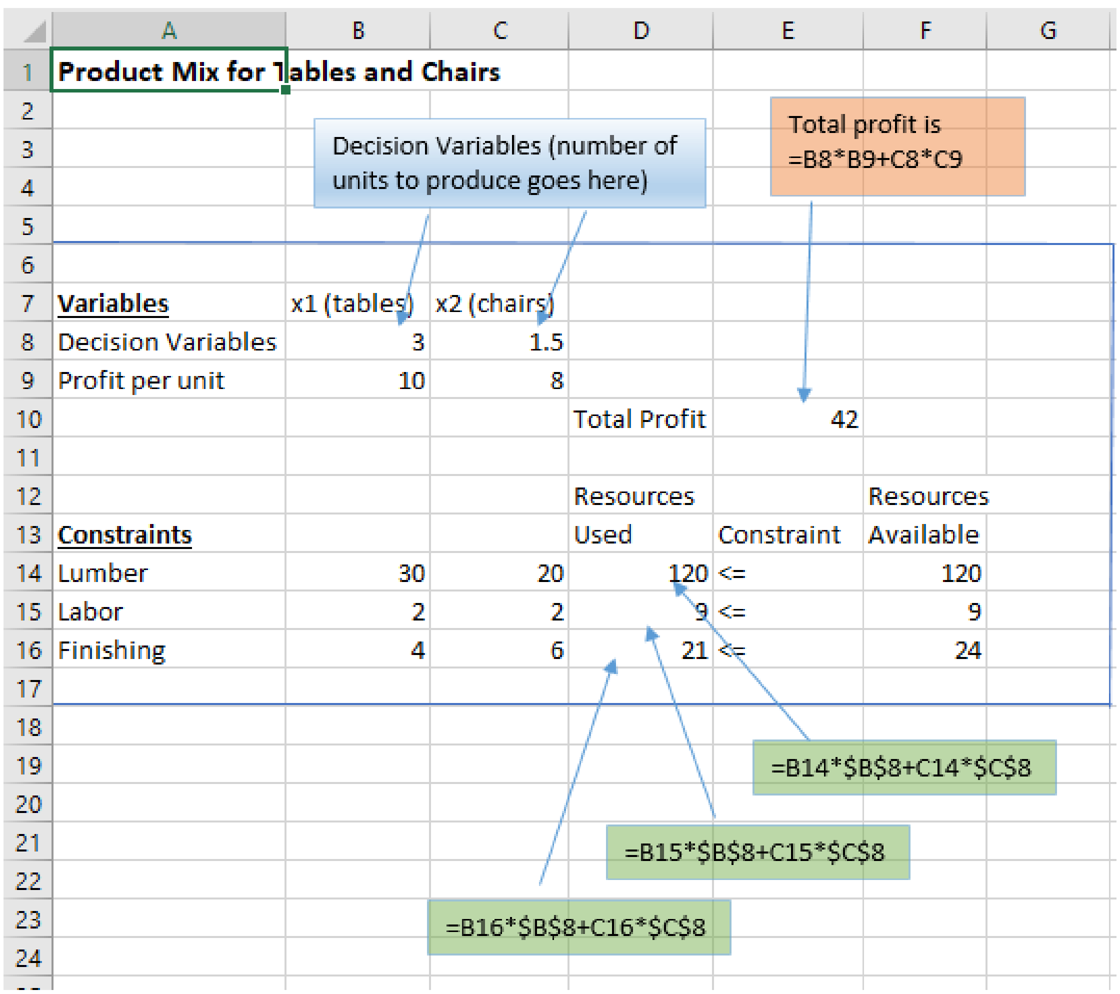


The resulting optimal solution is shown on the spreadsheet in Figure T5-7. The decision variables in cells B8 and C8 have now been changed to the values of x1 = 3 and x2 = 1.5, which occur when the profit is maximized at $42 in cell E10.

All of the resources available for Lumber and Labor have been used, 120 and 9 respectively. These are called binding constraints. The Finishing constraint uses only 21 of 24 hours available and thus is non-binding in the optimal solution.

At the bottom of the spreadsheet notice a tab labeled <Sensitivity Report 1>. Click on this tab to get the results of the sensitivity analysis that Solver has completed.

Figure T5-7
Optimal Solution for the Product Mix Problem



Note, if you are having problems, for more information on Solver review the YouTube video, Linear programming with Excel Solver https://youtu.be/RicajFzoenk Product Mix with 2 products and 2 resource constraints.

Sensitivity Analysis

The sensitivity report is shown in Figure T5-8. The <Variable Cells> section shows the range of the unit profit coefficients that x1 and x2 can take, one at a time, without changing the optimal solution. Since the reduced cost of these variables is both zero in the optimal solution, the following interpretation can be made. In the original formulation, the coefficient of x1 was 10. It can be increased by 2 to 12 or decreased by 2 to 8 and still have the optimal solution of x1 = 3. Likewise, the unit profit coefficient of x2 was 8 and it can be increased to 10 and decreased to 6.7777 (a decrease of 1.3333) without changing the optimal solution of x2 = 1.5. The usefulness of this analysis, for example, is the following. If a price increase for tables is considered which will change the unit profit coefficient from 10 to 11, we can determine how this will affect the optimal solution to the problem. Answer: it will have no effect on x1 = 3, the number of tables produced. However, the profit itself will actually increase due to the price increase.

The <Constraints> part of the Sensitivity Report shows shadow prices for allowable changes in the constraints. Lumber used has a shadow price of $0.2 or 20 cents. Since lumber is a binding constraint, we could increase the total profit by buying more lumber at the right price and amount. The shadow price is the marginal increase in the objective function (total profit) for each additional board foot of lumber that we can buy. If each additional board foot of lumber costs less than 20 cents, it is advisable to buy it, provided we can sell more tables or chairs, since profit would increase. There are limits to the number of board feet we could buy to increase or decrease from the current value of 120 board feet as shown in the report (allowable increase of 15 to 135 and allowable decrease of 15 to 105, for the shadow price to apply).

Likewise, labor is a binding constraint and has a shadow price of $2 per hour. Each additional hour of labor we could add has a marginal increase of $2 on total profit. It is also advantageous to acquire more labor than we are currently have at that price, provided we could sell the additional tables and chairs. There are also allowable increases and decreases in labor for this shadow price to apply.

Figure T5-8
Sensitivity Report

Variable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
$B$8	Decision Variables x1 (tables)	3	0	10	2	2
$C$8	Decision Variables x2 (chairs)	1.5	0	8	2	1.333333333

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
$D$14	Lumber Used	120	0.2	120	15	15
$D$15	Labor Used	9	2	9	0.6	1
$D$16	Finishing Used	21	0	24	1E+30	3

Finally the Finishing capacity of 24 hours is not all used in the optimal solution, only 21 hours are used. It is a non-binding constraint and the shadow price is 0 as it should be since there would be no increase in profit at the margin for more finishing capacity. The allowable increase is 1E+30 which is a very large number and effectively infinity. In other words we can add as much capacity for finishing as we want without affecting the optimal solution, since we are not using all the capacity we already have. The allowable decrease is 3 hours of finishing capacity at which point this constraint will become binding.

Sensitivity Analysis is useful to management to show the flexibility of the optimal solution to changes in the resources available or the unit profit coefficients. These are the critical constants used to define the original problem that can be changed by management.

For more information on using Sensitivity analysis review the YouTube video: LP Sensitivity Analysis - Interpreting Excel's Solver Report https://youtu.be/FzY333RdrBM. It starts with a Solver solution and shows how to use the sensitivity report.

TRANSPORTATION PROBLEM USING SOLVER

The transportation problem is a special case of a linear programming problem. It is a widely used application that is easily optimized with Excel Solver. The general problem is to determine the best transportation plan to ship products between factories and warehouses, given specific supply and demand constraints. The formulation of the transportation problem is somewhat different in terms of entering the data and formulas into the Excel spreadsheet and therefore requires some elaboration.

Formulation

Suppose there are three factories that can ship products to three different warehouses at different unit costs for each route chosen as shown in Figure T5-9. Also shown in the bottom and far right margins respectively are the total number of units required at each warehouse and the total amount available to ship from each factory.

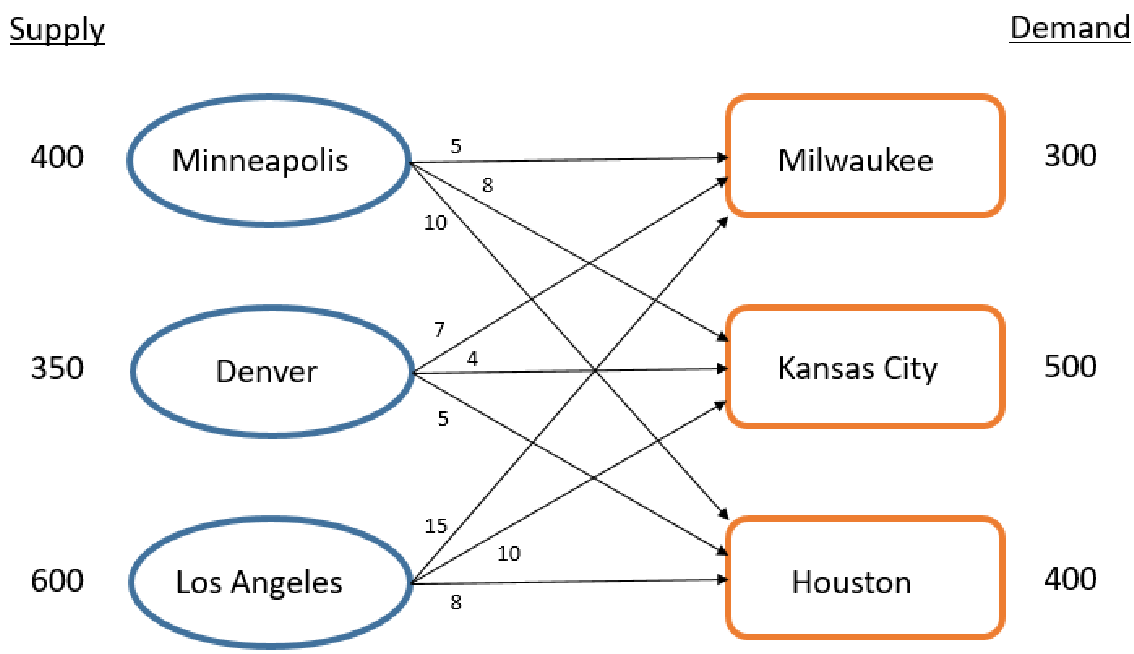
Figure T5-9
Unit costs of shipping from factories to warehouses and requirements

To Warehouses

From Factories	1.Milwaukee	2.Kansas City	3.Houston	Total units available
1.Minneapolis	5	8	10	400
2.Denver	7	4	5	350
3.Los Angeles	15	10	8	600
Total units required	300	500	400	

In Figure T5-10 we illustrate this transportation problem as a network. Nodes show each of the three origins and the three destinations. The network arcs (arrows) represents the flow of units originating in factories (supply) to be delivered to warehouses (demand). The arcs in the network are labeled with the cost per unit of each transportation route from Figure T5-9. This representation is useful to visualize the transportation problem as a network of shipping routes from supply to demand.

Figure T5-10
Transportation Network



The transportation problem can be written in mathematical form. There are nine unknown variables corresponding to each of the nine cells (or arcs) which represent the shipments from the source factory to the destination warehouse. These variables are labeled X_{ij} corresponding to the amount shipped from factory i (i=1,2,3) to warehouse j (j=1,2,3). For example, X_{22} is the amount shipped from factory 2 (Denver) to warehouse 2 (Kansas City). In a similar way, the unit cost of shipping one unit from factory i to warehouse j is C_{ij}. It costs $4 to ship one unit from factory 2 to warehouse 2.

The total cost of shipping all units, C is just the amount shipped multiplied by the unit cost of shipping in each cell and added over all 9 cells. We wish to minimize C. Below we show this generalized and compact version of the problem.

Min C = $\sum_{i=1}^{i=3}\sum_{j=1}^{j=3} \boldsymbol{CijXij}$

Subject to:

$$\sum_{j=1}^{j=3} \boldsymbol{Xij} \le \boldsymbol{Ai}$$

$$\sum_{i=1}^{i=3} \boldsymbol{Xij} = \boldsymbol{Bj}$$
All $\boldsymbol{Xij} \geq \boldsymbol{0}$

The first constraint requires that the amount shipped from each factory i when summed over all three warehouses j (j=1,2,3) be ≤ the amount available at that factory $\boldsymbol{Ai}$

The second constraint requires that the total amount shipped to each warehouse j is summed over all three factories i (i=1,2,3) and must be equal to the demand $\boldsymbol{Bj}$ at that warehouse. All the demand orders at warehouses must be filled therefore we use the = signs for $\boldsymbol{B_j}$.

The third constraint indicates that all variables must be non-negative.

The complete version of the problem is as follows. As you can see, the formulation satisfies all the conditions of any LP problem, just with many variables and constraints.

Min $5X_{11} + 8X_{12} + 10X_{13} + 7X_{21} + 4X_{22} + 5X_{23} + 15X_{31} + 10X_{32} + 8X_{33}$

Subject to:

$$X_{11} + X_{12} + X_{13} \leq 400$$
$$X_{21} + X_{22} + X_{23} \leq 350$$
$$X_{31} + X_{32} + X_{33} \leq 600$$
$$X_{11} + X_{21} + X_{31} = 300$$
$$X_{12} + X_{22} + X_{32} = 500$$
$$X_{13} + X_{23} + X_{33} = 400$$
$$X_{11} \geq 0, X_{12} \geq 0, X_{13} \geq 0, X_{21} \geq 0, X_{22} \geq 0, X_{23} \geq 0, X_{31} \geq 0, X_{32} \geq 0, X_{33} \geq 0$$

Excel Formulation

To set up Excel, we use a short-hand method to enter variables and constraints from what we have been doing. First, copy the information from Figure T5-9 and paste it directly into an Excel worksheet as shown on the top of Figure T5-11. We label the copied information "unit cost table" referring to the unit costs shown in blue. The entire table is then copied again and pasted below in the spreadsheet with the label, "Amount shipped". In this table all the cell values are changed from the cost values to values of the amount shipped, and are set to zero (0) as a starting point. We will be multiplying the unit cost table (Cij) by the amounts shipped (Xij) table and summed over all rows and columns to get the total cost of shipping, which we want to minimize. Solver will determine the optimal amount to be shipped from each factory to each warehouse to minimize total shipping costs.

To continue setting up this problem, we create the cells F13, F14 and F15 and insert their related formulas to indicate the total amount actually shipped from each factory, since it can be less than the amount available at the factories. The initial amounts shipped are entered as zero in all cells. Solver will determine these amounts in the optimal solution.

We also create a new row with cells B17, C17 and D17 and insert their related formulas to indicate the total units received at each warehouse, which we will require to equal the total

units required. The total shipped from all three factories must just equal demand at each warehouse.

Finally, we add the new label, "Total Transportation Cost", and the cell E19 that is set equal to zero for the initial total cost. This is the objective function cell that we will be minimizing. A special formula from Excel provides an abbreviated form of multiplication and addition to calculate the Total Transportation Cost. We use the SumProduct formula to multiply the Cij unit costs in the top matrix with the corresponding Xij amount shipped in the bottom matrix and then adding the product of XijCij over all 9 cells. This formula combines ranges that we designate; one range in the unit cost matrix and the second range in the amount shipped matrix as SumProduct(B5:D7,B13:D15) which is equal to the objective function, the total cost to be minimized in E19.

Figure T5-11
Excel Transportation Problem formulation

	A	B	C	D	E	F	G
1							
2	**Unit cost matrix**		To Warehouses				
3	From Factories	1.Milwaukee	2.Kansas City	3.Houston	Total units available		
5	1.Minneapolis	5	8	10	400		
6	2.Denver	7	4	5	350		
7	3.Los Angeles	15	10	8	600		
8	Total units required	300	500	400			
9							
10	**Amount Shipped**		To Warehouses				
11	From				Total units available	Total units	
12	Factories	1.Milwaukee	2.Kansas City	3.Houston		Shipped	
13	1.Minneapolis	0	0	0	400	0	
14	2.Denver	0	0	0	350	0	
15	3.Los Angeles	0	0	0	600	0	
16	Total units required	300	500	400		=SUM(B15:D15)	
17	Total units received	0	0	0			
18							
19			Total Transportation Cost		0		
20							
21		=SUM(B13:B15)	=SUMPRODUCT(B5:D7,B13:D15)				
22							
23							

Solver Solution

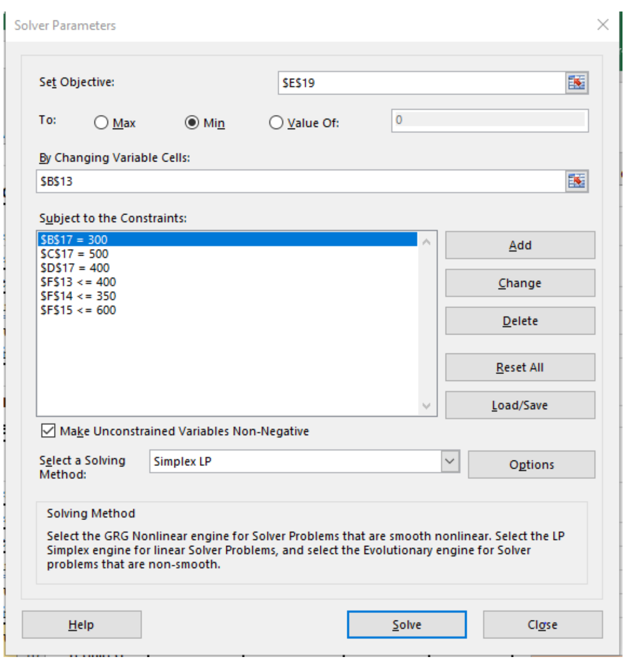
Save your formulation, as a standard practice, for later use in case Excel crashes when Solving. We can now use Solver to find the optimal solution for the transportation formulation. After clicking on Data and Solver, the Solver menu will appear. Be sure that $E$19 is in the <Set Objective> box. Also check that Min is selected for minimization of this problem. Enter the <By Changing Variables Cells> as $B$13:$D$15. This is the range of all the 9 cells in the amount shipped matrix at the bottom of the spreadsheet and are the Xij variables in the problem.

Enter the six constraints for this problem, three to ensure the amount shipped is less than the total available from a factory and three to ensure that all warehouse demand is met. This is done by selecting the Add button and adding constraints, one at a time, starting with $B$17 =300. Note that all the warehouse demand constraints are equal signs which are selected in the middle box when adding a constraint. These demands must be met at each warehouse, since we want to ship what the customer ordered. The factory constraints starting with $F$13 are <= constraints since each factory has an upper limit on what it can ship.

In order to get a feasible solution, the sum of the units available at all three factories (1350 units in this case) must be greater than the sum of the units required (1200 units in this case), otherwise the factories won't be able to meet demand, and Solver will not find a feasible solution.

The resulting Solver menu is shown in Figure T5-12. Make sure the <Make Unconstrained Variables Non-Negative> box is checked and Select the Simplex Method in the next box. Press Solve to get the optimal solution.

Figure T5-12
Solver Menu for Transportation Problem



The optimal solution from the lower half of the Excel sheet is shown below in Figure T5-13. The optimal number of units to ship from each factory to each warehouse is shown. For example, 300 units should be shipped from Minneapolis to Milwaukee and 400 units from Los Angeles to Houston, along with the other shipments shown in the matrix. The minimum cost for the problem is $7400 to fulfill all the demands at the warehouses.

Figure T5-13
Optimal Solution to the Transportation Problem

10	**Amount Shipped**		To Warehouses			
11	From				Total units	Total units
12	Factories	1.Milwaukee	2.Kansas City	3.Houston	available	Shipped
13	1.Minneapolis	300	100	0	400	400
14	2.Denver	0	350	0	350	350
15	3.Los Angeles	0	50	400	600	450
16	Total units required	300	500	400		=SUM(B15:D15)
17	Total units received	300	500	400		
18						
19			Total Transportation Cost		7400	
20						

The sensitivity report from this problem can be read the same way as before. For example, the optimal solution ships 300 units from Minneapolis to Milwaukee. The sensitivity report shows that the cost of this shipment is currently 5, but can increase to 11 or decrease to 0 without changing the shipment size of 300 units. Increasing the unit cost from 5 to 11, however would change the total cost from 7400 to 9200 by adding $1800 (unit cost increase of 6*300 units). Sensitivity changes are also available for all 9 decision variables.

Likewise the sensitivity report tells us the shadow price for all six constraints. For the constraint of 300 shipped to Milwaukee, the shadow price is $7. This indicates the minimum cost solution would be increased by $7 for each additional unit shipped to Milwaukee. This holds for an increase of 100 from 300 to 400 units. Since the increase in the minimum cost solution is $7 more for each additional unit shipped to Milwaukee, the company would need to charge $7 more per unit from the customer just to break even.

MIXED INTEGER PROGAMMING USING SOLVER

An integer programming problem has one or more variables required to be integers. This is desirable when rounding off the solution is not suitable and the variables must be integers in the optimal solution. If an integer decision variable is required to be 0 or 1, it is called a binary variable. These are usually indicator variables. When X=1, an entity (machine, person or project) is selected, otherwise it is not selected when X=0. Integer programming problems can also have mixed variables, some integer and others continuous. For example, if a machine has

a fixed cost when it is used and a variable cost per unit when used, the problem will require an integer variable to reflect the fixed cost and a continuous variable for the variable cost per unit.

Integer programming problems are more complex. They cannot be solved by the Simplex method, since they are non-linear in nature. Hence the simplex method will not find an optimal solution, but it can still be useful in a modified form. A common method is called branch and bound (B&B). The B&B method starts with a relaxed solution which ignores all the integer constraints. If an integer solution is found by the relaxed Simplex method, there is no need to go further. Otherwise, the B&B method will create two sub-problems to be solved with the Simplex method. For example, if an integer variable X1 in the optimal relaxed solution is 4.6, one sub-problem would have X1 <=4 and the other sub-problem would have X1>=5. Then, these two sub-problems are solved with the Simplex method and we check to see if all required variables are integers, otherwise keep branching. Bounds are set to insure certain parts of the solution space need not be searched to help limit the number of branches needed. The B&B method keeps branching until all integer variables are found to within a small tolerance.

Mixed Integer: Machine Fixed Cost

The first example we formulate is a mixed integer problem. Suppose a company receives an order for 1200 units of a product. The company has three machines that it can use to make the product with different efficiencies and capacities as shown below in Figure T5-14.

Figure T5-14
Mixed Integer example

	Machine 1	Machine 2	Machine 3
Variable cost/unit	3.60	4.17	2.20
Fixed cost	300	200	500
Capacity	600	1000	900

If a machine is used, it incurs a fixed setup cost to produce the product along with the variable cost per unit multiplied by the number of units produced. The company wants to minimize its costs of production to make the 1200 units from all three machines.

<u>Decision Variables</u>
Let Xi = number of units produced on machine i (i=1,2, 3)
Let Yi = 1 if machine i is used and 0 otherwise.
The total cost to be minimized is

Min Cost = 3.60*X1 + 4.17*X2 +2.20*X3 + 300Y1 + 200 Y2 + 500 Y3

The variable cost is multiplied by the number of units produced by each machine and the fixed cost is incurred (when Yi =1) only when machine i is used. We need to constrain Xi to zero when Yi = 0 to insure machine i is not used unless the setup cost is incurred.

Constraints

X1 + X2 + X3 = 1200   We need this to satisfy the order of 1200 units.
X1 <= 600*Y1   rewrite this constraint as X1 - 600*Y1 <=0
X2 <= 1000*Y2   rewrite this constraint as X2 - 1000*Y2 <=0
X3<= 900*Y3   rewrite this constraint as X3 - 900*Y3 <=0

X1, X2, X3 >=0, Y1, Y2, Y3 = (0,1 integers).

The second, third, and fourth constraints above insure that Xi will be positive only if Yi=1 (machine 1 is used), otherwise Xi = 0 when Yi = 0 (machine i is not used).

We can now enter this problem into Excel. It will be similar to previous problems except we need to specify the three integer variables (Yi) as binary variables.

The problem is shown in Figure T5-15. The first step is to insert the variables and constraints labels in Column A. Then Columns B to G are labeled for the six variables (3 continuous and 3 binary) and their initial values are set equal to zero. The unit costs for each variable are entered in row 4. The variable unit costs are entered as usual under X1, X2 and X3. Note the fixed costs are entered under the Y_i variables. They will be incurred when Y_i =1, otherwise they will be zero. Then the Total cost equation is entered in cell I5 which will be the objective function to minimize.

Next, the four constraints are entered. The constraint in row 9 insures that the demand of 1200 units is met by the sum of units produced by all three machines. The remaining three constraints indicate that the fixed cost of a machine is only charged (Y_i =1) when the machine is used to produce the product and not charged when the machine is not used (Y_i = 0).

Figure T5-15
Mixed Integer Fixed Cost Machine Selection.

	A	B	C	D	E	F	G	H	I	J	K	L
1												
2	**Variables**	X1	X2	X3	Y1	Y2	Y3					
3	Decision Variables	300	0	900	1	0	1					
4	Cost per unit	3.6	4.17	2.2	300	200	500					
5							Total cost	3860				
6												
7								constraint		right hand		
8	**Constraints**							formula		side		
9	Meet demand	1	1	1	0	0	0	1200	"="	1200		
10	Machine 1	1			-600			-300	<=	0		
11	Machine 2		1			-1000		0	<=	0		
12	Machine 3			1			-900	0	<=	0		
13		>=0	>=0	>=0	0,1	0,1	0,1					
14												
15												
16												

=B4*B3+C4*C3+D4*D3+E4*E3+F4*F3+G4*G3

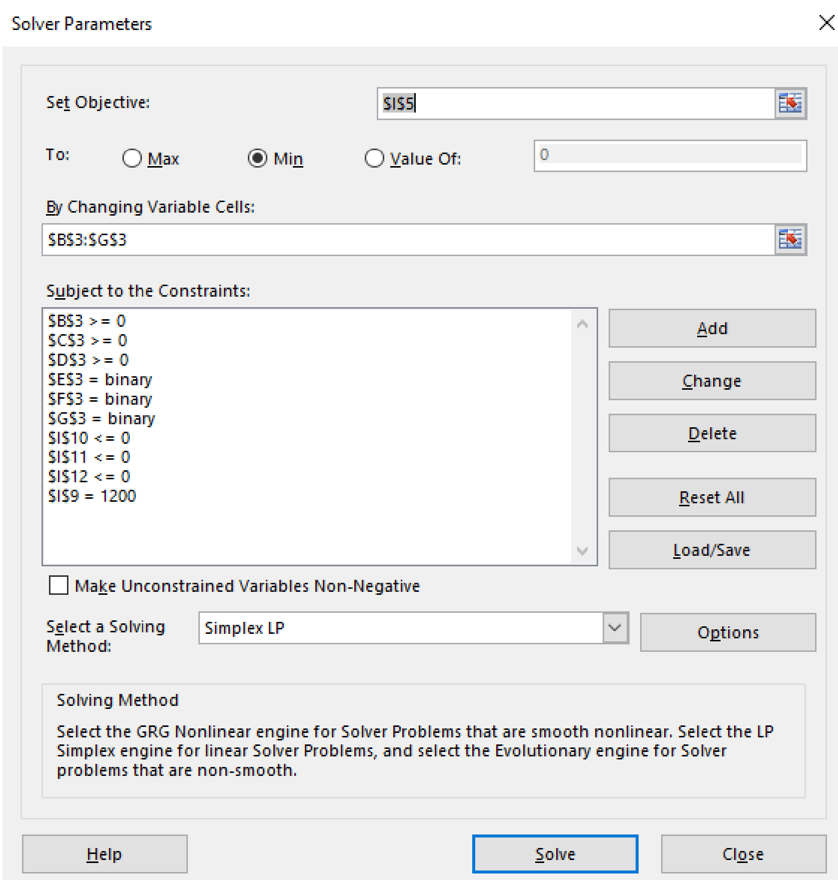
=B9*B3+C9*C3+D9*D3

=B10*B3+E10*E3

Solver Solution

When Solver is activated, the following menu appears in Figure T5-16. Enter the <Set Objective> cell as $I$5, if not already entered, and click on <Min> button. The <Changing Variable Cells> are $B$3:$G$3. The constraints are entered as shown in Figure T5-16.

Figure T5-16
Solver Menu Screen



Notice, because this problem has binary integer variables in cells E3, F3 and G3 we make the bin (i.e., binary) selection when adding these three constraints. The variables B3, C3, and D3 must be constrained to be non-negative, since we will not be constraining all variables to be non-negative as we have done before. Uncheck the box <Make Unconstrained Variable Non-negative>

Set "Select a Solving Method" to <Simplex LP>. Select the <Options> button. On the next screen set the <Integer Optimality (%)> to "5" to indicate that solver can stop when it is within 5% of the relaxed solution without integer constraints. This will limit the number of Simplex iterations when the exact integer solution is not found quickly. Also select <Max Time (Seconds)> to "60". Then Select <OK> and <Solve>. Always save your formulation right before you run Solver to revise it later, in case your computer does crash.

The solution will appear in the Excel Spreadsheet as follows:

X1 = 300, X2 = 0, X3 = 900, Y1 = 1, Y2 = 0, Y3 = 1

Total cost is $3860.

There is no sensitivity analysis, since this is a non-linear problem.

DISCUSSION QUESTIONS

1. Define the following terms:
  a. Constraint
  b. Objective function
  c. Extreme point
  d. Optimal solution
  e. Feasible solution

2. What information is provided by shadow prices?
3. The simplex method is an adjacent-extreme-point method. Explain.
4. What is the purpose of the branch-and-bound method?
5. Define what is meant by a mixed-integer problem.
6. What is the purpose of using binary variables?
7. How is the transportation problem different from other L.P. problems?

SOLVED PROBLEMS

S1. The x-ray department of a hospital has two machines, A and B, which can be used to develop x-ray film. The maximum daily processing capacity of these machines is A <= 80 films and B <= 100 films. The department must plan to process exactly 140 films per day. Each x-ray processed on machine A takes 0.1 hours of labor and each x-ray processed on machine B takes .15 hours. There are 20 hours of labor available to process the x-rays. The operating costs per film are $3 for machine A and $4 for machine B. To minimize costs, how many films per day should each machine process?

a. Solve this problem by Excel Solver
b. How much would you be willing to pay for additional capacity on machines A and B or additional labor?

Solution

a.The solution is shown below. It calls for 80 films to be processed on machine A and 60 on machine B. The capacity on machine A is fully utilized, but machine B is not. Labor is also not fully utilized. The minimum cost is $480.

	A	B	C	D	E	F
1	xray problem					
2		number of films				
3	**Variables**	A	B			
4	decision variables	80	60			
5	Coefficients	3	4	total cost	480	
6						
7	**Constraints**			utilized		Capacity
8	films required	1	1	140	"="	140
9	capacity on A	1		80	<=	80
10	capacity on B		1	60	<=	100
11	Labor	0.1	0.15	17	<=	20
12						

b. The sensitivity analysis indicates that the shadow price on machine B and labor are 0 which is as they should be. There is no change in the minimum cost since these constraints are not binding. Machine A which is fully utilized has a shadow price of -1 which means the total cost can be decreased by 1 for every additional hour of capacity on machine A up to an increase of 60 hours. The number of films required has a shadow price of 4 which means the total cost is increased by $4 for every additional film processed up to 20 more.

S2 The x-ray department has decided to add another machine C and will now consider fixed costs for all three machines along with the variable costs. The capacity of machine C is 60 films daily, its variable cost is $5 per film, it takes .1 hours of labor to process a film and its fixed cost is $250. The fixed costs to set up machine A is $300 and $200 to set up machine B. The three machines must be able to process a total of 200 films a day and the labor available will be increased to 30 hours. Reformulate the problem with binary variables for the fixed costs of machine A, B and C.

Solution

The solution now uses all three machines with the number of films processed by A=80, B=100 and C=20. The new machine is only partially used with 20 films vs 60 films capacity, probably because of its higher costs. Only 25 hours of labor is used.

	A	B	C	D	E	F	G	H	I	J
	X-Ray problem with fixed costs									
		number of films			Fixed setup costs					
	Variables	A	B	C	Y1	Y2	Y3			
	decision variables	80	100	20	1	1	1			
	Coefficients	3	4	5	300	200	250	total cost	1490	
	Constraints							utilized		capacity
	films required	1	1	1				200	"="	200
	capacity on A	1			-80			0	<=	0
0	capacity on B		1			-100		0	<=	0
1	capacity on C			1			-60	-40	<=	0
2	labor	0.1	0.15	0.1				25	<=	30
3										

PROBLEMS

1. Solve the following problem

   Objective: max $3X_1 + 4X_2$

   Subject to:

   $$4X_1 + 2X_2 \leq 8$$
   $$2X_1 + 5X_2 \leq 10$$

   $$X_1 \geq 0, X_2 \geq 0$$

   a. Find a graphical solution to this problem by graphing the constraints and the objective function. What are all the four extreme points in this problem and which adjacent extreme points would be examined by the Simplex method in reaching the optimal solution?

   b. What is the optimal solution using Excel Solver? Find the optimal values of X_1 and X_2 and the objective function. Is it the same as the solution from the graphical method?

2. Solve the following problem.

   Objective: min $6X_1 + X_2$

   Subject to:

   $$2X_1 + X_2 \geq 6$$
   $$4X_1 - X_2 \geq 6$$

   $$X_1 \geq 0, X_2 \geq 0$$

   a. Find the optimal solution using Excel Solver. What are the optimal values of X_1 and X_2 and the objective function?

3. The Energex company makes two energy drinks: AwakeX and SuperX. Each drink requires caffeine and two types or resources that are available in limited supply: syrup and flavoring. The AwakeX drink takes 2 ounces of syrup and 1 ounce of flavoring per 12 ounce canned drink. The SuperX drink requires 2.5 ounces of syrup and 2 ounces of flavoring per drink. Energex has only 1000 ounces of syrup available and 1500 ounces of flavoring. AwakeX contributes \$1.00 per drink to profit and SuperX contributes \$1.50 per drink to profit. The company wishes to maximize profit.

   a. Formulate this problem as a linear programming problem.

   b. Find the optimal solution with Solver.

4. A company manufactures two types of toys: trucks and cars. Each truck requires 1 minute of molding time, 2 minutes of painting time, and 1 minute of packing time. Each car requires 2 minutes of molding time, 1 minute of painting time, and 2 minutes of packing time. There are a maximum of 300 minutes of molding time, 400 minutes of painting time, and 400 minutes of packing time available each day. Both cars and trucks contribute $1 per unit to profit.

c. Formulate this problem.

b. What is the optimal solution using Excel Solver? How many trucks and cars should be produced and what is the maximum profit?

c. Which constraints are binding and what are the associated shadow prices and allowable increases and decreases?

d. If you could purchase additional capacity of molding for 20 cents per minute, would you do it to increase profits?

5. The Realistic Picture Frame Company makes four different types of ready-made picture frames: rustic, modern, French, and Roman. Each frame takes the following amounts of resources in wood, labor, and machine time as shown by the unit production technology matrix below. The profit contributions per unit for each frame type are: rustic = $1.50, modern = $1.25, French = $.95, and Roman = $ 1.10. At the present time, we have available 1000 board feet of wood, 500 hours of labor, and 100 hours of machine time.

	Resources Used Per Unit Produced			
	Rustic	Modern	French	Roman
Wood, board feet	1.0	1.5	2.0	2.0
Labor, hours	1.0	.9	.7	.6
Machine, hours	.3	.2	.1	.1

a. Formulate the problem as a linear programming problem. What are the formulas for the objective function and the constraint on wood?

b. Find the optimal solution to this problem using Excel Solver. What is the optimal number of frames of each type to produce, and how much profit contribution is made?

c. Which constraints are binding in the optimal solution?

d. How much would you pay in order to make one more unit of wood, labor, or machine time available?

6. A berry farm can choose to plant four different types of strawberries. Some of the berry plants provide more fruit, but they also require more land and labor. The yields from each type of berry plant and the amount of land and labor required are shown below.

Berry Type	A	B	C	D
Yield (lb/plant)	2	2.5	3	1.8
Land (plants/acre)	10,000	8000	6000	12,000
Labor hours/plant)	.1	.15	.2	.1

At the present time, berries are selling for 50 cents a pound, and we have 10 acres of land and 10,000 hours of labor available for planting and picking the berries. The Berry farm wishes to maximize revenue from the plantings using all the 10 acres. Due to the need for disease resistance of plants the farm will not plant more than 3 acres of any one strawberry type. Hint: Formulate this problem with decision variables equal to the number of acres planted of each type of berry.

a. Formulate the problem as a linear programming problem. What are the formulas for the objective function to maximize revenue and the constraint on Labor hours?

b. How many berries of each type should be planted to maximize revenue?

c. More labor can be obtained for $5 an hour, and more land can be rented for $8,000 per acre. Should additional resources be obtained? If yes, how much?

7. Gasoline is blended from several stocks to obtain minimum required octane ratings. Suppose an order for 1000 gallons of 80-octane gasoline has been received. There are three blending stocks available to meet this order. Stock A has 95 octane and costs $1.20 per gallon. Stock B has 70 octane and cost $1 per gallon. Stock C has 85 octane and costs $1.10 per gallon. There are only 500 gallons of stock A, 900 gallons of stock B, and 600 gallons of stock C available. The three stocks are mixed together to provide the 80-octane requirement. Assume that octane numbers combine in proportion to the volume of the stocks which are blended together. How much stock of each type should be used to minimize the total cost of filling the order?

a. Formulate this problem. Hint: The constraint for blending in proportional volumes of A, B and C is: $\frac{95A+70B+85C}{A+B+C} = 80$ which equates to 15A-10B+5C = 0

b. What is the optimal solution using Excel Solver

c. How much would you be willing to pay for an additional gallon of Stock A, B or C?

8. Brand X animal feed is made from a combination of wheat and corn. Wheat contains 10 percent protein, 40 percent starch, and 50 percent fiber by weight. Corn contains 15 percent protein, 50 percent starch, and 35 percent fiber by weight. The recipe for brand X feed calls for a minimum of 45 percent starch and a minimum of 40 percent fiber; there is no restriction

on protein. If a batch of at least 800 pounds of brand X is produced, how many pounds of corn and how many pounds of wheat should be used to minimize costs? Wheat and corn combine in proportion to their weights when blended together to provide the required percentages of starch and fiber. Corn costs 5 cents per pound and wheat costs 3 cents per pound.

a. What are blending constraints for this problem? Hint: there is one blending constraint for starch and another for fiber. Express the constraints in the >= format.

b. What is the optimal solution using Excel Solver. How much wheat and corn is used and what is the minimum cost?

c. How much will it cost to produce each additional pound of animal feed beyond the 800 pounds required by the problem?

9. **Transportation Problem. Wholesaler ships to Zones.** A wholesale warehouse company has 2 different locations: Chicago and Los Angeles that ship to a retail zones on the East Coast and in the South. The cost of shipping one unit from each warehouse to each zone is shown below along with the maximum amounts available at each warehouse and the amount required that must be shipped to each zone.

	East Coast	South	Available
Chicago	6	4	1000
Los Angeles	10	3	800
Total Required	600	900	

a. Write the equations for this as a transportation problem

b. Using Solver what is the minimum cost solution?

c. How much in total is shipped from Chicago and from Los Angeles?

10. **Transportation Problem -- Small Electronics Company.** A small electronics company wants to send a sensor that it makes from three factories to three warehouses. The factories are in Atlanta, Chicago and San Diego and the warehouses are in Newark, Dallas and Phoenix. The cost of shipping from each factory to each warehouse is shown below along with the capacities at each factory and the requirements at each warehouse.

	Newark	**Dallas**	**Phoenix**	Available
Atlanta	6.50	7.00	8.50	**900**
Chicago	6.00	6.50	7.50	**1000**
San Diego	9.00	7.50	5.00	**1200**
Required	**600**	**450**	**800**	

a. What is the optimal solution to this problem?

b. What are the shadow prices for shipping more to Newark, Dallas or Phoenix?

c. If we could charge $8 per unit for each additional unit shipped to the Phoenix warehouse, how much profit would we make for each additional unit?

d. If you could reduce the unit cost of shipping from Atlanta to Phoenix from $8.5 to $6, would you start using that route in the optimal solution?

11. **Transportation Problem.** A plumbing supply company has two warehouses and four large commercial customers that it serves with various plumbing supplies. The warehouses are in Houston and Minneapolis. The customers are in New York, Miami, Los Angeles and Seattle. The cost of shipping from each warehouse to each customer, the capacity of warehouses and the requirements of customers that must be satisfied are shown below.

	New York	Miami	Los Angeles	Seattle	Available
Houston	10	9	15	16	**400**
Minneapolis	8	11	12	14	**300**
Required	**100**	**150**	**160**	**90**	

a. What are the optimal shipping amounts and the minimum cost solution using Solver?

b. How much does shipping one more unit to the Miami customers increase the total cost of transportation in the network?

12. Integer LP. In problem 4, production of toys, the optimal solution is to make 166.67 trucks and 66.67 cars. Solve this problem again requiring that both the number of toys and the number of trucks must be integers.

a. What is the integer solution optimal number of trucks and cars to make?

b. Does the number of trucks and cars differ significantly from rounding off the solution?

c. How much does profit decrease?

13. Mixed Integer Binary LP problem. A candy store makes two kinds of chocolate candy, regular and supreme. The regular candy costs $2.00 per pound for the ingredients and the supreme costs $3.00 per pound. Regular candy has a variable labor cost of $1.00 per pound

and supreme candy has a variable labor cost of 1.50 per pound. The regular candy sells for \$7.00 a pound and the supreme candy sells for \$9.00 per pound. It costs 200 to set up the mixer for regular candy and 300 to set it up for supreme candy. The store would like to maximize profit from these two products to produce 1000 pounds of candy. However, no more than 600 pounds of regular candy and no more than 800 pounds of supreme candy are planned for production.

a. Formulate this problem as a mixed integer LP problem.

b. How much candy of each type should be produced to maximize profit?

14. Mixed Integer Binary LP. The Gadget Phone Company makes a type of cellular phone and just received an order for 1500 phones. The phone can be made on either of two machines, or both of them, that each have limited capacities to make the phone. Machine 1 can make up to 1000 units and Machine 2 can make up to 800 units of the product. Machine 1 has a variable cost of \$100 per phone and a fixed setup cost of \$800. Machine 2 has a variable cost of \$200 per phone, a fixed setup cost of \$1200. How many phones should be made on each machine to minimize the total cost of producing the 1500 phones?

a. Formulate this problem as a mixed integer LP problem.

b. How many phones of each type should be produced to minimize cost and what is the total cost of production?

c. What are the binding constraints in the optimal solution?