Productivity and cost

Suppose a firm's only variable input is labor, L , purchased at a constant wage rate, w. Also suppose that the firm's output follows the short-run production function $Q = f(L)$ which is monotonically increasing in L: labor's marginal product is positive ($f'(L) > 0$) so that an additional worker always generates additional output. Under these assumptions, variable cost is simply the wage rate times the number of workers: $VC = wL$.

Marginal cost is the change in variable cost associated with the next unit of output: $MC = \frac{dVC}{dQ}$. Since variable cost is a function of output, and output is a function of labor input, we can use the chain rule to find this derivative: $\frac{dVC}{dQ} = \frac{dVC}{dL} \cdot \frac{dL}{dQ}$. From the expression for VC, the first term is simply the wage, w. That is, adding one more worker adds that worker's wage to variable cost. The assumption that Q is monotonically increasing in L assures us that $\frac{dL}{dQ} = 1\Big/\frac{dQ}{dL} = 1/f'(L)$: the number of workers required to produce one more unit of output is the inverse of the increase in output created by an additional worker. Combining these two results, we see that $MC = w/f'(L)$. That is, MC is the wage rate divided by labor's marginal product, so that marginal cost and marginal product are inversely related. Stated differently, marginal cost can be found in two steps. First, determine how many additional workers are required to produce an additional unit of output. This is equal to $1/f'(L)$, the inverse of labor's marginal product. Second, multiply this number of workers by the wage rate paid to each one in order to determine marginal cost—the increase in cost associated with one additional unit of output.

For example, suppose that labor's marginal product is 5, and the wage is \$10. Since the next worker could produce five additional units of output, it would take only one-fifth of a worker—or one worker working $1/5^{th}$ of an hour—to produce the next one unit of output. This amount of labor would cost \$10 x 1/5 = \$2, so the marginal cost is \$2.

We can also determine the slope of the marginal cost curve as the slope of $w/f'(L)$, recalling that marginal cost is itself the slope of variable cost. That is, we require $\frac{d^2VC(Q)}{dQ^2} = \frac{d}{dQ}\left(\frac{w}{f'(L)}\right) = w\frac{d}{dQ}\left(\frac{1}{f'(L)}\right)$.

We begin this rather daunting-looking task by the device of a substitute variable and application of the chain rule. If we let $z = f'(L)$, then what we require is $w\frac{dz^{-1}}{dQ}$. However, since z is a function of L and L is a function of Q, we must use the chain rule twice to obtain the desired result:

$$\frac{dz^{-1}}{dQ} = \frac{dz^{-1}}{dz} \cdot \frac{dz}{dL} \cdot \frac{dL}{dQ} = -\frac{1}{z^2} \cdot \frac{dz}{dL} \cdot \frac{dL}{dQ}.$$

We can work this term by term: $-\frac{1}{z^2} = -\frac{1}{(f'(L))^2}$. For the second term, $\frac{dz}{dL} = \frac{df'(L)}{dL} = f''(L)$ and finally $\frac{dL}{dQ} = \frac{1}{f'(L)}$. Combining all these results, $\frac{dz^{-1}}{dQ} = -\frac{1}{(f'(L))^2} \cdot f''(L) \cdot \frac{1}{f'(L)} = -\frac{f''(L)}{(f'(L))^3}$.

Finally, remembering the wage rate, we have the result that we require: $\frac{d^2VC(Q)}{dQ^2} = -\frac{wf''(L)}{(f'(L))^3}$.

The wage is clearly positive, as is the denominator by our earlier assumption. So, the slope of marginal cost is positive if and only if $f''(L)$ is negative. That is, if marginal product is diminishing, marginal cost will be increasing, as asserted in the text.