CHAPTER 29

Markov Chains

Chapter 16 focused on decision making in the face of uncertainty about *one* future event (learning the true state of nature). However, some decisions need to take into account uncertainty about *many* future events. We now begin laying the groundwork for decision making in this broader context.

In particular, this chapter presents probability models for processes that *evolve over time* in a probabilistic manner. Such processes are called *stochastic processes.* After briefly introducing general stochastic processes in the first section, the remainder of the chapter focuses on a special kind called a *Markov chain.* Markov chains have the special property that probabilities involving how the process will evolve in the future depend only on the present state of the process, and so are independent of events in the past. Many processes fit this description, so Markov chains provide an especially important kind of probability model.

For example, Chap. 17 mentioned that *continuous-time Markov chains* (described in Sec. 29.8) are used to formulate most of the basic models of *queueing theory*. Markov chains also provided the foundation for the study of *Markov decision models* in Chap. 19. There are a wide variety of other applications of Markov chains as well. A considerable number of books and articles present some of these applications. One is Selected Reference 4, which describes applications in such diverse areas as the classification of customers, DNA sequencing, the analysis of genetic networks, the estimation of sales demand over time, and credit rating. Selected Reference 6 focuses on applications in finance and Selected Reference 3 describes applications for analyzing baseball strategy. The list goes on and on, but let us turn now to a description of stochastic processes in general and Markov chains in particular.

29.1 STOCHASTIC PROCESSES

A **stochastic process** is defined as an indexed collection of random variables $\{X_t\}$, where the index t runs through a given set T. Often T is taken to be the set of nonnegative integers, and X_t represents a measurable characteristic of interest at time t. For example, X_t might represent the inventory level of a particular product at the end of week t.

Stochastic processes are of interest for describing the behavior of a system operating over some period of time. A stochastic process often has the following structure.

> The current status of the system can fall into any one of $M + 1$ mutually exclusive categories called **states.** For notational convenience, these states are labeled $0, 1, \ldots, M$. The random variable X_t represents the *state of the system* at time t, so its only possible values are $0, 1, \ldots, M$. The system is observed at particular points of time, labeled $t = 0, 1, 2, \ldots$. Thus, the stochastic process $\{X_t\} = \{X_0, X_1, X_2, \ldots\}$ provides a mathematical representation of how the status of the physical system evolves over time.

This kind of process is referred to as being a *discrete time* stochastic process with a *finite state space.* Except for Sec. 29.8, this will be the only kind of stochastic process considered in this chapter. (Section 29.8 describes a certain *continuous time* stochastic process.)

A Weather Example

The weather in the town of Centerville can change rather quickly from day to day. However, the chances of being dry (no rain) tomorrow are somewhat larger if it is dry today than if it rains today. In particular, the probability of being dry tomorrow is **0.8** if it is dry today, but is only **0.6** if it rains today. These probabilities do not change if information about the weather before today is also taken into account.

The evolution of the weather from day to day in Centerville is a stochastic process. Starting on some initial day (labeled as day 0), the weather is observed on each day t, for $t = 0, 1, 2, \ldots$. The state of the system on day t can be either

State $0 =$ Day t is dry

or

State $1 =$ Day t has rain.

Thus, for $t = 0, 1, 2, \ldots$, the random variable X_t takes on the values,

$$X_t = \begin{cases} 0 & \text{if day } t \text{ is dry} \\ 1 & \text{if day } t \text{ has rain.} \end{cases}$$

The stochastic process $\{X_t\} = \{X_0, X_1, X_2, \ldots\}$ provides a mathematical representation of how the status of the weather in Centerville evolves over time.

An Inventory Example

Dave's Photography Store has the following inventory problem. The store stocks a particular model camera that can be ordered weekly. Let $D_1, D_2, \ldots$ represent the *demand* for this camera (the number of units that would be sold if the inventory is not depleted) during the first week, second week, . . . , respectively, so the random variable D_t (for $t = 1, 2, \ldots$) is

$D_t =$ number of cameras that would be sold in week t if the inventory is not depleted. (This number includes lost sales when the inventory is depleted.)

It is assumed that the D_t are independent and identically distributed random variables having a *Poisson distribution* with a mean of 1. Let X_0 represent the number of cameras on hand at the outset, X_1 the number of cameras on hand at the end of week 1, X_2 the number of cameras on hand at the end of week 2, and so on, so the random variable X_t (for $t = 0, 1, 2, \ldots$) is

$X_t =$ number of cameras on hand at the end of week t.

Assume that $X_0 = 3$, so that week 1 begins with three cameras on hand.

$$\{X_t\} = \{X_0, X_1, X_2, \ldots\}$$

is a stochastic process where the random variable X_t represents the state of the system at time t, namely,

State at time t = number of cameras on hand at the end of week t.

As the owner of the store, Dave would like to learn more about how the status of this stochastic process evolves over time while using the current ordering policy described below.

At the end of each week t (Saturday night), the store places an order that is delivered in time for the next opening of the store on Monday. The store uses the following order policy:

If $X_t = 0$, order 3 cameras.
If $X_t > 0$, do not order any cameras.

Thus, the inventory level fluctuates between a minimum of zero cameras and a maximum of three cameras, so the possible states of the system at time t (the end of week t) are

Possible states = 0, 1, 2, or 3 cameras on hand.

Since each random variable X_t ($t = 0, 1, 2, \ldots$) represents the state of the system at the end of week t, its only possible values are 0, 1, 2, or 3. The random variables X_t are dependent and may be evaluated iteratively by the expression

$$X_{t+1} = \begin{cases} \max\{3 - D_{t+1}, 0\} & \text{if} \quad X_t = 0 \\ \max\{X_t - D_{t+1}, 0\} & \text{if} \quad X_t \geq 1, \end{cases}$$

for $t = 0, 1, 2, \ldots$.

These examples are used for illustrative purposes throughout many of the following sections. Section 29.2 further defines the particular type of stochastic process considered in this chapter.

29.2 MARKOV CHAINS

Assumptions regarding the joint distribution of $X_0, X_1, \ldots$ are necessary to obtain analytical results. One assumption that leads to analytical tractability is that the stochastic process is a Markov chain, which has the following key property:

A stochastic process $\{X_t\}$ is said to have the **Markovian property** if $P\{X_{t+1} = j | X_0 = k_0, X_1 = k_1, \ldots, X_{t-1} = k_{t-1}, X_t = i\} = P\{X_{t+1} = j | X_t = i\}$, for $t = 0, 1, \ldots$ and every sequence $i, j, k_0, k_1, \ldots, k_{t-1}$.

In words, this Markovian property says that the conditional probability of any future "event," given any past "events" and the present state $X_t = i$, is *independent* of the past events and depends only upon the present state.

A stochastic process $\{X_t\}$ ($t = 0, 1, \ldots$) is a **Markov chain** if it has the *Markovian property.*

The conditional probabilities $P\{X_{t+1} = j | X_t = i\}$ for a Markov chain are called (one-step) **transition probabilities.** If, for each i and j,

$$P\{X_{t+1} = j | X_t = i\} = P\{X_1 = j | X_0 = i\}, \qquad \text{for all } t = 1, 2, \ldots,$$

then the (one-step) transition probabilities are said to be *stationary.* Thus, having **stationary transition probabilities** implies that the transition probabilities do not change

over time. The existence of stationary (one-step) transition probabilities also implies that, for each i, j, and n ($n = 0, 1, 2, \ldots$),

$$P\{X_{t+n} = j \mid X_t = i\} = P\{X_n = j \mid X_0 = i\}$$

for all $t = 0, 1, \ldots$. These conditional probabilities are called ***n*-step transition probabilities.**

To simplify notation with stationary transition probabilities, let

$$p_{ij} = P\{X_{t+1} = j \mid X_t = i\},$$
$$p_{ij}^{(n)} = P\{X_{t+n} = j \mid X_t = i\}.$$

Thus, the n-step transition probability $p_{ij}^{(n)}$ is just the conditional probability that the system will be in state j after exactly n steps (time units), given that it starts in state i at any time t. When $n = 1$, note that $p_{ij}^{(1)} = p_{ij}$[1].

Because the $p_{ij}^{(n)}$ are conditional probabilities, they must be nonnegative, and since the process must make a transition into some state, they must satisfy the properties

$$p_{ij}^{(n)} \geq 0, \qquad \text{for all } i \text{ and } j;\ n = 0, 1, 2, \ldots,$$

and

$$\sum_{j=0}^{M} p_{ij}^{(n)} = 1 \qquad \text{for all } i;\ n = 0, 1, 2, \ldots.$$

A convenient way of showing all the n-step transition probabilities is the *n-step transition matrix*

$$\mathbf{P}^{(n)} = \begin{array}{c} \text{State} \\ 0 \\ 1 \\ \vdots \\ M \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & \cdots & M \end{array} \\ \begin{bmatrix} p_{00}^{(n)} & p_{01}^{(n)} & \cdots & p_{0M}^{(n)} \\ p_{10}^{(n)} & p_{11}^{(n)} & \cdots & p_{1M}^{(n)} \\ \cdots & \cdots & \cdots & \cdots \\ p_{M0}^{(n)} & p_{M1}^{(n)} & \cdots & p_{MM}^{(n)} \end{bmatrix} \end{array}$$

Note that the transition probability in a particular row and column is for the transition *from* the row state *to* the column state. When $n = 1$, we drop the superscript n and simply refer to this as the *transition matrix.*

The Markov chains to be considered in this chapter have the following properties:

1. A finite number of states.
2. Stationary transition probabilities.

We also will assume that we know the initial probabilities $P\{X_0 = i\}$ for all i.

Formulating the Weather Example as a Markov Chain

For the weather example introduced in the preceding section, recall that the evolution of the weather in Centerville from day to day has been formulated as a stochastic process $\{X_t\}$ ($t = 0, 1, 2, \ldots$) where

$$X_t = \begin{cases} 0 & \text{if day } t \text{ is dry} \\ 1 & \text{if day } t \text{ has rain.} \end{cases}$$

[1]For $n = 0$, $p_{ij}^{(0)}$ is just $P\{X_0 = j \mid X_0 = i\}$ and hence is 1 when $i = j$ and is 0 when $i \neq j$.

$$P\{X_{t+1} = 0 \mid X_t = 0\} = 0.8,$$
$$P\{X_{t+1} = 0 \mid X_t = 1\} = 0.6.$$

Furthermore, because these probabilities do not change if information about the weather before today (day t) is also taken into account,

$$P\{X_{t+1} = 0 \mid X_0 = k_0, X_1 = k_1, \ldots, X_{t-1} = k_{t-1}, X_t = 0\} = P\{X_{t+1} = 0 \mid X_t = 0\}$$
$$P\{X_{t+1} = 0 \mid X_0 = k_0, X_1 = k_1, \ldots, X_{t-1} = k_{t-1}, X_t = 1\} = P\{X_{t+1} = 0 \mid X_t = 1\}$$

for $t = 0, 1, \ldots$ and every sequence $k_0, k_1, \ldots, k_{t-1}$. These equations also must hold if $X_{t+1} = 0$ is replaced by $X_{t+1} = 1$. (The reason is that states 0 and 1 are mutually exclusive and the only possible states, so the probabilities of the two states must sum to 1.) Therefore, the stochastic process has the *Markovian property,* so the process is a Markov chain.

Using the notation introduced in this section, the (one-step) transition probabilities are

$$p_{00} = P\{X_{t+1} = 0 \mid X_t = 0\} = 0.8,$$
$$p_{10} = P\{X_{t+1} = 0 \mid X_t = 1\} = 0.6$$

for all $t = 1, 2, \ldots$, so these are *stationary* transition probabilities. Furthermore,

$$p_{00} + p_{01} = 1, \quad \text{so} \quad p_{01} = 1 - 0.8 = 0.2,$$
$$p_{10} + p_{11} = 1, \quad \text{so} \quad p_{11} = 1 - 0.6 = 0.4.$$

Therefore, the (one-step) transition matrix is

$$\mathbf{P} = \begin{array}{c} \text{State} \\ 0 \\ 1 \end{array}\!\!\begin{array}{c} \begin{array}{cc} 0 & 1 \end{array} \\ \begin{bmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{bmatrix} \end{array} = \begin{array}{c} \text{State} \\ 0 \\ 1 \end{array}\!\!\begin{array}{c} \begin{array}{cc} 0 & 1 \end{array} \\ \begin{bmatrix} 0.8 & 0.2 \\ 0.6 & 0.4 \end{bmatrix} \end{array}$$

where these transition probabilities are for the transition *from* the row state *to* the column state. Keep in mind that state 0 means that the day is dry, whereas state 1 signifies that the day has rain, so these transition probabilities give the probability of the state the weather will be in tomorrow, given the state of the weather today.

The state transition diagram in Fig. 29.1 graphically depicts the same information provided by the transition matrix. The two nodes (circle) represent the two possible states for the weather, and the arrows show the possible transitions (including back to the same state) from one day to the next. Each of the transition probabilities is given next to the corresponding arrow.

The n-step transition matrices for this example will be shown in the next section.

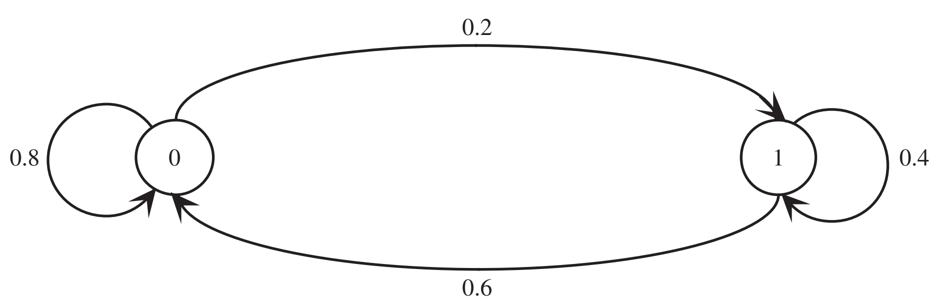


FIGURE 29.1
The state transition diagram for the weather example.

Formulating the Inventory Example as a Markov Chain

Returning to the inventory example developed in the preceding section, recall that X_t is the number of cameras in stock at the end of week t (before ordering any more), so X_t represents the *state of the system* at time t (the end of week t). Given that the current state is $X_t = i$, the expression at the end of Sec. 29.1 indicates that X_{t+1} depends only on D_{t+1} (the demand in week $t + 1$) and X_t. Since X_{t+1} is independent of any past history of the inventory system prior to time t, the stochastic process $\{X_t\}$ $(t = 0, 1, \ldots)$ has the *Markovian property* and so is a Markov chain.

Now consider how to obtain the (one-step) transition probabilities, i.e., the elements of the (one-step) *transition matrix*

$$\mathbf{P} = \begin{array}{c} \text{State} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} p_{00} & p_{01} & p_{02} & p_{03} \\ p_{10} & p_{11} & p_{12} & p_{13} \\ p_{20} & p_{21} & p_{22} & p_{23} \\ p_{30} & p_{31} & p_{32} & p_{33} \end{bmatrix} \end{array}$$

given that D_{t+1} has a Poisson distribution with a mean of 1. Thus,

$$P\{D_{t+1} = n\} = \frac{(1)^n e^{-1}}{n!}, \qquad \text{for } n = 0, 1, \ldots,$$

so (to three significant digits)

$$P\{D_{t+1} = 0\} = e^{-1} = 0.368,$$
$$P\{D_{t+1} = 1\} = e^{-1} = 0.368,$$
$$P\{D_{t+1} = 2\} = \frac{1}{2}e^{-1} = 0.184,$$
$$P\{D_{t+1} \geq 3\} = 1 - P\{D_{t+1} \leq 2\} = 1 - (0.368 + 0.368 + 0.184) = 0.080.$$

For the first row of **P,** we are dealing with a transition from state $X_t = 0$ to some state X_{t+1}. As indicated at the end of Sec. 29.1,

$$X_{t+1} = \max\{3 - D_{t+1}, 0\} \qquad \text{if} \qquad X_t = 0.$$

Therefore, for the transition to $X_{t+1} = 3$ or $X_{t+1} = 2$ or $X_{t+1} = 1$,

$$p_{03} = P\{D_{t+1} = 0\} = 0.368,$$
$$p_{02} = P\{D_{t+1} = 1\} = 0.368,$$
$$p_{01} = P\{D_{t+1} = 2\} = 0.184.$$

A transition from $X_t = 0$ to $X_{t+1} = 0$ implies that the demand for cameras in week $t + 1$ is 3 or more after 3 cameras are added to the depleted inventory at the beginning of the week, so

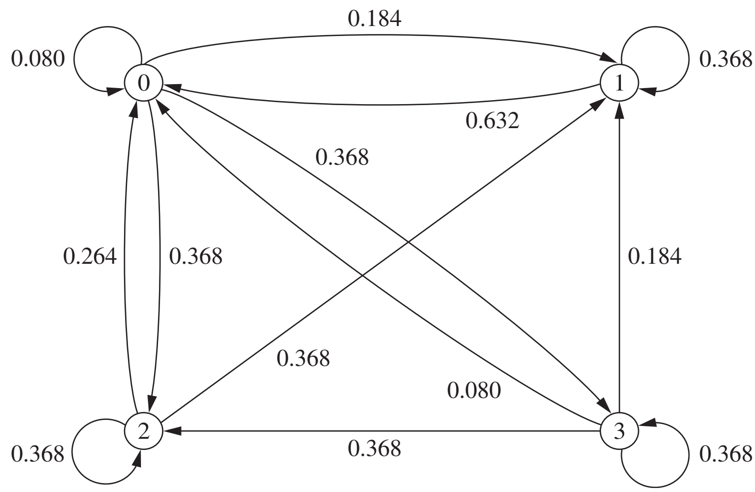
$$p_{00} = P\{D_{t+1} \geq 3\} = 0.080.$$

For the other rows of **P,** the formula at the end of Sec. 29.1 for the next state is

$$X_{t+1} = \max\{X_t - D_{t+1}, 0\} \qquad \text{if} \qquad X_t \geq 1.$$

This implies that $X_{t+1} \leq X_t$, so $p_{12} = 0$, $p_{13} = 0$, and $p_{23} = 0$. For the other transitions,

$$p_{11} = P\{D_{t+1} = 0\} = 0.368,$$
$$p_{10} = P\{D_{t+1} \geq 1) = 1 - P\{D_{t+1} = 0\} = 0.632,$$
$$p_{22} = P\{D_{t+1} = 0\} = 0.368,$$
$$p_{21} = P\{D_{t+1} = 1\} = 0.368,$$
$$p_{20} = P\{D_{t+1} \geq 2\} = 1 - P\{D_{t+1} \leq 1\} = 1 - (0.368 + 0.368) = 0.264.$$



■ **FIGURE 29.2**
The state transition diagram for the inventory example.

For the last row of **P,** week $t + 1$ begins with 3 cameras in inventory, so the calculations for the transition probabilities are exactly the same as for the first row. Consequently, the complete transition matrix (to three significant digits) is

$$\mathbf{P} = \begin{array}{c} \text{State} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & \quad 1 & \quad 2 & \quad 3 \end{array} \\ \begin{bmatrix} 0.080 & 0.184 & 0.368 & 0.368 \\ 0.632 & 0.368 & 0 & 0 \\ 0.264 & 0.368 & 0.368 & 0 \\ 0.080 & 0.184 & 0.368 & 0.368 \end{bmatrix} \end{array}$$

The information given by this transition matrix can also be depicted graphically with the state transition diagram in Fig. 29.2. The four possible states for the number of cameras on hand at the end of a week are represented by the four nodes (circles) in the diagram. The arrows show the possible transitions from one state to another, or sometimes from a state back to itself, when the camera store goes from the end of one week to the end of the next week. The number next to each arrow gives the probability of that particular transition occurring next when the camera store is in the state at the base of the arrow.

Additional Examples of Markov Chains

A Stock Example. Consider the following model for the value of a stock. At the end of a given day, the price is recorded. If the stock has gone up, the probability that it will go up tomorrow is **0.7.** If the stock has gone down, the probability that it will go up tomorrow is only **0.5.** (For simplicity, we will count the stock staying the same as a decrease.) This is a Markov chain, where the possible states for each day are as follows:

State 0: The stock increased on this day.
State 1: The stock decreased on this day.

The transition matrix that shows each probability of going from a particular state today to a particular state tomorrow is given by

$$\mathbf{P} = \begin{array}{c} \text{State} \\ 0 \\ 1 \end{array} \begin{array}{c} \begin{array}{cc} 0 & 1 \end{array} \\ \begin{bmatrix} 0.7 & 0.3 \\ 0.5 & 0.5 \end{bmatrix} \end{array}$$

The form of the state transition diagram for this example is exactly the same as for the weather example shown in Fig. 29.1, so we will not repeat it here. The only difference is that the transition probabilities in the diagram are slightly different (0.7 replaces 0.8, 0.3 replaces 0.2, and 0.5 replaces both 0.6 and 0.4 in Fig. 29.1).

A Second Stock Example. Suppose now that the stock market model is changed so that the stock's going up tomorrow depends upon whether it increased today *and* yesterday. In particular, if the stock has increased for the past two days, it will increase tomorrow with probability **0.9.** If the stock increased today but decreased yesterday, then it will increase tomorrow with probability **0.6.** If the stock decreased today but increased yesterday, then it will increase tomorrow with probability **0.5.** Finally, if the stock decreased for the past two days, then it will increase tomorrow with probability **0.3.** If we define the state as representing whether the stock goes up or down today, the system is no longer a Markov chain. However, we can transform the system to a Markov chain by defining the states as follows:[2]

State 0: The stock increased both today and yesterday.
State 1: The stock increased today and decreased yesterday.
State 2: The stock decreased today and increased yesterday.
State 3: The stock decreased both today and yesterday.

This leads to a four-state Markov chain with the following transition matrix:

$$\mathbf{P} = \begin{array}{c} \text{State} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0.9 & 0 & 0.1 & 0 \\ 0.6 & 0 & 0.4 & 0 \\ 0 & 0.5 & 0 & 0.5 \\ 0 & 0.3 & 0 & 0.7 \end{bmatrix} \end{array}$$

Figure 29.3 shows the state transition diagram for this example. An interesting feature of the example revealed by both this diagram and all the values of 0 in the transition matrix is that so many of the transitions from state i to state j are impossible in one step. In other words, $p_{ij} = 0$ for 8 of the 16 entries in the transition matrix. However, check out how it always is possible to go from any state i to any state j (including $j = i$) in two steps. The same holds true for three steps, four steps, and so forth. Thus, $p_{ij}^{(n)} > 0$ for n $= 2, 3, \ldots$ for all i and j.

A Gambling Example. Another example involves gambling. Suppose that a player has \$1 and with each play of the game wins \$1 with probability $p > 0$ or loses \$1 with probability $1 - p > 0$. The game ends when the player either accumulates \$3 or goes broke. This game is a Markov chain with the states representing the player's current holding of money, that is, 0, \$1, \$2, or \$3, and with the transition matrix given by

$$\mathbf{P} = \begin{array}{c} \text{State} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1-p & 0 & p & 0 \\ 0 & 1-p & 0 & p \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

[2]We again are counting the stock staying the same as a decrease. This example demonstrates that Markov chains are able to incorporate arbitrary amounts of history, but at the cost of significantly increasing the number of states.

FIGURE 29.3 The state transition diagram for the second stock example.

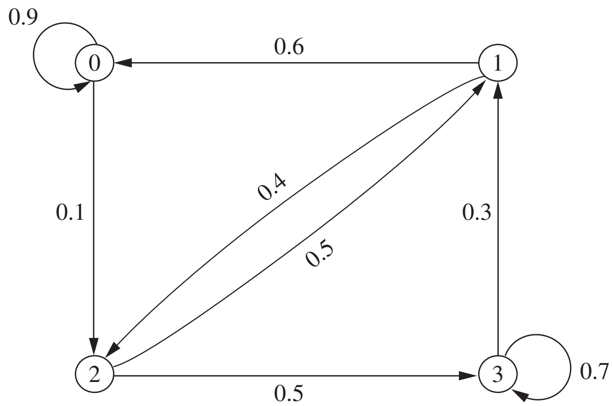
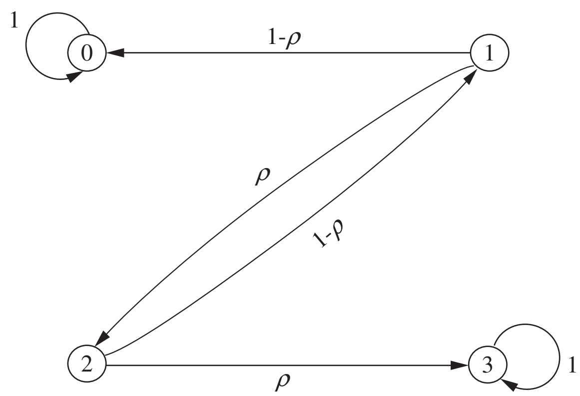


FIGURE 29.4 The state transition diagram for the gambling example.



The state transition diagram for this example is shown in Fig. 29.4. This diagram demonstrates that once the process enters either state 0 or state 3, it will stay in that state forever after, since $p_{00} = 1$ and $p_{33} = 1$. States 0 and 3 are examples of what are called an **absorbing state** (a state that is never left once the process enters that state). We will focus on analyzing absorbing states in Sec. 29.7.

Note that in both the inventory and gambling examples, the numeric labeling of the states that the process reaches coincides with the physical expression of the system—i.e., actual inventory levels and the player's holding of money, respectively—whereas the numeric labeling of the states in the weather and stock examples has no physical significance.

29.3 CHAPMAN-KOLMOGOROV EQUATIONS

Section 29.2 introduced the n-step transition probability $p_{ij}^{(n)}$. The following *Chapman-Kolmogorov equations* provide a method for computing these n-step transition probabilities:

$$p_{ij}^{(n)} = \sum_{k=0}^{M} p_{ik}^{(m)} p_{kj}^{(n-m)}, \qquad \begin{aligned} &\text{for all } i = 0, 1, \ldots, M, \\ &j = 0, 1, \ldots, M, \\ &\text{and any } m = 1, 2, \ldots, n-1, \\ &n = m+1, m+2, \ldots .^3 \end{aligned}$$

[3] These equations also hold in a trivial sense when $m = 0$ or $m = n$, but $m = 1, 2, \ldots, n-1$ are the only interesting cases.

These equations point out that in going from state i to state j in n steps, the process will be in some state k after exactly m (less than n) steps. Thus, $p_{ik}^{(m)}\, p_{kj}^{(n-m)}$ is just the conditional probability that, given a starting point of state i, the process goes to state k after m steps and then to state j in $n - m$ steps. Therefore, summing these conditional probabilities over all possible k must yield $p_{ij}^{(n)}$. The special cases of $m = 1$ and $m = n - 1$ lead to the expressions

$$p_{ij}^{(n)} = \sum_{k=0}^{M} p_{ik} p_{kj}^{(n-1)}$$

and

$$p_{ij}^{(n)} = \sum_{k=0}^{M} p_{ik}^{(n-1)} p_{kj},$$

for all states i and j. These expressions enable the n-step transition probabilities to be obtained from the one-step transition probabilities recursively. This recursive relationship is best explained in matrix notation (see Appendix 4). For $n = 2$, these expressions become

$$p_{ij}^{(2)} = \sum_{k=0}^{M} p_{ik} p_{kj}, \qquad \text{for all states } i \text{ and } j,$$

where the $p_{ij}^{(2)}$ are the elements of a matrix $\mathbf{P}^{(2)}$. Also note that these elements are obtained by multiplying the matrix of one-step transition probabilities by itself; i.e.,

$$\mathbf{P}^{(2)} = \mathbf{P} \cdot \mathbf{P} = \mathbf{P}^2.$$

In the same manner, the above expressions for $p_{ij}^{(n)}$ when $m = 1$ and $m = n - 1$ indicate that the matrix of n-step transition probabilities is

$$\begin{aligned} \mathbf{P}^{(n)} &= \mathbf{P}\mathbf{P}^{(n-1)} = \mathbf{P}^{(n-1)}\mathbf{P} \\ &= \mathbf{P}\mathbf{P}^{n-1} = \mathbf{P}^{n-1}\mathbf{P} \\ &= \mathbf{P}^n. \end{aligned}$$

Thus, the n-step transition probability matrix $\mathbf{P}^n$ can be obtained by computing the nth power of the one-step transition matrix $\mathbf{P}$.

n-Step Transition Matrices for the Weather Example

For the weather example introduced in Sec. 29.1, we now will use the above formulas to calculate various n-step transition matrices from the (one-step) transition matrix $\mathbf{P}$ that was obtained in Sec. 29.2. To start, the two-step transition matrix is

$$\mathbf{P}^{(2)} = \mathbf{P} \cdot \mathbf{P} = \begin{bmatrix} 0.8 & 0.2 \\ 0.6 & 0.4 \end{bmatrix} \begin{bmatrix} 0.8 & 0.2 \\ 0.6 & 0.4 \end{bmatrix} = \begin{bmatrix} 0.76 & 0.24 \\ 0.72 & 0.28 \end{bmatrix}.$$

Thus, if the weather is in state 0 (dry) on a particular day, the probability of being in state 0 two days later is 0.76 and the probability of being in state 1 (rain) then is 0.24. Similarly, if the weather is in state 1 now, the probability of being in state 0 two days later is 0.72 whereas the probability of being in state 1 then is 0.28.

The probabilities of the state of the weather three, four, or five days into the future also can be read in the same way from the three-step, four-step, and five-step transition matrices calculated to three significant digits below.

$$\mathbf{P}^{(3)} = \mathbf{P}^3 = \mathbf{P} \cdot \mathbf{P}^2 = \begin{bmatrix} 0.8 & 0.2 \\ 0.6 & 0.4 \end{bmatrix} \begin{bmatrix} 0.76 & 0.24 \\ 0.72 & 0.28 \end{bmatrix} = \begin{bmatrix} 0.752 & 0.248 \\ 0.744 & 0.256 \end{bmatrix}$$

$$\mathbf{P}^{(4)} = \mathbf{P}^4 = \mathbf{P} \cdot \mathbf{P}^3 = \begin{bmatrix} 0.8 & 0.2 \\ 0.6 & 0.4 \end{bmatrix} \begin{bmatrix} 0.752 & 0.248 \\ 0.744 & 0.256 \end{bmatrix} = \begin{bmatrix} 0.75 & 0.25 \\ 0.749 & 0.251 \end{bmatrix}$$

$$\mathbf{P}^{(5)} = \mathbf{P}^5 = \mathbf{P} \cdot \mathbf{P}^4 = \begin{bmatrix} 0.8 & 0.2 \\ 0.6 & 0.4 \end{bmatrix} \begin{bmatrix} 0.75 & 0.25 \\ 0.749 & 0.251 \end{bmatrix} = \begin{bmatrix} 0.75 & 0.25 \\ 0.75 & 0.25 \end{bmatrix}$$

Note that the five-step transition matrix has the interesting feature that the two rows have identical entries (after rounding to three significant digits). This reflects the fact that the probability of the weather being in a particular state is essentially independent of the state of the weather five days before. Thus, the probabilities in either row of this five-step transition matrix are referred to as the *steady-state probabilities* of this Markov chain.

We will expand further on the subject of the steady-state probabilities of a Markov chain, including how to derive them more directly, at the beginning of Sec. 29.5.

n-Step Transition Matrices for the Inventory Example

Returning to the inventory example included in Sec. 29.1, we now will calculate its n-step transition matrices to three decimal places for $n = 2$, 4, and 8. To start, its one-step transition matrix $\mathbf{P}$ obtained in Sec. 29.2 can be used to calculate the two-step transition matrix $\mathbf{P}^{(2)}$ as follows:

$$\mathbf{P}^{(2)} = \mathbf{P}^2 = \begin{bmatrix} 0.080 & 0.184 & 0.368 & 0.368 \\ 0.632 & 0.368 & 0 & 0 \\ 0.264 & 0.368 & 0.368 & 0 \\ 0.080 & 0.184 & 0.368 & 0.368 \end{bmatrix} \begin{bmatrix} 0.080 & 0.184 & 0.368 & 0.368 \\ 0.632 & 0.368 & 0 & 0 \\ 0.264 & 0.368 & 0.368 & 0 \\ 0.080 & 0.184 & 0.368 & 0.368 \end{bmatrix}$$

$$= \begin{bmatrix} 0.249 & 0.286 & 0.300 & 0.165 \\ 0.283 & 0.252 & 0.233 & 0.233 \\ 0.351 & 0.319 & 0.233 & 0.097 \\ 0.249 & 0.286 & 0.300 & 0.165 \end{bmatrix}.$$

For example, given that there is one camera left in stock at the end of a week, the probability is 0.283 that there will be no cameras in stock 2 weeks later, that is, $p_{10}^{(2)} = 0.283$. Similarly, given that there are two cameras left in stock at the end of a week, the probability is 0.097 that there will be three cameras in stock 2 weeks later, that is, $p_{23}^{(2)} = 0.097$.

The four-step transition matrix can also be obtained as follows:

$$\mathbf{P}^{(4)} = \mathbf{P}^4 = \mathbf{P}^{(2)} \cdot \mathbf{P}^{(2)}$$

$$= \begin{bmatrix} 0.249 & 0.286 & 0.300 & 0.165 \\ 0.283 & 0.252 & 0.233 & 0.233 \\ 0.351 & 0.319 & 0.233 & 0.097 \\ 0.249 & 0.286 & 0.300 & 0.165 \end{bmatrix} \begin{bmatrix} 0.249 & 0.286 & 0.300 & 0.165 \\ 0.283 & 0.252 & 0.233 & 0.233 \\ 0.351 & 0.319 & 0.233 & 0.097 \\ 0.249 & 0.286 & 0.300 & 0.165 \end{bmatrix}$$

$$= \begin{bmatrix} 0.289 & 0.286 & 0.261 & 0.164 \\ 0.282 & 0.285 & 0.268 & 0.166 \\ 0.284 & 0.283 & 0.263 & 0.171 \\ 0.289 & 0.286 & 0.261 & 0.164 \end{bmatrix}.$$

For example, given that there is one camera left in stock at the end of a week, the probability is 0.282 that there will be no cameras in stock 4 weeks later, that is, $p_{10}^{(4)} = 0.282$. Similarly, given that there are two cameras left in stock at the end of a week, the probability is 0.171 that there will be three cameras in stock 4 weeks later, that is, $p_{23}^{(4)} = 0.171$.

The transition probabilities for the number of cameras in stock 8 weeks from now can be read in the same way from the eight-step transition matrix calculated below.

$$\mathbf{P}^{(8)} = \mathbf{P}^8 = \mathbf{P}^{(4)} \cdot \mathbf{P}^{(4)}$$

$$= \begin{bmatrix} 0.289 & 0.286 & 0.261 & 0.164 \\ 0.282 & 0.285 & 0.268 & 0.166 \\ 0.284 & 0.283 & 0.263 & 0.171 \\ 0.289 & 0.286 & 0.261 & 0.164 \end{bmatrix} \begin{bmatrix} 0.289 & 0.286 & 0.261 & 0.164 \\ 0.282 & 0.285 & 0.268 & 0.166 \\ 0.284 & 0.283 & 0.263 & 0.171 \\ 0.289 & 0.286 & 0.261 & 0.164 \end{bmatrix}$$

$$= \begin{array}{c} \text{State} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} \;\;0\;\; & \;\;\;1\;\;\; & \;\;\;2\;\;\; & \;\;\;3\;\; \end{array} \\ \begin{bmatrix} 0.286 & 0.285 & 0.264 & 0.166 \\ 0.286 & 0.285 & 0.264 & 0.166 \\ 0.286 & 0.285 & 0.264 & 0166 \\ 0.286 & 0.285 & 0.264 & 0.166 \end{bmatrix} \end{array}$$

Like the five-step transition matrix for the weather example, this matrix has the interesting feature that its rows have identical entries (after rounding). The reason once again is that probabilities in any row are the *steady-state probabilities* for this Markov chain, i.e., the probabilities of the state of the system after enough time has elapsed that the initial state is no longer relevant.

Your IOR Tutorial includes a procedure for calculating $\mathbf{P}^{(n)} = \mathbf{P}^n$ for any positive integer $n \leq 99$.

Unconditional State Probabilities

Recall that one- or n-step transition probabilities are *conditional* probabilities; for example, $P\{X_n = j \mid X_0 = i\} = p_{ij}^{(n)}$. Assume that n is small enough that these conditional probabilities are not yet *steady-state* probabilities. In this case, if the *unconditional* probability $P\{X_n = j\}$ is desired, it is necessary to specify the probability distribution of the initial state, namely, $P\{X_0 = i\}$ for $i = 0, 1, \ldots, M$. Then

$$P\{X_n = j\} = P\{X_0 = 0\}\, p_{0j}^{(n)} + P\{X_0 = 1\}p_{1j}^{(n)} + \cdots + P\{X_0 = M\}p_{Mj}^{(n)}.$$

In the inventory example, it was assumed that initially there were 3 units in stock, that is, $X_0 = 3$. Thus, $P\{X_0 = 0\} = P\{X_0 = 1\} = P\{X_0 = 2\} = 0$ and $P\{X_0 = 3\} = 1$. Hence, the (unconditional) probability that there will be three cameras in stock 2 weeks after the inventory system began is $P\{X_2 = 3\} = (1)p_{33}^{(2)} = 0.165$.

29.4 CLASSIFICATION OF STATES OF A MARKOV CHAIN

We have just seen near the end of the preceding section that the n-step transition probabilities for the inventory example converge to steady-state probabilities after a sufficient number of steps. However, this is not true for all Markov chains. The long-run properties of a Markov chain depend greatly on the characteristics of its states and transition matrix. To further describe the properties of Markov chains, it is necessary to present some concepts and definitions concerning these states.

State j is said to be **accessible** from state i if $p_{ij}^{(n)} > 0$ for some $n \geq 0$. (Recall that $p_{ij}^{(n)}$ is just the conditional probability of being in state j after n steps, starting in state i.) Thus, state j being accessible from state i means that it is possible for the system to enter state j eventually when it starts from state i. This is clearly true for the weather example (see Fig. 29.1) since $p_{ij} > 0$ for all i and j. In the inventory example (see Fig. 29.2), $p_{ij}^{(2)} > 0$ for all i and j, so every state is accessible from every other state. In general, a sufficient condition for *all* states to be accessible is that there exists a value of n for which $p_{ij}^{(n)} > 0$ for all i and j.

In the gambling example given at the end of Sec. 29.2 (see Fig. 29.4), state 2 is not accessible from state 3. This can be deduced from the context of the game (once the player reaches state 3, the player never leaves this state), which implies that $p_{32}^{(n)} = 0$ for all $n \geq 0$. However, even though state 2 is *not* accessible from state 3, state 3 *is* accessible from state 2 since, for $n = 1$, the transition matrix given at the end of Sec. 29.2 indicates that $p_{23} = p > 0$.

If state j is accessible from state i and state i is accessible from state j, then states i and j are said to **communicate.** In both the weather and inventory examples, all states communicate. In the gambling example, states 2 and 3 do not. (The same is true of states 1 and 3, states 1 and 0, and states 2 and 0.) In general,

1. Any state communicates with itself (because $p_{ii}^{(0)} = P\{X_0 = i \mid X_0 = i\} = 1$).
2. If state i communicates with state j, then state j communicates with state i.
3. If state i communicates with state j and state j communicates with state k, then state i communicates with state k.

Properties 1 and 2 follow from the definition of states communicating, whereas property 3 follows from the Chapman-Kolmogorov equations.

As a result of these three properties of communication, the states may be partitioned into one or more separate **classes** such that those states that communicate with each other are in the same class. (A class may consist of a single state.) If there is only one class, i.e., all the states communicate, the Markov chain is said to be **irreducible.** In both the weather and inventory examples, the Markov chain is irreducible. In both of the stock examples in Sec. 29.2, the Markov chain also is irreducible. However, the gambling example contains three classes. Observe in Fig. 29.4 how state 0 forms a class, state 3 forms a class, and states 1 and 2 form a class.

Recurrent States and Transient States

It is often useful to talk about whether a process entering a state will ever return to this state. Here is one possibility.

> A state is said to be a **transient** state if, upon entering this state, the process *might never return* to this state again. Therefore, state i is transient if and only if there exists a state j ($j \neq i$) that is accessible from state i but not vice versa, that is, state i is not accessible from state j.

Thus, if state i is transient and the process visits this state, there is a positive probability (perhaps even a probability of 1) that the process will later move to state j and so will never return to state i. Consequently, a transient state will be visited only a finite number of times. To illustrate, consider the gambling example presented at the end of Sec. 29.2. Its state transition diagram shown in Fig. 29.4 indicates that both states 1 and 2 are transient states since the process will leave these states sooner or later to enter either state 0 or state 3 and then will remain in that state forever.

When starting in state i, another possibility is that the process *definitely* will return to this state.

> A state is said to be a **recurrent** state if, upon entering this state, the process *definitely will return* to this state again. Therefore, a state is recurrent if and only if it is not transient.

Since a recurrent state definitely will be revisited after each visit, it will be visited infinitely often if the process continues forever. For example, all the states in the state transition diagrams shown in Figs. 29.1, 29.2, and 29.3 are recurrent states because the process always will return to each of these states. Even for the gambling example, states 0 and 3 are recurrent states because the process will keep returning immediately to one of these states forever once the process enters that state. Note in Fig. 29.4 how the process eventually will enter either state 0 or state 3 and then will never leave that state again.

If the process enters a certain state and then stays in this state at the next step, this is considered a *return* to this state. Hence, the following kind of state is a special type of recurrent state.

> A state is said to be an **absorbing** state if, upon entering this state, the process *never will leave* this state again. Therefore, state i is an absorbing state if and only if $p_{ii} = 1$.

As just noted, both states 0 and 3 for the gambling example fit this definition, so they both are absorbing states as well as a special type of recurrent state. We will discuss absorbing states further in Sec. 29.7.

Recurrence is a class property. That is, all states in a class are either recurrent or transient. Furthermore, in a finite-state Markov chain, not all states can be transient. Therefore, all states in an irreducible finite-state Markov chain are recurrent. Indeed, one can identify an irreducible finite-state Markov chain (and therefore conclude that all states are recurrent) by showing that all states of the process communicate. It has already been pointed out that a sufficient condition for *all* states to be accessible (and therefore communicate with each other) is that there exists a value of n for which $p_{ij}^{(n)} > 0$ for all i and j. Thus, all states in the inventory example (see Fig. 29.2) are recurrent, since $p_{ij}^{(2)}$ is positive for all i and j. Similarly, both the weather example and the first stock example contain only recurrent states, since p_{ij} is positive for all i and j. By calculating $p_{ij}^{(2)}$ for all i and j in the second stock example in Sec. 29.2 (see Fig. 29.3), it follows that all states are recurrent since $p_{ij}^{(2)} > 0$ for all i and j.

As another example, suppose that a Markov chain has the following transition matrix:

$$
\mathbf{P} = \begin{array}{c} \text{State} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array}
\begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\
\begin{bmatrix}
\frac{1}{4} & \frac{3}{4} & 0 & 0 & 0 \\
\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & \frac{1}{3} & \frac{2}{3} & 0 \\
1 & 0 & 0 & 0 & 0
\end{bmatrix} \end{array}
$$

Note that state 2 is an absorbing state (and hence a recurrent state) because if the process enters state 2 (row 3 of the matrix), it will never leave. State 3 is a transient state because if the process is in state 3, there is a positive probability that it will never return. The probability is $\frac{1}{3}$ that the process will go from state 3 to state 2 on the first step. Once the process is in state 2, it remains in state 2. State 4 also is a transient state because if the process starts in state 4, it immediately leaves and can never return. States 0 and 1 are recurrent states. To see this, observe from $\mathbf{P}$ that if the process starts in either of

these states, it can never leave these two states. Furthermore, whenever the process moves from one of these states to the other one, it always will return to the original state eventually.

Periodicity Properties

Another useful property of Markov chains is *periodicities.* The **period** of state i is defined to be the integer t ($t > 1$) such that $p_{ii}^{(n)} = 0$ for all values of n other than $t, 2t, 3t, \ldots$ and t is the smallest integer with this property. In the gambling example (end of Section 29.2), starting in state 1, it is possible for the process to enter state 1 only at times 2, 4, . . . , so state 1 has period 2. The reason is that the player can break even (be neither winning nor losing) only at times 2, 4, . . . , which can be verified by calculating $p_{11}^{(n)}$ for all n and noting that $p_{11}^{(n)} = 0$ for n odd. You also can see in Fig. 29.4 that the process always takes two steps to return to state 1 until the process gets absorbed in either state 0 or state 3. (The same conclusion also applies to state 2.)

If there are two consecutive numbers s and $s + 1$ such that the process can be in state i at times s and $s + 1$, the state is said to have period 1 and is called an **aperiodic** state.

Just as recurrence is a class property, it can be shown that periodicity is a class property. That is, if state i in a class has period t, then all states in that class have period t. In the gambling example, state 2 also has period 2 because it is in the same class as state 1 and we noted above that state 1 has period 2.

It is possible for a Markov chain to have both a recurrent class of states and a transient class of states where the two classes have different periods greater than 1.

In a finite-state Markov chain, recurrent states that are aperiodic are called **ergodic** states. A Markov chain is said to be *ergodic* if all its states are ergodic states. You will see next that a key long-run property of a Markov chain that is both irreducible and ergodic is that its n-step transition probabilities will converge to steady-state probabilities as n grows large.

29.5 LONG-RUN PROPERTIES OF MARKOV CHAINS

Steady-State Probabilities

While calculating the n-step transition probabilities for both the weather and inventory examples in Sec. 29.3, we noted an interesting feature of these matrices. If n is large enough ($n = 5$ for the weather example and $n = 8$ for the inventory example), all the rows of the matrix have identical entries, so the probability that the system is in each state j no longer depends on the initial state of the system. In other words, there is a limiting probability that the system will be in each state j after a large number of transitions, and this probability is independent of the initial state. These properties of the long-run behavior of finite-state Markov chains do, in fact, hold under relatively general conditions, as summarized below.

For any irreducible ergodic Markov chain, $\lim_{n\to\infty} p_{ij}^{(n)}$ exists and is independent of i. Furthermore,

$$\lim_{n\to\infty} p_{ij}^{(n)} = \pi_j > 0,$$

where the π_j uniquely satisfy the following **steady-state equations**

$$\pi_j = \sum_{i=0}^{M} \pi_i p_{ij}, \qquad \text{for } j = 0, 1, \ldots, M,$$

$$\sum_{j=0}^{M} \pi_j = 1.$$

If you prefer to work with a system of equations in matrix form, this system (excluding the sum = 1 equation) also can be expressed as

$$\pi = \pi \mathbf{P},$$

where $\pi = (\pi_0, \pi_1, \ldots, \pi_M)$.

The π_j are called the **steady-state probabilities** of the Markov chain. The term *steady-state* probability means that the probability of finding the process in a certain state, say j, after a large number of transitions tends to the value π_j, independent of the probability distribution of the initial state. It is important to note that the steady-state probability does *not* imply that the process settles down into one state. On the contrary, the process continues to make transitions from state to state, and at any step n the transition probability from state i to state j is still p_{ij}.

The π_j can also be interpreted as *stationary probabilities* (not to be confused with stationary transition probabilities) in the following sense. If the *initial* probability of being in state j is given by π_j (that is, $P\{X_0 = j\} = \pi_j$) for all j, then the probability of finding the process in state j at time $n = 1, 2, \ldots$ is also given by π_j (that is, $P\{X_n = j\} = \pi_j$).

Note that the steady-state equations consist of $M + 2$ equations in $M + 1$ unknowns. Because it has a unique solution, at least one equation must be redundant and can, therefore, be deleted. It cannot be the equation

$$\sum_{j=0}^{M} \pi_j = 1,$$

because $\pi_j = 0$ for all j will satisfy the other $M + 1$ equations. Furthermore, the solutions to the other $M + 1$ steady-state equations have a unique solution up to a multiplicative constant, and it is the final equation that forces the solution to be a probability distribution.

Application to the Weather Example. The weather example introduced in Sec. 29.1 and formulated in Sec. 29.2 has only two states (dry and rain), so the above steady-state equations become

$$\begin{aligned} \pi_0 &= \pi_0 p_{00} + \pi_1 p_{10}, \\ \pi_1 &= \pi_0 p_{01} + \pi_1 p_{11}, \\ 1 &= \pi_0 \quad\;\; + \pi_1. \end{aligned}$$

The intuition behind the first equation is that, in steady state, the probability of being in state 0 after the next transition must equal (1) the probability of being in state 0 now *and* then staying in state 0 after the next transition *plus* (2) the probability of being in state 1 now *and* next making the transition to state 0. The logic for the second equation is the same, except in terms of state 1. The third equation simply expresses the fact that the probabilities of these mutually exclusive states must sum to 1.

Referring to the transition probabilities given in Sec. 29.2 for this example, these equations become

$$\begin{aligned} \pi_0 &= 0.8\pi_0 + 0.6\pi_1, &\quad \text{so} \quad & 0.2\pi_0 = 0.6\pi_1, \\ \pi_1 &= 0.2\pi_0 + 0.4\pi_1, &\quad \text{so} \quad & 0.6\pi_1 = 0.2\pi_0, \\ 1 &= \pi_0 \quad + \pi_1. \end{aligned}$$

Note that one of the first two equations is redundant since both equations reduce to $\pi_0 = 3\pi_1$. Combining this result with the third equation immediately yields the following steady-state probabilities:

$$\pi_0 = 0.75, \qquad \pi_1 = 0.25$$

These are the same probabilities as obtained in each row of the five-step transition matrix calculated in Sec. 29.3 because five transitions proved enough to make the state probabilities essentially independent of the initial state.

Application to the Inventory Example. The inventory example introduced in Sec. 29.1 and formulated in Sec. 29.2 has four states. Therefore, in this case, the steady-state equations can be expressed as

$$\begin{aligned} \pi_0 &= \pi_0 p_{00} + \pi_1 p_{10} + \pi_2 p_{20} + \pi_3 p_{30}, \\ \pi_1 &= \pi_0 p_{01} + \pi_1 p_{11} + \pi_2 p_{21} + \pi_3 p_{31}, \\ \pi_2 &= \pi_0 p_{02} + \pi_1 p_{12} + \pi_2 p_{22} + \pi_3 p_{32}, \\ \pi_3 &= \pi_0 p_{03} + \pi_1 p_{13} + \pi_2 p_{23} + \pi_3 p_{33}, \\ 1 &= \pi_0 \quad + \pi_1 \quad + \pi_2 \quad + \pi_3. \end{aligned}$$

Substituting values for p_{ij} (see the transition matrix in Sec. 29.2) into these equations leads to the equations

$$\begin{aligned} \pi_0 &= 0.080\pi_0 + 0.632\pi_1 + 0.264\pi_2 + 0.080\pi_3, \\ \pi_1 &= 0.184\pi_0 + 0.368\pi_1 + 0.368\pi_2 + 0.184\pi_3, \\ \pi_2 &= 0.368\pi_0 \qquad\qquad\quad + 0.368\pi_2 + 0.368\pi_3, \\ \pi_3 &= 0.368\pi_0 \qquad\qquad\qquad\qquad\qquad + 0.368\pi_3, \\ 1 &= \quad \pi_0 + \quad \pi_1 + \quad \pi_2 + \quad \pi_3. \end{aligned}$$

Solving the last four equations simultaneously provides the solution

$$\pi_0 = 0.286, \qquad \pi_1 = 0.285, \qquad \pi_2 = 0.263, \qquad \pi_3 = 0.166,$$

which is essentially the result that appears in matrix $\mathbf{P}^{(8)}$ in Sec. 29.3. Thus, after many weeks the probability of finding zero, one, two, and three cameras in stock at the end of a week tends to 0.286, 0.285, 0.263, and 0.166, respectively.

More about Steady-State Probabilities. Your IOR Tutorial includes a procedure for solving the steady-state equations to obtain the steady-state probabilities.

There are other important results concerning steady-state probabilities. In particular, if i and j are recurrent states belonging to different classes, then

$$p_{ij}^{(n)} = 0, \qquad \text{for all } n.$$

This result follows from the definition of a class.

Similarly, if j is a transient state, then

$$\lim_{n \to \infty} p_{ij}^{(n)} = 0, \qquad \text{for all } i.$$

Thus, the probability of finding the process in a transient state after a large number of transitions tends to zero.

Expected Average Cost per Unit Time

The preceding subsection dealt with irreducible finite-state Markov chains whose states were ergodic (recurrent and aperiodic). If the requirement that the states be aperiodic is relaxed, then the limit

$$\lim_{n\to\infty} p_{ij}^{(n)}$$

may not exist. To illustrate this point, consider the two-state transition matrix

$$\mathbf{P} = \begin{array}{c c} \text{State} & \begin{matrix} 0 & 1 \end{matrix} \\ \begin{matrix} 0 \\ 1 \end{matrix} & \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \end{array}.$$

If the process starts in state 0 at time 0, it will be in state 0 at times 2, 4, 6, . . . and in state 1 at times 1, 3, 5, ... Thus, $p_{00}^{(n)} = 1$ if n is even and $p_{00}^{(n)} = 0$ if n is odd, so that

$$\lim_{n\to\infty} p_{00}^{(n)}$$

does not exist. However, the following limit always exists for an irreducible (finite-state) Markov chain:

$$\lim_{n\to\infty} \left(\frac{1}{n} \sum_{k=1}^{n} p_{ij}^{(k)} \right) = \pi_j,$$

where the π_j satisfy the steady-state equations given in the preceding subsection.

This result is important in computing the *long-run average cost per unit time* associated with a Markov chain. Suppose that a cost (or other penalty function) $C(X_t)$ is incurred when the process is in state X_t at time t, for $t = 0, 1, 2, \ldots$. Note that $C(X_t)$ is a random variable that takes on any one of the values $C(0), C(1), \ldots, C(M)$ and that the function $C(\cdot)$ is independent of t. The expected average cost incurred over the first n periods is given by

$$E\left[\frac{1}{n} \sum_{t=1}^{n} C(X_t) \right].$$

By using the result that

$$\lim_{n\to\infty} \left(\frac{1}{n} \sum_{k=1}^{n} p_{ij}^{(k)} \right) = \pi_j,$$

it can be shown that the (long-run) *expected average cost per unit time* is given by

$$\lim_{n\to\infty} E\left[\frac{1}{n} \sum_{t=1}^{n} C(X_t) \right] = \sum_{j=0}^{M} \pi_j C(j).$$

Application to the Inventory Example. To illustrate, consider the inventory example introduced in Sec. 29.1, where the solution for the π_j was obtained in an earlier subsection. Suppose the camera store finds that a storage charge is being allocated for each camera remaining on the shelf at the end of the week. The cost is charged as follows:

$$C(x_t) = \begin{cases} 0 & \text{if} \quad x_t = 0 \\ 2 & \text{if} \quad x_t = 1 \\ 8 & \text{if} \quad x_t = 2 \\ 18 & \text{if} \quad x_t = 3 \end{cases}$$

Using the steady-state probabilities found earlier in this section, the long-run expected average storage cost per week can then be obtained from the preceding equation, i.e.,

$$\lim_{n\to\infty} E\left[\frac{1}{n}\sum_{t=1}^{n} C(X_t)\right] = 0.286(0) + 0.285(2) + 0.263(8) + 0.166(18) = 5.662.$$

Note that an alternative measure to the (long-run) expected average cost per unit time is the (long-run) *actual average cost per unit time.* It can be shown that this latter measure also is given by

$$\lim_{n\to\infty} \left[\frac{1}{n}\sum_{t=1}^{n} C(X_t)\right] = \sum_{j=0}^{M} \pi_j C(j)$$

for essentially all paths of the process. Thus, either measure leads to the same result. These results can also be used to interpret the meaning of the π_j. To do so, let

$$C(X_t) = \begin{cases} 1 & \text{if} \quad X_t = j \\ 0 & \text{if} \quad X_t \neq j. \end{cases}$$

The (long-run) expected fraction of times the system is in state j is then given by

$$\lim_{n\to\infty} E\left[\frac{1}{n}\sum_{t=1}^{n} C(X_t)\right] = \lim_{n\to\infty} E(\text{fraction of times system is in state } j) = \pi_j.$$

Similarly, π_j can also be interpreted as the (long-run) actual fraction of times that the system is in state j.

Expected Average Cost per Unit Time for Complex Cost Functions

In the preceding subsection, the cost function was based solely on the state that the process is in at time t. In many important problems encountered in practice, the cost may also depend upon some other random variable.

For example, in the inventory example introduced in Sec. 29.1, suppose that the costs to be considered are the ordering cost and the penalty cost for unsatisfied demand (storage costs are so small they will be ignored). It is reasonable to assume that the number of cameras ordered to arrive at the beginning of week t depends only upon the state of the process X_{t-1} (the number of cameras in stock) when the order is placed at the end of week $t - 1$. However, the cost of unsatisfied demand in week t will also depend upon the demand D_t. Therefore, the total cost (ordering cost plus cost of unsatisfied demand) for week t is a function of X_{t-1} and D_t, that is, $C(X_{t-1}, D_t)$.

Under the assumptions of this example, it can be shown that the (long-run) *expected average cost per unit time* is given by

$$\lim_{n\to\infty} E\left[\frac{1}{n}\sum_{t=1}^{n} C(X_{t-1}, D_t)\right] = \sum_{j=0}^{M} k(j)\, \pi_j,$$

where

$$k(j) = E[C(j, D_t)],$$

and where this latter (conditional) expectation is taken with respect to the probability distribution of the random variable D_t, given the state j. Similarly, the (long-run) actual average cost per unit time is given by

$$\lim_{n\to\infty} \left[\frac{1}{n} \sum_{t=1}^{n} C(X_{t-1}, D_t)\right] = \sum_{j=0}^{M} k(j)\pi_j.$$

Now let us assign numerical values to the two components of $C(X_{t-1}, D_t)$ in this example, namely, the ordering cost and the penalty cost for unsatisfied demand. If $z > 0$ cameras are ordered, the cost incurred is $(10 + 25z)$ dollars. If no cameras are ordered, no ordering cost is incurred. For each unit of unsatisfied demand (lost sales), there is a penalty of \$50. Therefore, given the ordering policy described in Sec. 29.1, the cost in week t is given by

$$C(X_{t-1}, D_t) = \begin{cases} 10 + (25)(3) + 50 \max\{D_t - 3, 0\} & \text{if} \quad X_{t-1} = 0 \\ 50 \max\{D_t - X_{t-1}, 0\} & \text{if} \quad X_{t-1} \geq 1, \end{cases}$$

for $t = 1, 2, \ldots$. Hence,

$$C(0, D_t) = 85 + 50 \max\{D_t - 3, 0\},$$

so that

$$\begin{aligned} k(0) &= E[C(0, D_t)] = 85 + 50E(\max\{D_t - 3, 0\}) \\ &= 85 + 50[P_D(4) + 2P_D(5) + 3P_D(6) + \cdots], \end{aligned}$$

where $P_D(i)$ is the probability that the demand equals i, as given by a Poisson distribution with a mean of 1, so that $P_D(i)$ becomes negligible for i larger than about 6. Since $P_D(4) = 0.015$, $P_D(5) = 0.003$, and $P_D(6) = 0.001$, we obtain $k(0) = 86.2$. Also using $P_D(2) = 0.184$ and $P_D(3) = 0.061$, similar calculations lead to the results

$$\begin{aligned} k(1) = E[C(1, D_t)] &= 50E(\max\{D_t - 1, 0\}) \\ &= 50[P_D(2) + 2P_D(3) + 3P_D(4) + \cdots] \\ &= 18.4, \\ k(2) = E[C(2, D_t)] &= 50E(\max\{D_t - 2, 0\}) \\ &= 50[P_D(3) + 2P_D(4) + 3P_D(5) + \cdots] \\ &= 5.2, \end{aligned}$$

and

$$\begin{aligned} k(3) = E[C(3, D_t)] &= 50E(\max\{D_t - 3, 0\}) \\ &= 50[P_D(4) + 2P_D(5) + 3P_D(6) + \cdots] \\ &= 1.2. \end{aligned}$$

Thus, the (long-run) expected average cost per week is given by

$$\sum_{j=0}^{3} k(j)\pi_j = 86.2(0.286) + 18.4(0.285) + 5.2(0.263) + 1.2(0.166) = \$31.46.$$

This is the cost associated with the particular ordering policy described in Sec. 29.1. The cost of other ordering policies can be evaluated in a similar way to identify the policy that minimizes the expected average cost per week.

The results of this subsection were presented only in terms of the inventory example. However, the (nonnumerical) results still hold for other problems as long as the following conditions are satisfied:

1. $\{X_t\}$ is an irreducible (finite-state) Markov chain.
2. Associated with this Markov chain is a sequence of random variables $\{D_t\}$ which are independent and identically distributed.
3. For a fixed $m = 0, \pm 1, \pm 2, \ldots$, a cost $C(X_t, D_{t+m})$ is incurred at time t, for $t = 0, 1, 2, \ldots$.
4. The sequence $X_0, X_1, X_2, \ldots, X_t$ must be independent of D_{t+m}

In particular, if these conditions are satisfied, then

$$\lim_{n\to\infty} E\left[\frac{1}{n}\sum_{t=1}^{n} C(X_t, D_{t+m})\right] = \sum_{j=0}^{M} k(j)\pi_j,$$

where

$$k(j) = E[C(j, D_{t+m})],$$

and where this latter conditional expectation is taken with respect to the probability distribution of the random variable D_t, given the state j. Furthermore,

$$\lim_{n\to\infty} \left[\frac{1}{n}\sum_{t=1}^{n} C(X_t, D_{t+m})\right] = \sum_{j=0}^{M} k(j)\pi_j$$

for essentially all paths of the process.

29.6 FIRST PASSAGE TIMES

Section 29.3 dealt with finding n-step transition probabilities from state i to state j. It is often desirable to also make probability statements about the number of transitions made by the process in going from state i to state j *for the first time.* This length of time is called the **first passage time** in going from state i to state j. When $j = i$, this first passage time is just the number of transitions until the process returns to the initial state i. In this case, the first passage time is called the **recurrence time** for state i.

To illustrate these definitions, reconsider the inventory example introduced in Sec. 29.1, where X_t is the number of cameras on hand at the end of week t, where we start with $X_0 = 3$. Suppose that it turns out that

$$X_0 = 3, \quad X_1 = 2, \quad X_2 = 1, \quad X_3 = 0, \quad X_4 = 3, \quad X_5 = 1.$$

In this case, the first passage time in going from state 3 to state 1 is 2 weeks, the first passage time in going from state 3 to state 0 is 3 weeks, and the recurrence time for state 3 is 4 weeks.

In general, the first passage times are random variables. The probability distributions associated with them depend upon the transition probabilities of the process. In particular, let $f_{ij}^{(n)}$ denote the probability that the first passage time from state i to j is equal to n. For $n > 1$, this first passage time is n if the first transition is from state i to some state k $(k \neq j)$ and then the first passage time from state k to state j is $n - 1$. Therefore, these probabilities satisfy the following recursive relationships:

$$f_{ij}^{(1)} = p_{ij}^{(1)} = p_{ij},$$

$$f_{ij}^{(2)} = \sum_{k\neq j} p_{ik} f_{kj}^{(1)},$$

$$f_{ij}^{(n)} = \sum_{k\neq j} p_{ik} f_{kj}^{(n-1)}.$$

Thus, the probability of a first passage time from state i to state j in n steps can be computed recursively from the one-step transition probabilities.

In the inventory example, the probability distribution of the first passage time in going from state 3 to state 0 is obtained from these recursive relationships as follows:

$$
\begin{aligned}
f_{30}^{(1)} &= p_{30} = 0.080,\\
f_{30}^{(2)} &= p_{31}f_{10}^{(1)} + p_{32}f_{20}^{(1)} + p_{33}f_{30}^{(1)}\\
&= 0.184(0.632) + 0.368(0.264) + 0.368(0.080) = 0.243,\\
&\vdots
\end{aligned}
$$

where the p_{3k} and $f_{k0}^{(1)} = p_{k0}$ are obtained from the (one-step) transition matrix given in Sec. 29.2.

For fixed i and j, the $f_{ij}^{(n)}$ are nonnegative numbers such that

$$\sum_{n=1}^{\infty} f_{ij}^{(n)} \le 1.$$

Unfortunately, this sum may be strictly less than 1, which implies that a process initially in state i may never reach state j. When the sum does equal 1, $f_{ij}^{(n)}$ (for $n = 1, 2, \ldots$) can be considered as a probability distribution for the random variable, the first passage time.

Although obtaining $f_{ij}^{(n)}$ for all n may be tedious, it is relatively simple to obtain the expected first passage time from state i to state j. Denote this expectation by μ_{ij}, which is defined by

$$
\mu_{ij} = \begin{cases} \infty & \text{if } \sum_{n=1}^{\infty} f_{ij}^{(n)} < 1 \\ \sum_{n=1}^{\infty} n f_{ij}^{(n)} & \text{if } \sum_{n=1}^{\infty} f_{ij}^{(n)} = 1. \end{cases}
$$

Whenever

$$\sum_{n=1}^{\infty} f_{ij}^{(n)} = 1,$$

μ_{ij} uniquely satisfies the equation

$$\mu_{ij} = 1 + \sum_{k \ne j} p_{ik}\mu_{kj}.$$

This equation recognizes that the first transition from state i can be to either state j or to some other state k. If it is to state j, the first passage time is 1. Given that the first transition is to some state k ($k \ne j$) instead, which occurs with probability p_{ik}, the conditional expected first passage time from state i to state j is $1 + \mu_{kj}$. Combining these facts, and summing over all the possibilities for the first transition, leads directly to this equation.

For the inventory example, these equations for the μ_{ij} can be used to compute the expected time until the cameras are out of stock, given that the process is started when three cameras are available. This expected time is just the expected first passage time μ_{30}. Since all the states are recurrent, the system of equations leads to the expressions

$$\mu_{30} = 1 + p_{31}\mu_{10} + p_{32}\mu_{20} + p_{33}\mu_{30},$$

$$\mu_{20} = 1 + p_{21}\mu_{10} + p_{22}\mu_{20} + p_{23}\mu_{30},$$
$$\mu_{10} = 1 + p_{11}\mu_{10} + p_{12}\mu_{20} + p_{13}\mu_{30},$$

or

$$\mu_{30} = 1 + 0.184\mu_{10} + 0.368\mu_{20} + 0.368\mu_{30},$$
$$\mu_{20} = 1 + 0.368\mu_{10} + 0.368\mu_{20},$$
$$\mu_{10} = 1 + 0.368\mu_{10}.$$

The simultaneous solution to this system of equations is

$$\mu_{10} = 1.58 \text{ weeks},$$
$$\mu_{20} = 2.51 \text{ weeks},$$
$$\mu_{30} = 3.50 \text{ weeks},$$

so that the expected time until the cameras are out of stock is 3.50 weeks. Thus, in making these calculations for μ_{30}, we also obtain μ_{20} and μ_{10}.

For the case of μ_{ij} where $j = i$, μ_{ii} is the expected number of transitions until the process returns to the initial state i, and so is called the **expected recurrence time** for state i. After obtaining the steady-state probabilities $(\pi_0, \pi_1, \ldots, \pi_M)$ as described in the preceding section, these expected recurrence times can be calculated immediately as

$$\mu_{ii} = \frac{1}{\pi_i}, \qquad \text{for } i = 0, 1, \ldots, M.$$

Thus, for the inventory example, where $\pi_0 = 0.286$, $\pi_1 = 0.285$, $\pi_2 = 0.263$, and $\pi_3 = 0.166$, the corresponding expected recurrence times are

$$\mu_{00} = \frac{1}{\pi_0} = 3.50 \text{ weeks}, \qquad \mu_{22} = \frac{1}{\pi_2} = 3.80 \text{ weeks},$$

29.7 ABSORBING STATES

It was pointed out in Sec. 29.4 that a state k is called an *absorbing state* if $p_{kk} = 1$, so that once the chain visits k it remains there forever. If k is an absorbing state, and the process starts in state i, the probability of *ever* going to state k is called the **probability of absorption** into state k, given that the system started in state i. This probability is denoted by f_{ik}.

When there are two or more absorbing states in a Markov chain, and it is evident that the process will be absorbed into one of these states, it is desirable to find these probabilities of absorption. These probabilities can be obtained by solving a system of linear equations that considers all the possibilities for the first transition and then, given the first transition, considers the conditional probability of absorption into state k. In particular, if the state k is an absorbing state, then the set of absorption probabilities f_{ik} satisfies the system of equations

$$f_{ik} = \sum_{j=0}^{M} p_{ij} f_{jk}, \qquad \text{for } i = 0, 1, \ldots, M,$$

subject to the conditions

$$f_{kk} = 1,$$
$$f_{ik} = 0, \qquad \text{if state } i \text{ is recurrent and } i \neq k.$$

Absorption probabilities are important in random walks. A **random walk** is a Markov chain with the property that if the system is in a state *i*, then in a single transition the system either remains at *i* or moves to one of the two states immediately adjacent to *i*. For example, a random walk often is used as a model for situations involving gambling.

A Second Gambling Example. To illustrate the use of absorption probabilities in a random walk, consider a gambling example similar to that presented in Sec. 29.2. However, suppose now that two players (A and B), each having \$2, agree to keep playing the game and betting \$1 at a time until one player is broke. The probability of A winning a single bet is $\frac{1}{3}$, so B wins the bet with probability $\frac{2}{3}$. The number of dollars that player A has before each bet (0, 1, 2, 3, or 4) provides the states of a Markov chain with transition matrix

$$\mathbf{P} = \begin{array}{c} \text{State} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{2}{3} & 0 & \frac{1}{3} & 0 & 0 \\ 0 & \frac{2}{3} & 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{2}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}.$$

Starting from state 2, the probability of absorption into state 0 (A losing all her money) can be obtained by solving for f_{20} from the system of equations given at the beginning of this section,

$$\begin{aligned} f_{00} &= 1 \qquad \text{(since state 0 is an absorbing state)}, \\ f_{10} &= \frac{2}{3} f_{00} + \frac{1}{3} f_{20}, \\ f_{20} &= \frac{2}{3} f_{10} + \frac{1}{3} f_{30}, \\ f_{30} &= \frac{2}{3} f_{20} + \frac{1}{3} f_{40}, \\ f_{40} &= 0 \qquad \text{(since state 4 is an absorbing state)}. \end{aligned}$$

This system of equations yields

$$f_{20} = \frac{2}{3}\left(\frac{2}{3} + \frac{1}{3} f_{20}\right) + \frac{1}{3}\left(\frac{2}{3} f_{20}\right) = \frac{4}{9} + \frac{4}{9} f_{20},$$

which reduces to $f_{20} = \frac{4}{5}$ as the probability of absorption into state 0.

Similarly, the probability of A finishing with \$4 (B going broke) when starting with \$2 (state 2) is obtained by solving for f_{24} from the system of equations,

$$\begin{aligned} f_{04} &= 0 \qquad \text{(since state 0 is an absorbing state)}, \\ f_{14} &= \frac{2}{3} f_{04} + \frac{1}{3} f_{24}, \\ f_{24} &= \frac{2}{3} f_{14} + \frac{1}{3} f_{34}, \\ f_{34} &= \frac{2}{3} f_{24} + \frac{1}{3} f_{44}, \\ f_{44} &= 1 \qquad \text{(since state 0 is an absorbing state)}. \end{aligned}$$

This yields

$$f_{24} = \frac{2}{3}\left(\frac{1}{3} f_{24}\right) + \frac{1}{3}\left(\frac{2}{3} f_{24} + \frac{1}{3}\right) = \frac{4}{9} f_{24} + \frac{1}{9},$$

so $f_{24} = \frac{1}{5}$ is the probability of absorption into state 4.

A Credit Evaluation Example. There are many other situations where absorbing states play an important role. Consider a department store that classifies the balance of a customer's bill as fully paid (state 0), 1 to 30 days in arrears (state 1), 31 to 60 days in arrears (state 2), or bad debt (state 3). The accounts are checked *monthly* to determine the state of each customer. In general, credit is not extended and customers are expected to pay their bills promptly. Occasionally, customers miss the deadline for paying their bill. If this occurs when the balance is within 30 days in arrears, the store views the customer as being in state 1. If this occurs when the balance is between 31 and 60 days in arrears, the store views the customer as being in state 2. Customers that are more than 60 days in arrears are put into the bad-debt category (state 3), and then bills are sent to a collection agency.

After examining data over the past several years on the month by month progression of individual customers from state to state, the store has developed the following transition matrix:[4]

State \ State	0: Fully Paid	1: 1 to 30 Days in Arrears	2: 31 to 60 Days in Arrears	3: Bad Debt
0: fully paid	1	0	0	0
1: 1 to 30 days in arrears	0.7	0.2	0.1	0
2: 31 to 60 days in arrears	0.5	0.1	0.2	0.2
3: bad debt	0	0	0	1

Although each customer ends up in state 0 or 3, the store is interested in determining the probability that a customer will end up as a bad debt given that the account belongs to the 1 to 30 days in arrears state, and similarly, given that the account belongs to the 31 to 60 days in arrears state.

To obtain this information, the set of equations presented at the beginning of this section must be solved to obtain f_{13} and f_{23}. By substituting, the following two equations are obtained:

$$\begin{aligned} f_{13} &= p_{10}f_{03} + p_{11}f_{13} + p_{12}f_{23} + p_{13}f_{33}, \\ f_{23} &= p_{20}f_{03} + p_{21}f_{13} + p_{22}f_{23} + p_{23}f_{33}. \end{aligned}$$

Noting that $f_{03} = 0$ and $f_{33} = 1$, we now have two equations in two unknowns, namely,

$$\begin{aligned} (1 - p_{11})f_{13} &= p_{13} + p_{12}f_{23}, \\ (1 - p_{22})f_{23} &= p_{23} + p_{21}f_{13}. \end{aligned}$$

Substituting the values from the transition matrix leads to

$$\begin{aligned} 0.8f_{13} &= 0.1f_{23}, \\ 0.8f_{23} &= 0.2 + 0.1f_{13}, \end{aligned}$$

and the solution is

$$\begin{aligned} f_{13} &= 0.032, \\ f_{23} &= 0.254. \end{aligned}$$

[4]Customers who are fully paid (in state 0) and then subsequently fall into arrears on new purchases are viewed as "new" customers who start in state 1.

Thus, approximately 3 percent of the customers whose accounts are 1 to 30 days in arrears end up as bad debts, whereas about 25 percent of the customers whose accounts are 31 to 60 days in arrears end up as bad debts.

29.8 CONTINUOUS TIME MARKOV CHAINS

In all the previous sections, we assumed that the time parameter t was discrete (that is, $t = 0, 1, 2, \ldots$). Such an assumption is suitable for many problems, but there are certain cases (such as for some queueing models considered in Chap. 17) where a continuous time parameter (call it t') is required, because the evolution of the process is being observed *continuously* over time. The definition of a Markov chain given in Sec. 29.2 also extends to such continuous processes. This section focuses on describing these "continuous time Markov chains" and their properties.

Formulation

As before, we label the possible **states** of the system as $0, 1, \ldots, M$. Starting at time 0 and letting the time parameter t' run continuously for $t' \geq 0$, we let the random variable $X(t')$ be the state of the system at time t'. Thus, $X(t')$ will take on one of its possible $(M + 1)$ values over some interval, $0 \leq t' < t_1$, then will jump to another value over the next interval, $t_1 \leq t' < t_2$, etc., where these transit points $(t_1, t_2, \ldots)$ are random points in time (*not* necessarily integer).

Now consider the three points in time (1) $t' = r$ (where $r \geq 0$), (2) $t' = s$ (where $s > r$), and (3) $t' = s + t$ (where $t > 0$), interpreted as follows:

$t' = r$ is a past time,
$t' = s$ is the current time,
$t' = s + t$ is t time units into the future.

Therefore, the state of the system now has been observed at times $t' = s$ and $t' = r$. Label these states as

$$X(s) = i \qquad \text{and} \qquad X(r) = x(r).$$

Given this information, it now would be natural to seek the probability distribution of the state of the system at time $t' = s + t$. In other words, what is

$$P\{X(s + t) = j \mid X(s) = i \text{ and } X(r) = x(r)\}, \qquad \text{for } j = 0, 1, \ldots, M?$$

Deriving this conditional probability often is very difficult. However, this task is considerably simplified if the stochastic process involved possesses the following key property.

A continuous time stochastic process $\{X(t');\ t' \geq 0\}$ has the **Markovian property** if

$$P\{X(t + s) = j \mid X(s) = i \text{ and } X(r) = x(r)\} = P\{X(t + s) = j \mid X(s) = i\},$$

for all $i, j = 0, 1, \ldots, M$ and for all $r \geq 0$, $s > r$, and $t > 0$.

Note that $P\{X(t + s) = j \mid X(s) = i\}$ is a **transition probability,** just like the transition probabilities for discrete time Markov chains considered in the preceding sections, where the only difference is that t now need not be an integer.

If the transition probabilities are independent of s, so that

$$P\{X(t + s) = j \mid X(s) = i\} = P\{X(t) = j \mid X(0) = i\}$$

for all $s > 0$, they are called **stationary transition probabilities.**

To simplify notation, we shall denote these stationary transition probabilities by

$$p_{ij}(t) = P\{X(t) = j \mid X(0) = i\},$$

where $p_{ij}(t)$ is referred to as the **continuous time transition probability function.** We assume that

$$\lim_{t \to 0} p_{ij}(t) = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j. \end{cases}$$

Now we are ready to define the continuous time Markov chains to be considered in this section.

> A continuous time stochastic process $\{X(t'); t' \geq 0\}$ is a **continuous time Markov chain** if it has the *Markovian property.*

We shall restrict our consideration to continuous time Markov chains with the following properties:

1. A finite number of states.
2. Stationary transition probabilities.

Some Key Random Variables

In the analysis of continuous time Markov chains, one key set of random variables is the following:

> Each time the process enters state i, the amount of time it spends in that state before moving to a different state is a random variable T_i, where $i = 0, 1, \ldots, M$.

Suppose that the process enters state i at time $t' = s$. Then, for any fixed amount of time $t > 0$, note that $T_i > t$ if and only if $X(t') = i$ for all t' over the interval $s \leq t' \leq s + t$. Therefore, the Markovian property (with stationary transition probabilities) implies that

$$P\{T_i > t + s \mid T_i > s\} = P\{T_i > t\}.$$

This is a rather unusual property for a probability distribution to possess. It says that the probability distribution of the *remaining* time until the process transits out of a given state always is the same, regardless of how much time the process has already spent in that state. In effect, the random variable is memoryless; the process forgets its history. There is only one (continuous) probability distribution that possesses this property—the *exponential distribution.* The exponential distribution has a single parameter, call it q, where the mean is $1/q$ and the cumulative distribution function is

$$P\{T_i \leq t\} = 1 - e^{-qt}, \qquad \text{for } t \geq 0.$$

(We described the properties of the exponential distribution in detail in Sec. 17.4.)

This result leads to an equivalent way of describing a continuous time Markov chain:

1. The random variable T_i has an exponential distribution with a mean of $1/q_i$.
2. When leaving state i, the process moves to a state j with probability p_{ij}, where the p_{ij} satisfy the conditions

$$p_{ii} = 0 \qquad \text{for all } i,$$

and

$$\sum_{j=0}^{M} p_{ij} = 1 \qquad \text{for all } i.$$

3. The next state visited after state i is independent of the time spent in state i.

Just as the one-step transition probabilities played a major role in describing discrete time Markov chains, the analogous role for a continuous time Markov chain is played by the transition intensities.

The **transition intensities** are

$$q_i = -\frac{d}{dt}p_{ii}(0) = \lim_{t\to 0} \frac{1 - p_{ii}(t)}{t}, \qquad \text{for } i = 0, 1, 2, \ldots, M,$$

and

$$q_{ij} = \frac{d}{dt}p_{ij}(0) = \lim_{t\to 0} \frac{p_{ij}(t)}{t} = q_i p_{ij}, \qquad \text{for all } j \neq i,$$

where $p_{ij}(t)$ is the *continuous time transition probability function* introduced at the beginning of the section and p_{ij} is the probability described in property 2 of the preceding paragraph. Furthermore, q_i as defined here turns out to still be the parameter of the exponential distribution for T_i as well (see property 1 of the preceding paragraph).

The intuitive interpretation of the q_i and q_{ij} is that they are *transition rates.* In particular, q_i is the *transition rate out of state* i in the sense that q_i is the expected number of times that the process leaves state i per unit of time spent in state i. (Thus, q_i is the reciprocal of the expected time that the process spends in state i per visit to state i; that is, $q_i = 1/E[T_i]$.) Similarly, q_{ij} is the *transition rate from state i to state j* in the sense that q_{ij} is the expected number of times that the process transits from state i to state j per unit of time spent in state i. Thus,

$$q_i = \sum_{j \neq i} q_{ij}.$$

Just as q_i is the parameter of the exponential distribution for T_i, each q_{ij} is the parameter of an exponential distribution for a related random variable described below:

Each time the process enters state i, the amount of time it will spend in state i before a transition to state j occurs (if a transition to some other state does not occur first) is a random variable T_{ij}, where $i, j = 0, 1, \ldots, M$ and $j \neq i$. The T_{ij} are independent random variables, where each T_{ij} has an *exponential distribution* with parameter q_{ij}, so $E[T_{ij}] = 1/q_{ij}$. The time spent in state i until a transition occurs (T_i) is the *minimum* (over $j \neq i$) of the T_{ij}. When the transition occurs, the probability that it is to state j is $p_{ij} = q_{ij}/q_i$.

Steady-State Probabilities

Just as the transition probabilities for a discrete time Markov chain satisfy the Chapman-Kolmogorov equations, the continuous time transition probability function also satisfies these equations. Therefore, for any states i and j and nonnegative numbers t and s $(0 \leq s \leq t)$,

$$p_{ij}(t) = \sum_{k=0}^{M} p_{ik}(s)p_{kj}(t - s).$$

A pair of states i and j are said to *communicate* if there are times t_1 and t_2 such that $p_{ij}(t_1) > 0$ and $p_{ji}(t_2) > 0$. All states that communicate are said to form a *class.* If all states form a single class, i.e., if the Markov chain is *irreducible* (hereafter assumed), then

$$p_{ij}(t) > 0, \qquad \text{for all } t > 0 \text{ and all states } i \text{ and } j.$$

Furthermore,

$$\lim_{t \to \infty} p_{ij}(t) = \pi_j$$

always exists and is independent of the initial state of the Markov chain, for $j = 0, 1, \ldots, M$. These limiting probabilities are commonly referred to as the **steady-state probabilities** (or *stationary probabilities*) of the Markov chain.

The π_j satisfy the equations

$$\pi_j = \sum_{i=0}^{M} \pi_i p_{ij}(t), \qquad \text{for } j = 0, 1, \ldots, M \text{ and every } t \geq 0.$$

However, the following **steady-state equations** provide a more useful system of equations for solving for the steady-state probabilities:

$$\pi_j q_j = \sum_{i \neq j} \pi_i q_{ij}, \qquad \text{for } j = 0, 1, \ldots, M.$$

and

$$\sum_{j=0}^{M} \pi_j = 1.$$

The steady-state equation for state j has an intuitive interpretation. The left-hand side ($\pi_j q_j$) is the *rate* at which the process *leaves* state j, since π_j is the (steady-state) probability that the process is in state j and q_j is the transition rate out of state j given that the process is in state j. Similarly, each term on the right-hand side ($\pi_i q_{ij}$) is the *rate* at which the process *enters* state j from state i, since q_{ij} is the transition rate from state i to state j given that the process is in state i. By summing over all $i \neq j$, the entire right-hand side then gives the rate at which the process enters state j from any other state. The overall equation thereby states that the rate at which the process leaves state j must equal the rate at which the process enters state j. Thus, this equation is analogous to the conservation of flow equations encountered in many engineering and science courses.

Because each of the first $M + 1$ *steady-state equations* requires that two rates be *in balance* (equal), these equations sometimes are called the **balance equations.**

Example. A certain shop has two identical machines that are operated continuously except when they are broken down. Because they break down fairly frequently, the top-priority assignment for a full-time maintenance person is to repair them whenever needed.

The time required to repair a machine has an exponential distribution with a mean of $\frac{1}{2}$ day. Once the repair of a machine is completed, the time until the next breakdown of that machine has an exponential distribution with a mean of 1 day. These distributions are independent.

Define the random variable $X(t')$ as

$X(t')$ = number of machines broken down at time t',

so the possible values of $X(t')$ are 0, 1, 2. Therefore, by letting the time parameter t' run continuously from time 0, the continuous time stochastic process $\{X(t'); t' \geq 0\}$ gives the evolution of the number of machines broken down.

Because both the repair time and the time until a breakdown have exponential distributions, $\{X(t'); t' \geq 0\}$ is a *continuous time Markov chain*[5] with states 0, 1, 2. Consequently, we can use the steady-state equations given in the preceding subsection to find the steady-state probability distribution of the number of machines broken down. To do this, we need to determine all the *transition rates,* i.e., the q_i and q_{ij} for $i, j = 0, 1, 2$.

The state (number of machines broken down) increases by 1 when a breakdown occurs and decreases by 1 when a repair occurs. Since both breakdowns and repairs occur one at a time, $q_{02} = 0$ and $q_{20} = 0$. The expected repair time is $\frac{1}{2}$ day, so the rate at which repairs are completed (when any machines are broken down) is 2 per day, which implies that $q_{21} = 2$ and $q_{10} = 2$. Similarly, the expected time until a particular operational machine breaks down is 1 day, so the rate at which it breaks down (when operational) is 1 per day, which implies that $q_{12} = 1$. During times when both machines are operational, breakdowns occur at the rate of $1 + 1 = 2$ per day, so $q_{01} = 2$.

These transition rates are summarized in the rate diagram shown in Fig. 29.5. These rates now can be used to calculate the *total transition rate* out of each state.

$$q_0 = q_{01} = 2$$
$$q_1 = q_{10} + q_{12} = 3$$
$$q_2 = q_{21} = 2$$

Plugging all the rates into the steady-state equations given in the preceding subsection then yields

Balance equation for state 0:	$2\pi_0 = 2\pi_1$
Balance equation for state 1:	$3\pi_1 = 2\pi_0 + 2\pi_2$
Balance equation for state 2:	$2\pi_2 = \pi_1$
Probabilities sum to 1:	$\pi_0 + \pi_1 + \pi_2 = 1$

Any one of the balance equations (say, the second) can be deleted as redundant, and the simultaneous solution of the remaining equations gives the steady-state distribution as

$$(\pi_0, \pi_1, \pi_2) = \left(\frac{2}{5}, \frac{2}{5}, \frac{1}{5}\right).$$

Thus, in the long run, both machines will be broken down simultaneously 20 percent of the time, and one machine will be broken down another 40 percent of the time.

[5]Proving this fact requires the use of two properties of the exponential distribution discussed in Sec. 17.4 (*lack of memory* and *the minimum of exponentials is exponential*), since these properties imply that the T_{ij} random variables introduced earlier do indeed have exponential distributions.

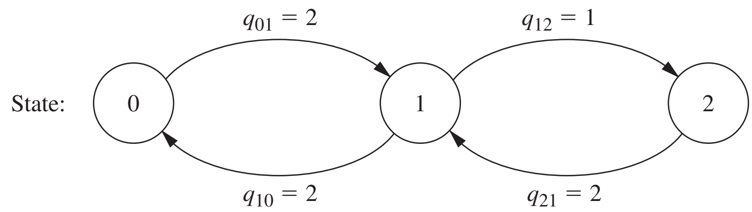


FIGURE 29.5
The rate diagram for the example of a continuous time Markov chain.

Chapter 17 (on queueing theory) features many more examples of continuous time Markov chains. In fact, most of the basic models of queueing theory fall into this category. The current example actually fits one of these models (the finite calling population variation of the *M/M/s* model included in Sec. 17.6).

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Automatic Procedures in IOR Tutorial:

Enter Transition Matrix
Chapman-Kolmogorov Equations
Steady-State Probabilities

"Ch. 29—Markov Chains" LINGO File for Selected Examples

See Appendix 1 for documentation of the software.

PROBLEMS

The symbol to the left of some of the problems (or their parts) has the following meaning.

C: Use the computer with the corresponding automatic procedures just listed (or other equivalent routines) to solve the problem.

29.2-1. Assume that the probability of rain tomorrow is 0.5 if it is raining today, and assume that the probability of its being clear (no rain) tomorrow is 0.9 if it is clear today. Also assume that these probabilities do not change if information is also provided about the weather before today.

(a) Explain why the stated assumptions imply that the *Markovian property* holds for the evolution of the weather.

(b) Formulate the evolution of the weather as a Markov chain by defining its states and giving its (one-step) transition matrix.

29.2-2. Consider the second version of the stock market model presented as an example in Sec. 29.2. Whether the stock goes up tomorrow depends upon whether it increased today *and* yesterday. If the stock increased today and yesterday, it will increase tomorrow with probability α_1. If the stock increased today and decreased yesterday, it will increase tomorrow with probability α_2. If the stock decreased today and increased yesterday, it will increase tomorrow with probability α_3. Finally, if the stock decreased today and yesterday, it will increase tomorrow with probability α_4.

(a) Construct the (one-step) transition matrix of the Markov chain.

(b) Explain why the states used for this Markov chain cause the mathematical definition of the Markovian property to hold even though what happens in the future (tomorrow) depends upon what happened in the past (yesterday) as well as the present (today).

29.2-3. Reconsider Prob. 29.2-2. Suppose now that whether or not the stock goes up tomorrow depends upon whether it increased today, yesterday, *and* the day before yesterday. Can this problem be formulated as a Markov chain? If so, what are the possible states? Explain why these states give the process the *Markovian property* whereas the states in Prob. 29.2-2 do not.

29.3-1. Reconsider Prob. 29.2-1.

C **(a)** Use the procedure *Chapman-Kolmogorov Equations* in your IOR Tutorial to find the n-step transition matrix $\mathbf{P}^{(n)}$ for $n = 2, 5, 10, 20$.

(b) The probability that it will rain today is 0.5. Use the results from part (*a*) to determine the probability that it will rain n days from now, for $n = 2, 5, 10, 20$.

C **(c)** Use the procedure *Steady-State Probabilities* in your IOR Tutorial to determine the steady-state probabilities of the state of the weather. Describe how the probabilities in the n-step transition matrices obtained in part (*a*) compare to these steady-state probabilities as n grows large.

29.3-2. Suppose that a communications network transmits binary digits, 0 or 1, where each digit is transmitted 10 times in succession. During each transmission, the probability is 0.995 that the digit entered will be transmitted accurately. In other words, the probability is 0.005 that the digit being transmitted will be recorded with the opposite value at the end of the transmission. For each transmission after the first one, the digit entered for transmission is the one that was recorded at the end of the preceding transmission. If X_0 denotes the binary digit entering the system, X_1 the binary digit recorded after the first transmission, X_2 the binary digit recorded after the second transmission, . . . , then $\{X_n\}$ is a Markov chain.

(a) Construct the (one-step) transition matrix.

C **(b)** Use your IOR Tutorial to find the 10-step transition matrix $\mathbf{P}^{(10)}$. Use this result to identify the probability that a digit entering the network will be recorded accurately after the last transmission.

C **(c)** Suppose that the network is redesigned to improve the probability that a single transmission will be accurate from 0.995 to 0.998. Repeat part (*b*) to find the new probability that a digit entering the network will be recorded accurately after the last transmission.

29.3-3. A particle moves on a circle through points that have been marked 0, 1, 2, 3, 4 (in a clockwise order). The particle starts at point 0. At each step it has probability 0.5 of moving one point clockwise (0 follows 4) and 0.5 of moving one point counterclockwise. Let X_n $(n \geq 0)$ denote its location on the circle after step n. $\{X_n\}$ is a Markov chain.

(a) Construct the (one-step) transition matrix.

C **(b)** Use your IOR Tutorial to determine the n-step transition matrix $\mathbf{P}^{(n)}$ for $n = 5, 10, 20, 40, 80$.

C **(c)** Use your IOR Tutorial to determine the steady-state probabilities of the state of the Markov chain. Describe how the probabilities in the n-step transition matrices obtained in part (*b*) compare to these steady-state probabilities as n grows large.

29.4-1. Given the following (one-step) transition matrices of a Markov chain, determine the classes of the Markov chain and whether they are recurrent.

$$
\textbf{(a)}\ \mathbf{P} = \begin{array}{c} \text{State} \\ 0 \\ 1 \\ 2 \\ 3 \end{array}
\begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0 & 0 & \frac{1}{3} & \frac{2}{3} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \end{array}
$$

$$\textbf{(b)}\ \mathbf{P} = \begin{array}{cc} \text{State} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0\\ 1\\ 2\\ 3 \end{array} & \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{bmatrix} \end{array}$$

29.4-2. Given each of the following (one-step) transition matrices of a Markov chain, determine the classes of the Markov chain and whether they are recurrent.

$$\textbf{(a)}\ \mathbf{P} = \begin{array}{cc} \text{State} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0\\ 1\\ 2\\ 3 \end{array} & \begin{bmatrix} 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \end{bmatrix} \end{array}$$

$$\textbf{(b)}\ \mathbf{P} = \begin{array}{cc} \text{State} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} 0\\ 1\\ 2 \end{array} & \begin{bmatrix} 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 0 \end{bmatrix} \end{array}$$

29.4-3. Given the following (one-step) transition matrix of a Markov chain, determine the classes of the Markov chain and whether they are recurrent.

$$\mathbf{P} = \begin{array}{cc} \text{State} & \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 0\\ 1\\ 2\\ 3\\ 4 \end{array} & \begin{bmatrix} \frac{1}{4} & \frac{3}{4} & 0 & 0 & 0 \\ \frac{3}{4} & \frac{1}{4} & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & \frac{3}{4} & \frac{1}{4} \\ 0 & 0 & 0 & \frac{1}{4} & \frac{3}{4} \end{bmatrix} \end{array}$$

29.4-4. Determine the period of each of the states in the Markov chain that has the following (one-step) transition matrix.

$$\mathbf{P} = \begin{array}{cc} \text{State} & \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{array}{c} 0\\ 1\\ 2\\ 3\\ 4\\ 5 \end{array} & \begin{bmatrix} 0 & 0 & 0 & \frac{2}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 & 0 & \frac{3}{4} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \end{bmatrix} \end{array}$$

29.4-5. Consider the Markov chain that has the following (one-step) transition matrix.

$$\mathbf{P} = \begin{array}{cc} \text{State} & \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 0\\ 1\\ 2\\ 3\\ 4 \end{array} & \begin{bmatrix} 0 & \frac{4}{5} & 0 & \frac{1}{5} & 0 \\ \frac{1}{4} & 0 & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{10} & \frac{2}{5} \\ 0 & 0 & 0 & 1 & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} & 0 \end{bmatrix} \end{array}$$

(a) Determine the classes of this Markov chain and, for each class, determine whether it is recurrent or transient.

(b) For each of the classes identified in part (*a*), determine the period of the states in that class.

29.5-1. Reconsider Prob. 29.2-1. Suppose now that the given probabilities, 0.5 and 0.9, are replaced by arbitrary values, α and β, respectively. Solve for the *steady-state probabilities* of the state of the weather in terms of α and β.

29.5-2. A transition matrix **P** is said to be doubly stochastic if the sum over each column equals 1; that is,

$$\sum_{i=0}^{M} p_{ij} = 1, \qquad \text{for all } j.$$

If such a chain is irreducible, aperiodic, and consists of $M + 1$ states, show that

$$\pi_j = \frac{1}{M+1}, \qquad \text{for } j = 0, 1, \ldots, M.$$

29.5-3. Reconsider Prob. 29.3-3. Use the results given in Prob. 29.5-2 to find the steady-state probabilities for this Markov chain. Then find what happens to these steady-state probabilities if, at each step, the probability of moving one point clockwise changes to 0.9 and the probability of moving one point counterclockwise changes to 0.1.

C **29.5-4.** The leading brewery on the West Coast (labeled *A*) has hired an OR analyst to analyze its market position. It is particularly concerned about its major competitor (labeled *B*). The analyst believes that brand switching can be modeled as a Markov chain using three states, with states *A* and *B* representing customers drinking beer produced from the aforementioned breweries and state *C* representing all other brands. Data are taken monthly, and the analyst has constructed the following (one-step) transition matrix from past data.

	A	*B*	*C*
A	0.8	0.15	0.05
B	0.25	0.7	0.05
C	0.15	0.05	0.8

What are the steady-state market shares for the two major breweries?

29.5-5. Consider the following blood inventory problem facing a hospital. There is need for a rare blood type, namely, type AB, Rh negative blood. The demand D (in pints) over any 3-day period is given by

$$P\{D = 0\} = 0.4, \qquad P\{D = 1\} = 0.3,$$
$$P\{D = 2\} = 0.2, \qquad P\{D = 3\} = 0.1.$$

Note that the expected demand is 1 pint, since $E(D) = 0.3(1) + 0.2(2) + 0.1(3) = 1$. Suppose that there are 3 days between deliveries. The hospital proposes a policy of receiving 1 pint at each delivery and using the oldest blood first. If more blood is required than is on hand, an expensive emergency delivery is made. Blood is

discarded if it is still on the shelf after 21 days. Denote the state of the system as the number of pints on hand just after a delivery. Thus, because of the discarding policy, the largest possible state is 7.

(a) Construct the (one-step) transition matrix for this Markov chain.

C **(b)** Find the steady-state probabilities of the state of the Markov chain.

(c) Use the results from part (*b*) to find the steady-state probability that a pint of blood will need to be discarded during a 3-day period. (*Hint*: Because the oldest blood is used first, a pint reaches 21 days only if the state was 7 and then $D = 0$.)

(d) Use the results from part (*b*) to find the steady-state probability that an emergency delivery will be needed during the 3-day period between regular deliveries.

C **29.5-6.** In the last subsection of Sec. 29.5, the (long-run) expected average cost per week (based on just ordering costs and unsatisfied demand costs) is calculated for the inventory example of Sec. 29.1. Suppose now that the ordering policy is changed to the following. Whenever the number of cameras on hand at the end of the week is 0 or 1, an order is placed that will bring this number up to 3. Otherwise, no order is placed.

Recalculate the (long-run) expected average cost per week under this new inventory policy.

29.5-7. Consider the inventory example introduced in Sec. 29.1, but with the following change in the ordering policy. If the number of cameras on hand at the end of each week is 0 or 1, two additional cameras will be ordered. Otherwise, no ordering will take place. Assume that the storage costs are the same as given in the second subsection of Sec. 29.5.

C **(a)** Find the steady-state probabilities of the state of this Markov chain.

(b) Find the long-run expected average storage cost per week.

29.5-8. Consider the following inventory policy for the certain product. If the demand during a period exceeds the number of items available, this unsatisfied demand is backlogged; i.e., it is filled when the next order is received. Let Z_n ($n = 0, 1, \ldots$) denote the amount of inventory on hand minus the number of units backlogged before ordering at the end of period n ($Z_0 = 0$). If Z_n is zero or positive, no orders are backlogged. If Z_n is negative, then $-Z_n$ represents the number of backlogged units and no inventory is on hand. At the end of period n, if $Z_n < 1$, an order is placed for $2m$ units, where m is the smallest integer such that $Z_n + 2m \geq 1$. Orders are filled immediately.

Let $D_1, D_2, \ldots$, be the demand for the product in periods 1, 2, . . . , respectively. Assume that the D_n are independent and identically distributed random variables taking on the values, 0, 1, 2, 3, 4, each with probability $\frac{1}{5}$. Let X_n denote the amount of stock on hand *after* ordering at the end of period n (where $X_0 = 2$), so that

$$X_n = \begin{cases} X_{n-1} - D_n + 2m & \text{if } X_{n-1} - D_n < 1 \\ X_{n-1} - D_n & \text{if } X_{n-1} - D_n \geq 1 \end{cases} \quad (n = 1, 2, \ldots),$$

when $\{X_n\}$ ($n = 0, 1, \ldots$) is a Markov chain. It has only two states, 1 and 2, because the only time that ordering will take place is when $Z_n = 0, -1, -2$, or -3, in which case 2, 2, 4, and 4 units are ordered, respectively, leaving $X_n = 2, 1, 2, 1$, respectively.

(a) Construct the (one-step) transition matrix.

(b) Use the steady-state equations to solve manually for the steady-state probabilities.

(c) Now use the result given in Prob. 29.5-2 to find the steady-state probabilities.

(d) Suppose that the ordering cost is given by $(2 + 2m)$ if an order is placed and zero otherwise. The holding cost per period is Z_n if $Z_n \geq 0$ and zero otherwise. The shortage cost per period is $-4Z_n$ if $Z_n < 0$ and zero otherwise. Find the (long-run) expected average cost per unit time.

29.5-9. An important unit consists of two components placed in parallel. The unit performs satisfactorily if one of the two components is operating. Therefore, only one component is operated at a time, but both components are kept operational (capable of being operated) as often as possible by repairing them as needed. An operating component breaks down in a given period with probability 0.2. When this occurs, the parallel component takes over, if it is operational, at the beginning of the next period. Only one component can be repaired at a time. The repair of a component starts at the beginning of the first available period and is completed at the end of the next period. Let X_t be a vector consisting of two elements U and V, where U represents the number of components that are operational at the end of period t and V represents the number of periods of repair that have been completed on components that are not yet operational. Thus, $V = 0$ if $U = 2$ or if $U = 1$ and the repair of the nonoperational component is just getting under way. Because a repair takes two periods, $V = 1$ if $U = 0$ (since then one nonoperational component is waiting to begin repair while the other one is entering its second period of repair) or if $U = 1$ and the nonoperational component is entering its second period of repair. Therefore, the state space consists of the four states (2, 0), (1, 0), (0, 1), and (1, 1). Denote these four states by 0, 1, 2, 3, respectively. $\{X_t\}$ ($t = 0, 1, \ldots$) is a Markov chain (assume that $X_0 = 0$) with the (one-step) transition matrix

$$\mathbf{P} = \begin{array}{c} \text{State} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & \;\;1 & \;\;2 & \;\;3 \end{array} \\ \begin{bmatrix} 0.8 & 0.2 & 0 & 0 \\ 0 & 0 & 0.2 & 0.8 \\ 0 & 1 & 0 & 0 \\ 0.8 & 0.2 & 0 & 0 \end{bmatrix} \end{array}.$$

C **(a)** What is the probability that the unit will be inoperable (because both components are down) after n periods, for $n = 2, 5, 10, 20$?

C **(b)** What are the steady-state probabilities of the state of this Markov chain?

(c) If it costs \$30,000 per period when the unit is inoperable (both components down) and zero otherwise, what is the (long-run) expected average cost per period?

29.6-1. A computer is inspected at the end of every hour. It is found to be either working (up) or failed (down). If the computer is found to be up, the probability of its remaining up for the next hour is 0.95. If it is down, the computer is repaired, which may require more than

1 hour. Whenever the computer is down (regardless of how long it has been down), the probability of its still being down 1 hour later is 0.5.

(a) Construct the (one-step) transition matrix for this Markov chain.
(b) Use the approach described in Sec. 29.6 to find the μ_{ij} (the expected first passage time from state i to state j) for all i and j.

29.6-2. A manufacturer has a machine that, when operational at the beginning of a day, has a probability of 0.1 of breaking down sometime during the day. When this happens, the repair is done the next day and completed at the end of that day.

(a) Formulate the evolution of the status of the machine as a Markov chain by identifying three possible states at the end of each day, and then constructing the (one-step) transition matrix.
(b) Use the approach described in Sec. 29.6 to find the μ_{ij} (the expected first passage time from state i to state j) for all i and j. Use these results to identify the expected number of full days that the machine will remain operational before the next breakdown after a repair is completed.
(c) Now suppose that the machine already has gone 20 full days without a breakdown since the last repair was completed. How does the expected number of full days *hereafter* that the machine will remain operational before the next breakdown compare with the corresponding result from part (*b*) when the repair had just been completed? Explain.

29.6-3. Reconsider Prob. 29.6-2. Now suppose that the manufacturer keeps a spare machine that only is used when the primary machine is being repaired. During a repair day, the spare machine has a probability of 0.1 of breaking down, in which case it is repaired the next day. Denote the state of the system by (x, y), where x and y, respectively, take on the values 1 or 0 depending upon whether the primary machine (x) and the spare machine (y) are operational (value of 1) or not operational (value of 0) at the end of the day. [*Hint*: Note that (0, 0) is not a possible state.]

(a) Construct the (one-step) transition matrix for this Markov chain.
(b) Find the *expected recurrence time* for the state (1, 0).

29.6-4. Consider the inventory example presented in Sec. 29.1 except that demand now has the following probability distribution:

$$P\{D = 0\} = \frac{1}{4}, \qquad P\{D = 2\} = \frac{1}{4},$$

$$P\{D = 1\} = \frac{1}{2}, \qquad P\{D \geq 3\} = 0.$$

The ordering policy now is changed to ordering just 2 cameras at the end of the week if none are in stock. As before, no order is placed if there are any cameras in stock. Assume that there is one camera in stock at the time (the end of a week) the policy is instituted.

(a) Construct the (one-step) transition matrix.
C **(b)** Find the probability distribution of the state of this Markov chain n weeks after the new inventory policy is instituted, for $n = 2, 5, 10$.
(c) Find the μ_{ij} (the expected first passage time from state i to state j) for all i and j.
C **(d)** Find the steady-state probabilities of the state of this Markov chain.
(e) Assuming that the store pays a storage cost for each camera remaining on the shelf at the end of the week according to the function $C(0) = 0$, $C(1) = \$2$, and $C(2) = \$8$, find the long-run expected average storage cost per week.

29.6-5. A production process contains a machine that deteriorates rapidly in both quality and output under heavy usage, so that it is inspected at the end of each day. Immediately after inspection, the condition of the machine is noted and classified into one of four possible states:

State	Condition
0	Good as new
1	Operable—minimum deterioration
2	Operable—major deterioration
3	Inoperable and replaced by a good-as-new machine

The process can be modeled as a Markov chain with its (one-step) transition matrix **P** given by

State	0	1	2	3
0	0	$\frac{7}{8}$	$\frac{1}{16}$	$\frac{1}{16}$
1	0	$\frac{3}{4}$	$\frac{1}{8}$	$\frac{1}{8}$
2	0	0	$\frac{1}{2}$	$\frac{1}{2}$
3	1	0	0	0

C **(a)** Find the steady-state probabilities.
(b) If the costs of being in states 0, 1, 2, 3, are 0, \$1,000, \$3,000, and \$6,000, respectively, what is the long-run expected average cost per day?
(c) Find the *expected recurrence time* for state 0 (i.e., the expected length of time a machine can be used before it must be replaced).

29.7-1. Consider the following gambler's ruin problem. A gambler bets \$1 on each play of a game. Each time, he has a probability p of winning and probability $q = 1 - p$ of losing the dollar bet. He will continue to play until he goes broke or nets a fortune of T dollars. Let X_n denote the number of dollars possessed by the gambler after the nth play of the game. Then

$$X_{n+1} = \begin{cases} X_n + 1 & \text{with probability } p \\ X_n - 1 & \text{with probability } q = 1 - p \end{cases} \qquad \text{for } 0 < X_n < T,$$

$$X_{n+1} = X_n, \qquad \text{for } X_n = 0, \text{ or } T.$$

$\{X_n\}$ is a Markov chain. The gambler starts with X_0 dollars, where X_0 is a positive integer less than T.

(a) Construct the (one-step) transition matrix of the Markov chain.
(b) Find the classes of the Markov chain.

(c) Let $T = 3$ and $p = 0.3$. Using the notation of Sec. 29.7, find $f_{10}, f_{1T}, f_{20}, f_{2T}$.

(d) Let $T = 3$ and $p = 0.7$. Find $f_{10}, f_{1T}, f_{20}, f_{2T}$.

29.7-2. A video cassette recorder manufacturer is so certain of its quality control that it is offering a complete replacement warranty if a recorder fails within 2 years. Based upon compiled data, the company has noted that only 1 percent of its recorders fail during the first year, whereas 5 percent of the recorders that survive the first year will fail during the second year. The warranty does not cover replacement recorders.

(a) Formulate the evolution of the status of a recorder as a Markov chain whose states include two absorption states that involve needing to honor the warranty or having the recorder survive the warranty period. Then construct the (one-step) transition matrix.

(b) Use the approach described in Sec. 29.7 to find the probability that the manufacturer will have to honor the warranty.

29.8-1. Reconsider the example presented at the end of Sec. 29.8. Suppose now that a third machine, identical to the first two, has been added to the shop. The one maintenance person still must maintain all the machines.

(a) Develop the *rate diagram* for this Markov chain.

(b) Construct the *steady-state equations.*

(c) Solve these equations for the *steady-state probabilities.*

29.8-2. The state of a particular continuous time Markov chain is defined as the number of jobs currently at a certain work center, where a maximum of two jobs are allowed. Jobs arrive individually. Whenever fewer than two jobs are present, the time until the next arrival has an exponential distribution with a mean of 2 days. Jobs are processed at the work center one at a time and then leave immediately. Processing times have an exponential distribution with a mean of 1 day.

(a) Construct the *rate diagram* for this Markov chain.

(b) Write the *steady-state equations.*

(c) Solve these equations for the *steady-state probabilities.*