Chapter 13

Application of the Laplace Transform in Circuit Analysis

1. Find the Laplace transform of the signal $t(0.5)^t\, u(t)$.

Solution: $$f(t) = t(0.5)^t\, u(t)$$

Consider $$L[(0.5)^t] = \int_0^{\infty} (0.5)^t\, e^{-St}\, dt$$

$$= \int_0^{\infty} (0.5\, e^{-S})^t\, dt$$

Let $0.5\, e^{-S} = x$

$$\int_0^{\infty} (x)^t\, dt = \left[\frac{xt}{\log x}\right]_0^{\infty} = \frac{x^{\infty}}{\log x} - \frac{1}{\log x}$$

Since $$x = 0.5\, e^{-S}$$

$$L\,[(0.5)^t] = \frac{-1}{\log 0.5\, e^{-S}}$$

If $$f(t) \xleftrightarrow{L} F(S), \text{ then } t\,.f(t) \xleftrightarrow{L} \frac{-d}{dS}\, F(S)$$

$\therefore$ $$L\,[t(0.5)^t] = \frac{-d}{dS}\left[\frac{-1}{\log(0.5\, e^{-S})}\right]$$

$$= \frac{d}{dS}\left[\frac{1}{\log(0.5\, e^{-S})}\right]$$

$$\frac{d}{dS}\left[\frac{1}{\log(0.5\,e^{-S})}\right] = \frac{\frac{1}{0.5\,e^{-S}}\,e^{-S}\,(-1)}{\left(\log(0.5\,e^{-S})\right)^2}$$

$$\therefore \qquad L\,[t(0.5)^t\,u(t)] = \frac{-e^{-S}}{\left(-0.5\,e^{-S}\right)\left[\log\left(0.5\,e^{-S}\right)\right]^2}$$

2. Find the Laplace transform of the waveform $x(t)$ shown in the figure.

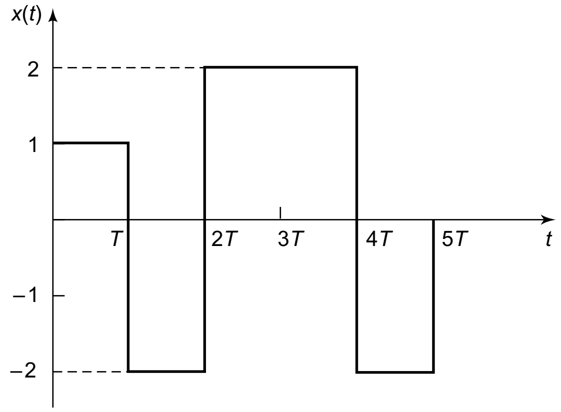

Fig. 13.1

Solution: By definition,

$$L\,[x(t)] = \int_0^{\infty} e^{-St}\,x(t)\,dt$$

$$= \int_0^{T} e^{-St}\,dt - \int_T^{2T} 2e^{-St}\,dt + \int_{2T}^{4T} 2e^{-St}\,dt - \int_{4T}^{5T} 2e^{-St}\,dt$$

$$= \left[\frac{-1}{S}e^{-St}\right]_0^T - \left[\frac{-2}{S}e^{-St}\right]_T^{2T} + \left[\frac{-2}{S}e^{-St}\right]_{2T}^{4T} - \left[\frac{-2}{S}e^{-St}\right]_{4T}^{5T}$$

$$= \left[\frac{1}{S} - \frac{1}{S}e^{-ST}\right] + \left[\frac{2}{S}e^{-2ST} - \frac{2}{S}e^{-ST}\right] - \left[\frac{2}{S}e^{-4ST} - \frac{2}{S}e^{-2ST}\right]$$

$$+ \left[\frac{2}{S}e^{-5ST} - \frac{2}{S}e^{-4ST}\right]$$

$$X(S) = \frac{1}{S}\;[1 - 3e^{-ST} + 4e^{-2ST} - 4e^{-4ST} + 2e^{-5ST}]$$

3. Find the signal $y(t)$, the Laplace transform of which is

$$Y(S) = \frac{S^3 + 7S^2 + 18S + 20}{S^2 + 5S + 6}$$

Solution: The fraction can be written as

$$Y(S) = 3 + 2 + \frac{25 + 8}{S^2 + 5S + 6}$$

Taking partial fraction for $\frac{2S + 8}{S^2 + 5S + 6}$, we get

$$\frac{2S + 8}{S^2 + 5S + 6} = \frac{A}{S + 3} + \frac{B}{S + 2}$$

$$A = \frac{2S + 8}{(S + 3)\,(S + 2)}\,(S + 3)\Big|_{S = -3} = -2$$

$$B = \frac{2S + 8}{(S + 3)\,(S + 2)}\,(S + 2)\Big|_{S = -2} = 4$$

$$Y(S) = S + 2 - \frac{2}{S + 3} + \frac{4}{S + 2}$$

Taking inverse transform both sides, we get

$$y(t) = \frac{d}{dt}\; S(t) + 2 \cdot \delta(t) - 2e^{-3t}\, u(t) + 4e^{-2t}\, u(t)$$

4. State and explain the initial value theorem of a Laplace transform. Using the initial value theorem, find the initial value of the signal corresponding to the Laplace transform

$$Y(S) = \frac{S + 1}{S\,(S + 2)}$$

Verify that the answer obtained is correct.

Solution: Refer section 13.4.

Initial value theorem states that

$$\underset{t \to 0}{\mathrm{Lt}}\; f(t) = \underset{t \to 0}{\mathrm{Lt}}\; SF(S)$$

Consider $\quad \underset{S \to \infty}{\mathrm{Lt}}\; SY(S) = \underset{S \to \infty}{\mathrm{Lt}}\; \frac{S + 1}{S + 2} = \frac{1 + 1/s}{1 + 2/s}$

$$\therefore \qquad \operatorname*{Lt}_{S \to \infty} SY(S) = 1$$

By taking inverse Laplace transform of $Y(S)$, we get $y(t)$

$$Y(S) = \frac{S+1}{S(S+2)} = \frac{A}{S} + \frac{B}{S+2}$$

$$A = SY(S)\Big|_{S=0} = \frac{S+1}{S+2}\Big|_{S=0} = \frac{1}{2}$$

$$B = (S+2)\,Y(S)\Big|_{S=-2} = \frac{S+1}{S}\Big|_{S=-2} = \frac{1}{2}$$

$$Y(S) = \frac{1}{2S} + \frac{1}{2\,(S+2)}$$

Taking inverse transform of both sides, we get

$$y(t) = \frac{1}{2}\,u(t) + \frac{1}{2}\,e^{-2t}\,u(t)$$

$$y(t)\Big|_{t=0} = \frac{1}{2} + \frac{1}{2} = 1$$

Hence, the initial value theorem is verified.

5. Find the inverse Laplace transform of

$$F(S) = \frac{1}{(S+2)^2}$$

Solution: The function can be written as

$$F(S) = F_1(S),\ F_2(S) = \frac{1}{S+2} \cdot \frac{1}{S+2}$$

We can use the convolution integral here:

$$f_1(t) = e^{-2t}\,u(t)$$

$$f_2(t) = e^{-2t}\,u(t)$$

By using convolution property, we have

$$f(t) = \int_0^t f_1(\tau)\,f_2(t-\tau)\,d\tau$$

$$= \int_0^t e^{-2\tau} \cdot e^{-2(t-\tau)}\, d\tau$$

$$= e^{-2t} \int_0^t d\tau$$

$$= e^{-2t} \left[\tau\right]_0^t$$

$$f(t) = te^{-2t}\, u(t)$$

6. For the circuit given, determine, the current in resistor 10 Ω when the switch is closed at $t = 0$, using Laplace transforms. Assume initial current through inductor is zero.

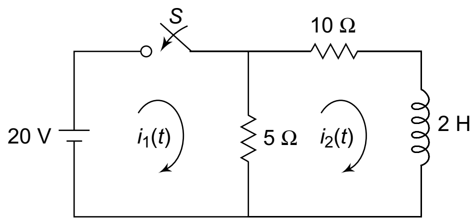

Fig. 13.2

Solution: When the switch is closed at $t = 0$, applying KVL in the circuit

$$5\,[i_1(t) - i_2(t)] = 20 \tag{1}$$

$$5\,[i_2(t) - i_1(t)] + 10i_2(t) + 2\,\frac{di_2(t)}{dt} = 0 \tag{2}$$

Taking Laplace transform for the above equation

$$5I_1(S) - 5I_2(S) = \frac{20}{S} \tag{3}$$

$$-\,5I_1(S) + 15\,I_2(S) + 2S\,I_2(S) = 0 \tag{4}$$

Add (3) and (4)

$$I_2(S)\,[10 + 2S] = \frac{20}{S}$$

$$I_2(S) = \frac{20}{S\,(2S+10)}$$

$$I_2(S) = \frac{10}{S\,(S+10)} = \frac{A}{S} + \frac{B}{S+5}$$

$$A = \frac{10}{S\,(S+5)}\,S\bigg|_{S=0} = 2$$

$$B = \frac{10}{S\,(S+5)}\,(S+5)\bigg|_{S=5} = -2$$

$$\therefore \qquad I_2(S) = \frac{2}{S} - \frac{2}{S+5}$$

Taking inverse Laplace transform on both sides, we get

$$i_2(t) = 2u(t) - 2e^{-5t}\,u(t)$$

Therefore, the current through 10 Ω resistor is

$$i_2(t) = 2(1 - e^{-5t})\,u(t) \text{ amp.}$$

7. The network shown in the figure is initially under steady-state condition with the switch in the position 1. The switch is moved from the position 1 to the position 2 at $t \neq 0$. Calculate the current $i(t)$ through R_1 after switching.

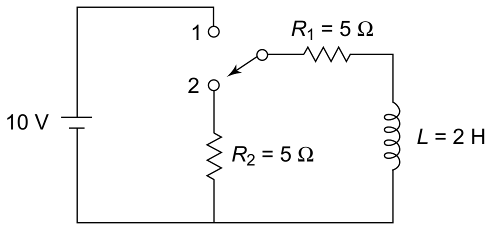

Fig. 13.3

Solution: When the switch is at the position 1, the steady-state current in the inductor

$$i = \frac{10}{5} = 2 \text{ A}$$

when the switch is at the position 2

$$Ri(t) + L\,\frac{di.(t)}{dt} = 0$$

Taking Laplace transforms on both sides

$$RI(S) + L\,[SI(S) - i(0)] = 0$$

$$(R + SL)\,I(S) = L\,i(0)$$

$$(10 + 2S)\, I(S) = 2 \times 2$$

$$I(S) = \frac{2}{S+5}$$

Taking inverse transform on both sides, we get

$$i(t) = 2e^{-5t}$$

8. In the network shown in the figure, the switch is closed at $t = 0$ and there is no initial charge on either of the capacitors. Find the current i by the Laplace transform method.

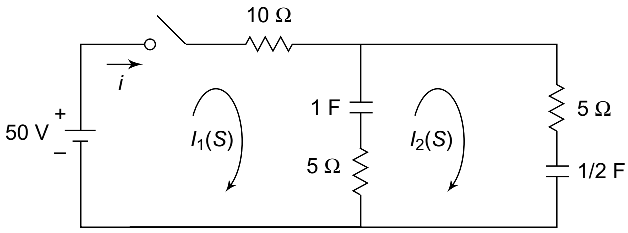

Fig. 13.4

Solution: Taking network transform, we have

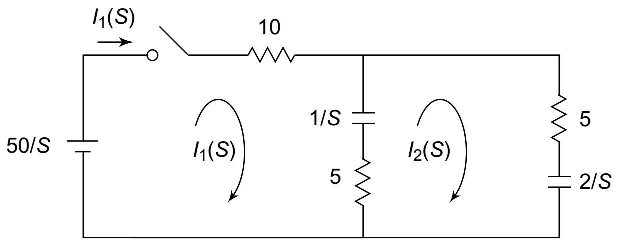

Fig. 13.5

On applying KVL to both loops, we get

$$\frac{50}{S} = 15I_1(S) - 5I_2(S) + \frac{I_1(S)}{S} - \frac{I_2(S)}{S} \tag{1}$$

$$0 = I_1(\text{S}) \left[-5 + \frac{1}{S}\right] + I_2(S)\left[10 + \frac{3}{S}\right] \tag{2}$$

From equation (2)

$$I_2(S) = \frac{\left(-5 + \dfrac{1}{S}\right)}{10 + \dfrac{3}{S}}\, I_1(S) \tag{3}$$

On substituting equation (3) in equation (1), we get

$$I_1(S) = \frac{500\,S + 150}{-2 + 35\,S + 125\,S^2}$$

$$I_1(S) = \frac{4\,(S + 0.25)}{(S + 0.328)\,(S - 0.04)} = \frac{A}{S + 0.328} + \frac{B}{S - 0.04}$$

From the above equation

$$A = \frac{4\,(S + 0.25)}{(S + 0.328)\,(S - 0.04)}\,(S + 0.328)\Bigg|_{S = -0.328}$$

$$A = 1.179$$

$$B = \frac{4\,(S + 0.25)}{(S + 0.328)\,(S - 0.04)}\,(S - 0.04)\Bigg|_{S = 0.04}$$

$$B = 3.15$$

$$I(S) = I_1(S) = \frac{1.179}{S + 0.328} + \frac{3.15}{S - 0.04}$$

By taking inverse Laplace transform

$$i(t) = 1.1179e^{-0.328t} + 3.15e^{+0.04t}$$

9. A sinusoidal voltage of 12 sin $8t$ volts is applied at $t = 0$ to a series circuit of $R = 4\ \Omega$ and $L = 1$ H. By Laplace transform method, determine the circuit current $i(t)$ for all $t \geq 0$. Assume zero initial conditions.

Solution:

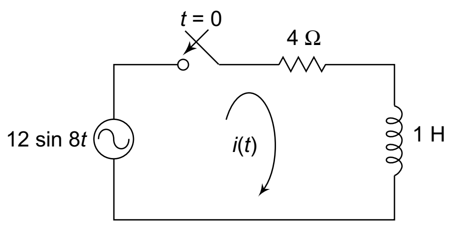

Fig. 13.6

Applying KVL in the circuit, for $t \geq 0$

$$12 \sin 8t = 4i(t) = L\frac{di\,(t)}{dt}$$

Taking Laplace transforms on both sides

$$\frac{12.8}{S^2+8^2} = 4I(S) + LSI(S)$$

$(\because\ i(0) = 0)$

$$I(S) = \frac{96}{(S^2+8^2)\,(S+4)}$$

Taking partial fractions

$$I(S) = \frac{AS+B}{S^2+8^2} + \frac{C}{S+4}$$

$$I(S) = \frac{-1.2S+4.8}{S^2+8^2} + \frac{1.2}{S+4}$$

$$I(S) = -1.2\frac{S}{S^2+8^2} + \frac{4.8}{8}\,\frac{8}{S^2+8^2} + \frac{1.2}{S+4}$$

Taking inverse transforms on both sides, we get

$$i(t) = (-\,1.2 \cos 8t + 0.6 \sin 8t + 1.2\, e^{-4t} \text{ amp.}$$

10. Consider the circuit shown in the figure, the switch is thrown from position 1 to 2 at time $t = 0$. Just before the switch is thrown, the initial conditions are
$i_L(0+) = 2\text{A}$, $v_C(0+) = 2$ V. Find the current $i(t)$ after the switch is thrown.

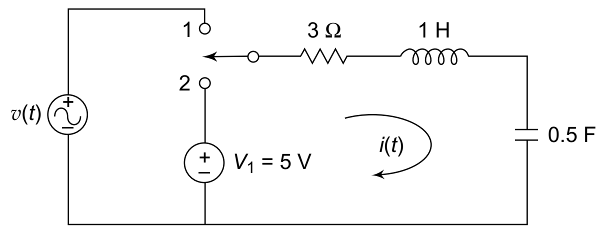

Fig. 13.7

Solution: At $t > 0$, the current $i(t)$ can be obtained by using KVL.

The loop equation is

$$3i(t) + 1\frac{di\,(t)}{dt} + \frac{1}{0.5}\int i(t)\,dt = 5V$$

Taking Laplace transforms on both sides

$$3I(S) + 1\,[SI(S) - i(0+)] + \frac{1}{0.5}\left[\frac{I\,(S)}{S}\right] + \left[\frac{v_C\,(0+)}{S}\right] = \frac{5}{S}$$

$$3I(S) + SI(S) - 2 + \frac{2}{S}\ I(S) + \frac{5}{S} = \frac{5}{S}$$

$$\left(3 + S + \frac{2}{S}\right) I(S) = \frac{5}{S} + 2 - \frac{2}{S}$$

$$\frac{3S + S^2 + 2}{S}\ I(S) = \frac{3 + 2S}{S}$$

$$I(S) = \frac{2S + 3}{S^2 + 3S + 2} = \frac{A}{S+1} + \frac{B}{S+2}$$

$$A = \frac{2\,S + 3}{(S+1)\,(S+2)}\,(S+1)\Bigg|_{S=-1} = 1$$

$$B = \frac{2\,S + 3}{(S+1)\,(S+2)}\,(S+2)\Bigg|_{S=-2} = 1$$

$$I(S) = \frac{1}{S+1} + \frac{1}{S+2}$$

Taking inverse Laplace transforms on both sides

$$i(t) = e^{-t}\,u(t) + e^{-2t}\,u(t)$$

Objective-Type Questions

1. The Laplace transformation of $f(t)$ is $F(S)$. Given $F(S) = \dfrac{\omega}{S^2 + \omega^2}$, the final value of $f(t)$ is
   (a) infinity
   (b) zero
   (c) one
   (d) none of the above
2. The Laplace transform of the function $i(t)$ is

   $I(S) = \dfrac{10S + 4}{S\,(S+1)\,(S^2 + 4S + 5)}$, its final value will be

   (a) $\dfrac{4}{5}$ (b) $\dfrac{5}{4}$
   (c) 4 (d) 5

3. If $L\,[f(t)] = \dfrac{2\,(S+1)}{S^2+2S+5}$, then $f(0+)$ and $f(\infty)$ are given by

   (a) 0, 2 respectively
   (b) 2, 0 respectively
   (c) 0, 1 respectively
   (d) $\dfrac{2}{5}$, 0 respectively

4. The Laplace transform of the waveform shown in the figure will be

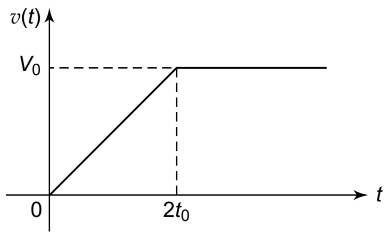

Fig. 13.8

   (a) $\dfrac{V_o}{t_o S^2}\left[1-e^{-t_o S}\right]-\dfrac{V_o}{S}\,e^{-2t_o S}$

   (b) $\dfrac{V_o}{t_o S^2}\left[1-e^{-t_o S}+e^{-2t_o S}\right]$

   (c) $\dfrac{V_o}{S}-\dfrac{V_o}{t_o\,S^2}\left(e^{-2t_o S}+e^{-3t_o S}\right)$

   (d) $\dfrac{V_o}{t_o\,S^2}\left(e^{-t_o S}\right)-\dfrac{V_o}{S}\,e^{-t_o S}$

5. Laplace transform of the function e^{-20t} is

   (a) $\dfrac{1}{S-20}$
   (b) $S + 20$
   (c) $S - 20$
   (d) $\dfrac{1}{S+20}$

6. The inverse transform of $\dfrac{2}{S+3}$ is

   (a) $2(t + 3)$
   (b) $2e^{-3t}$
   (c) e^{-3t}
   (d) $2e^{-t}$

7. Laplace transform of damped sine wave $e^{-3t} \sin 50t$ is

   (a) $\dfrac{1}{(S+3)^2 + 50^2}$

   (b) $\dfrac{S}{(S+3)^2 + 50^2}$

   (c) $\dfrac{50}{(S+3)^2 + 50^2}$

   (d) $\dfrac{50^2}{(S+3)^2 + 50^2}$