Appendix A

Algebra Review

A–1 Factorials

Definition and Properties of Factorials

The notation called factorial notation is used in probability. *Factorial notation* uses the exclamation point and involves multiplication. For example,

$$\begin{aligned} 5! &= 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120 \\ 4! &= 4 \cdot 3 \cdot 2 \cdot 1 = 24 \\ 3! &= 3 \cdot 2 \cdot 1 = 6 \\ 2! &= 2 \cdot 1 = 2 \\ 1! &= 1 \end{aligned}$$

In general, a factorial is evaluated as follows:

$$n! = n(n-1)(n-2)\cdots 3 \cdot 2 \cdot 1$$

Note that the factorial is the product of n factors, with the number decreased by 1 for each factor.

One property of factorial notation is that it can be stopped at any point by using the exclamation point. For example,

$$\begin{aligned} 5! &= 5 \cdot 4! && \text{since} \quad 4! = 4 \cdot 3 \cdot 2 \cdot 1 \\ &= 5 \cdot 4 \cdot 3! && \text{since} \quad 3! = 3 \cdot 2 \cdot 1 \\ &= 5 \cdot 4 \cdot 3 \cdot 2! && \text{since} \quad 2! = 2 \cdot 1 \\ &= 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \end{aligned}$$

Thus,

$$\begin{aligned} n! &= n(n-1)! \\ &= n(n-1)(n-2)! \\ &= n(n-1)(n-2)(n-3)! \qquad \text{etc.} \end{aligned}$$

Another property of factorials is

$$0! = 1$$

This fact is needed for formulas.

Operations with Factorials

Factorials cannot be added or subtracted directly. They must be multiplied out. Then the products can be added or subtracted.

Example A–1

Evaluate $3! + 4!$.

Solution

$$\begin{aligned} 3! + 4! &= (3 \cdot 2 \cdot 1) + (4 \cdot 3 \cdot 2 \cdot 1) \\ &= 6 + 24 = 30 \end{aligned}$$

Note: $3! + 4! \neq 7!$, since $7! = 5040$.

Example A–2

Evaluate $5! - 3!$.

Solution

$$\begin{aligned} 5! - 3! &= (5 \cdot 4 \cdot 3 \cdot 2 \cdot 1) - (3 \cdot 2 \cdot 1) \\ &= 120 - 6 = 114 \end{aligned}$$

Note: $5! - 3! \neq 2!$, since $2! = 2$.

Factorials cannot be multiplied directly. Again, you must multiply them out and then multiply the products.

Example A–3

Evaluate $3! \cdot 2!$.

Solution

$$3! \cdot 2! = (3 \cdot 2 \cdot 1) \cdot (2 \cdot 1) = 6 \cdot 2 = 12$$

Note: $3! \cdot 2! \neq 6!$, since $6! = 720$.

Finally, factorials cannot be divided directly unless they are equal.

Example A–4

Evaluate $6! \div 3!$.

Solution

$$\frac{6!}{3!} = \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1} = \frac{720}{6} = 120$$

Note: $\dfrac{6!}{3!} \neq 2! \qquad \text{since} \qquad 2! = 2$

But $\dfrac{3!}{3!} = \dfrac{3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1} = \dfrac{6}{6} = 1$

In division, you can take some shortcuts, as shown:

$$\frac{6!}{3!} = \frac{6 \cdot 5 \cdot 4 \cdot 3!}{3!} \quad \text{and} \quad \frac{3!}{3!} = 1$$
$$= 6 \cdot 5 \cdot 4 = 120$$
$$\frac{8!}{6!} = \frac{8 \cdot 7 \cdot 6!}{6!} \quad \text{and} \quad \frac{6!}{6!} = 1$$
$$= 8 \cdot 7 = 56$$

Another shortcut that can be used with factorials is cancellation, after factors have been expanded. For example,

$$\frac{7!}{(4!)(3!)} = \frac{7 \cdot 6 \cdot 5 \cdot 4!}{3 \cdot 2 \cdot 1 \cdot 4!}$$

Now cancel both instances of 4!. Then cancel the $3 \cdot 2$ in the denominator with the 6 in the numerator.

$$\frac{7 \cdot \overset{1}{\cancel{6}} \cdot 5 \cdot \overset{1}{\cancel{4!}}}{\underset{1}{\cancel{3}} \cdot \underset{1}{\cancel{2}} \cdot 1 \cdot \underset{1}{\cancel{4!}}} = 7 \cdot 5 = 35$$

Example A–5

Evaluate 10! ÷ (6!)(4!).

Solution

$$\frac{10!}{(6!)(4!)} = \frac{10 \cdot \overset{3}{\cancel{9}} \cdot \overset{1}{\cancel{8}} \cdot 7 \cdot \overset{1}{\cancel{6!}}}{\underset{1}{\cancel{4}} \cdot \underset{1}{\cancel{3}} \cdot \underset{1}{\cancel{2}} \cdot 1 \cdot \underset{1}{\cancel{6!}}} = 10 \cdot 3 \cdot 7 = 210$$

Exercises

Evaluate each expression.

A–1. 9!

A–2. 7!

A–3. 5!

A–4. 0!

A–5. 1!

A–6. 3!

A–7. $\frac{12!}{9!}$

A–8. $\frac{10!}{2!}$

A–9. $\frac{5!}{3!}$

A–10. $\frac{11!}{7!}$

A–11. $\frac{9!}{(4!)(5!)}$

A–12. $\frac{10!}{(7!)(3!)}$

A–13. $\frac{8!}{(4!)(4!)}$

A–14. $\frac{15!}{(12!)(3!)}$

A–15. $\frac{10!}{(10!)(0!)}$

A–16. $\frac{5!}{(3!)(2!)(1!)}$

A–17. $\frac{8!}{(3!)(3!)(2!)}$

A–18. $\frac{11!}{(7!)(2!)(2!)}$

A–19. $\frac{10!}{(3!)(2!)(5!)}$

A–20. $\frac{6!}{(2!)(2!)(2!)}$

A–2 Summation Notation

In mathematics, the symbol Σ (Greek capital letter sigma) means to add or find the sum. For example, ΣX means to add the numbers represented by the variable X. Thus, when X represents 5, 8, 2, 4, and 6, then ΣX means $5 + 8 + 2 + 4 + 6 = 25$.

Sometimes, a subscript notation is used, such as

$$\sum_{i=1}^{5} X_i$$

This notation means to find the sum of five numbers represented by X, as shown:

$$\sum_{i=1}^{5} X_i = X_1 + X_2 + X_3 + X_4 + X_5$$

When the number of values is not known, the unknown number can be represented by n, such as

$$\sum_{i=1}^{n} X_i = X_1 + X_2 + X_3 + \cdots + X_n$$

There are several important types of summation used in statistics. The notation ΣX^2 means to square each value before summing. For example, if the values of the X's are 2, 8, 6, 1, and 4, then

$$\Sigma X^2 = 2^2 + 8^2 + 6^2 + 1^2 + 4^2$$
$$= 4 + 64 + 36 + 1 + 16 = 121$$

The notation $(\Sigma X)^2$ means to find the sum of X's and then square the answer. For instance, if the values for X are 2, 8, 6, 1, and 4, then

$$(\Sigma X)^2 = (2 + 8 + 6 + 1 + 4)^2$$
$$= (21)^2 = 441$$

Another important use of summation notation is in finding the mean (shown in Section 3–1). The mean $\overline{X}$ is defined as

$$\overline{X} = \frac{\Sigma X}{n}$$

For example, to find the mean of 12, 8, 7, 3, and 10, use the formula and substitute the values, as shown:

$$\overline{X} = \frac{\Sigma X}{n} = \frac{12 + 8 + 7 + 3 + 10}{5} = \frac{40}{5} = 8$$

The notation $\Sigma(X - \overline{X})^2$ means to perform the following steps.

STEP 1 Find the mean.

STEP 2 Subtract the mean from each value.

STEP 3 Square the answers.

STEP 4 Find the sum.

Example A–6

Find the value of $\Sigma(X - \overline{X})^2$ for the values 12, 8, 7, 3, and 10 of X.

Solution

STEP 1 Find the mean.

$$\overline{X} = \frac{12 + 8 + 7 + 3 + 10}{5} = \frac{40}{5} = 8$$

STEP 2 Subtract the mean from each value.

$$12 - 8 = 4 \qquad 7 - 8 = -1 \qquad 10 - 8 = 2$$
$$8 - 8 = 0 \qquad 3 - 8 = -5$$

STEP 3 Square the answers.

$$4^2 = 16 \qquad (-1)^2 = 1 \qquad 2^2 = 4$$
$$0^2 = 0 \qquad (-5)^2 = 25$$

STEP 4 Find the sum.

$$16 + 0 + 1 + 25 + 4 = 46$$

Example A–7

Find $\Sigma(X - \overline{X})^2$ for the following values of X: 5, 7, 2, 1, 3, 6.

Solution

Find the mean.

$$\overline{X} = \frac{5 + 7 + 2 + 1 + 3 + 6}{6} = \frac{24}{6} = 4$$

Then the steps in Example A–6 can be shortened as follows:

$$\begin{aligned}\Sigma(X - \overline{X})^2 &= (5 - 4)^2 + (7 - 4)^2 + (2 - 4)^2 \\ &\quad + (1 - 4)^2 + (3 - 4)^2 + (6 - 4)^2 \\ &= 1^2 + 3^2 + (-2)^2 + (-3)^2 \\ &\quad + (-1)^2 + 2^2 \\ &= 1 + 9 + 4 + 9 + 1 + 4 = 28\end{aligned}$$

Exercises

For each set of values, find ΣX, ΣX^2, $(\Sigma X)^2$, and $\Sigma(X - \overline{X})^2$.

A–21. 9, 17, 32, 16, 8, 2, 9, 7, 3, 18

A–22. 4, 12, 9, 13, 0, 6, 2, 10

A–23. 5, 12, 8, 3, 4

A–24. 6, 2, 18, 30, 31, 42, 16, 5

A–25. 80, 76, 42, 53, 77

A–26. 123, 132, 216, 98, 146, 114

A–27. 53, 72, 81, 42, 63, 71, 73, 85, 98, 55

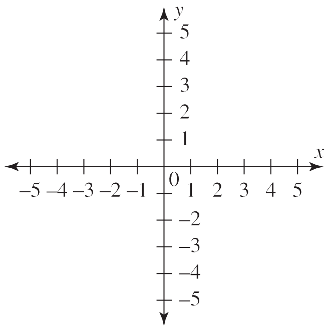
A–28. 43, 32, 116, 98, 120

A–29. 12, 52, 36, 81, 63, 74

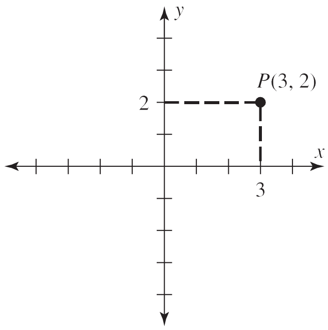
A–30. −9, −12, 18, 0, −2, −15

A–3 The Line

The following figure shows the *rectangular coordinate system,* or *Cartesian plane.* This figure consists of two axes: the horizontal axis, called the x axis, and the vertical axis, called the y axis. Each axis has numerical scales. The point of intersection of the axes is called the *origin.*

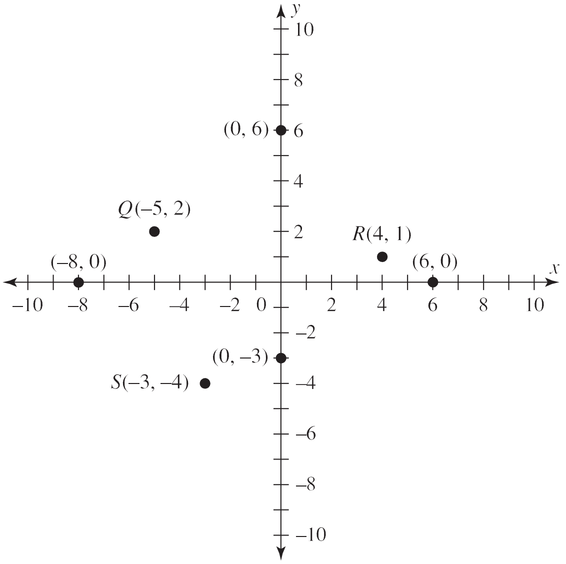


Points can be graphed by using coordinates. For example, the notation for point $P(3, 2)$ means that the x coordinate is 3 and the y coordinate is 2. Hence, P is located at the intersection of $x = 3$ and $y = 2$, as shown.



Other points, such as $Q(-5, 2)$, $R(4, 1)$, and $S(-3, -4)$, can be plotted, as shown in the next figure.

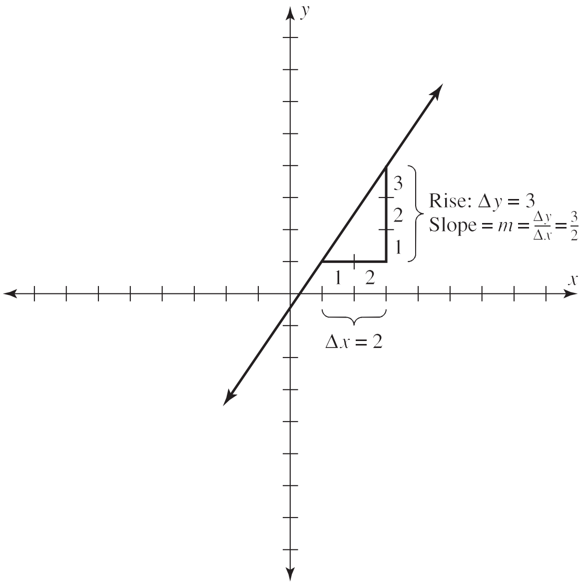
When a point lies on the y axis, the x coordinate is 0, as in $(0, 6)(0, -3)$, etc. When a point lies on the x axis, the y coordinate is 0, as in $(6, 0)(-8, 0)$, etc., as shown at the top of the next page.



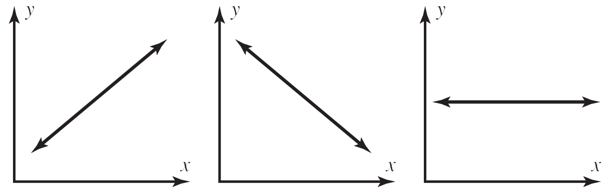
Two points determine a line. There are two properties of a line: its slope and its equation. The *slope m* of a line is determined by the ratio of the rise (called Δy) to the run (Δx).

$$m = \frac{\text{rise}}{\text{run}} = \frac{\Delta y}{\Delta x}$$

For example, the slope of the line shown below is $\frac{3}{2}$, or 1.5, since the height Δy is 3 units and the run Δx is 2 units.

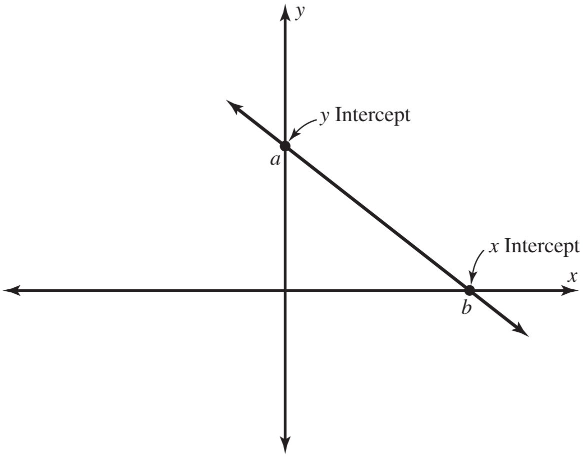


The slopes of lines can be positive, negative, or zero. A line going uphill from left to right has a positive slope. A line going downhill from left to right has a negative slope. And a line that is horizontal has a slope of zero.



(a) Positive slope **(b)** Negative slope **(c)** Zero slope

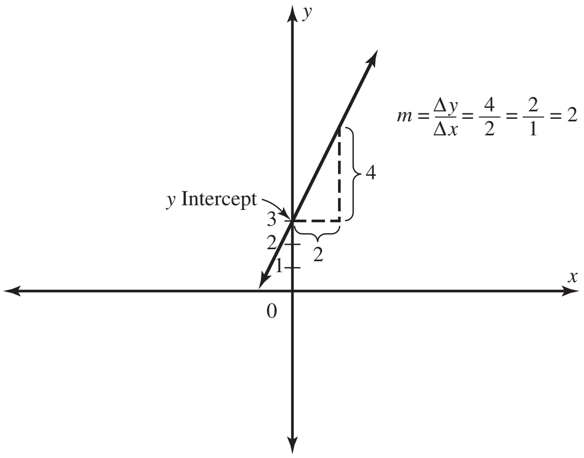
A point b where the line crosses the x axis is called the *x intercept* and has the coordinates (b, 0). A point a where the line crosses the y axis is called the *y intercept* and has the coordinates (0, a).



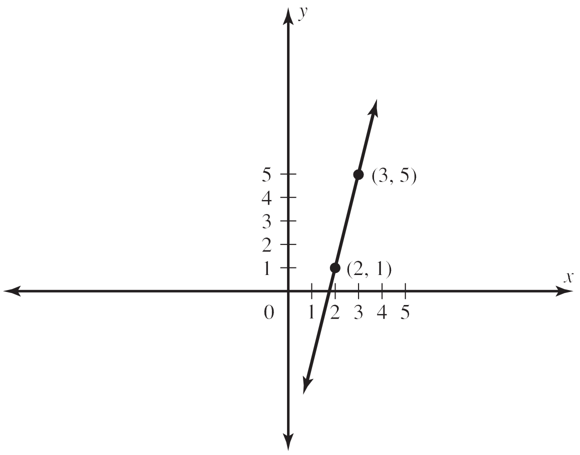
Every line has a unique equation of the form $y = a + bx$. For example, the equations

$$y = 5 + 3x$$
$$y = 8.6 + 3.2x$$
$$y = 5.2 - 6.1x$$

all represent different, unique lines. The number represented by a is the y intercept point; the number represented by b is the slope. The line whose equation is $y = 3 + 2x$ has a y intercept at 3 and a slope of 2, or $\frac{2}{1}$. This line can be shown as in the following graph.



If two points are known, then the graph of the line can be plotted. For example, to find the graph of a line passing through the points $P(2, 1)$ and $Q(3, 5)$, plot the points and connect them as shown below.



Given the equation of a line, you can graph the line by finding two points and then plotting them.

Example A–8

Plot the graph of the line whose equation is $y = 3 + 2x$.

Solution

Select any number as an x value, and substitute it in the equation to get the corresponding y value. Let $x = 0$. Then

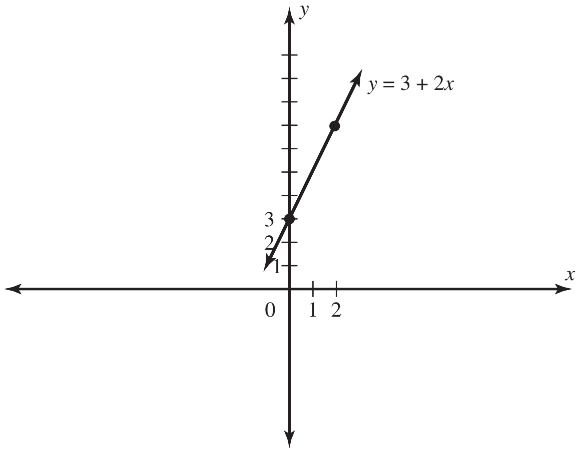
$$y = 3 + 2x = 3 + 2(0) = 3$$

Hence, when $x = 0$, then $y = 3$, and the line passes through the point (0, 3).

Now select any other value of x, say, $x = 2$.

$$y = 3 + 2x = 3 + 2(2) = 7$$

Hence, a second point is (2, 7). Then plot the points and graph the line.



Exercises

Plot the line passing through each set of points.

A–31. $P(3, 2)$, $Q(1, 6)$

A–32. $P(0, 5)$, $Q(8, 0)$

A–33. $P(-2, 4)$, $Q(3, 6)$

A–34. $P(-1, -2)$, $Q(-7, 8)$

A–35. $P(6, 3)$, $Q(10, 3)$

Find at least two points on each line, and then graph the line containing these points.

A–36. $y = 5 + 2x$

A–37. $y = -1 + x$

A–38. $y = 3 + 4x$

A–39. $y = -2 - 2x$

A–40. $y = 4 - 3x$