Teacher Edition

Glencoe Secondary Mathematics

ALIGNED TO THE

COMMON
CORE
STATE
STANDARDS

Algebra 1

McGraw Hill Education

Bothell, WA • Chicago, IL • Columbus, OH • New York, NY

TI-Nspire is a trademark of Texas Instruments Incorporated.
Texas Instruments images used with permission.

connectED.mcgraw-hill.com

McGraw-Hill Education

Copyright © 2012 by The McGraw-Hill Companies, Inc.

All rights reserved. No part of this publication may be reproduced or distributed in any form or by any means, or stored in a database or retrieval system, without the prior written consent of The McGraw-Hill Companies, Inc., including, but not limited to, network storage or transmission, or broadcast for distance learning.

Permission is granted to reproduce the material contained on pages 84-91 on the condition that such material be reproduced only for classroom use; be provided to students, teachers, or families without charge; and be used solely in conjunction with *Glencoe Algebra 1*.

Send all inquiries to:
McGraw-Hill Education
STEM Learning Solutions Center
8787 Orion Place
Columbus, OH 43240

ISBN: 978-0-07-661969-6 (*Teacher Edition*)
MHID: 0-07-661969-9 (*Teacher Edition*)

ISBN: 978-0-07-661900-9 (*Student Edition*)
MHID: 0-07-661900-1 (*Student Edition*)

Printed in the United States of America.

1 2 3 4 5 6 7 8 9 ONL 19 18 17 16 15 14 13 12 11

STEM

McGraw-Hill is committed to providing instructional materials in Science, Technology, Engineering, and Mathematics (STEM) that give students a solid foundation, one that prepares them for college and careers in the 21st Century.

Common Core State Standards

Table of Contents

Welcome to Glencoe Secondary Mathematics to the Common Core

How to Use This Supplement

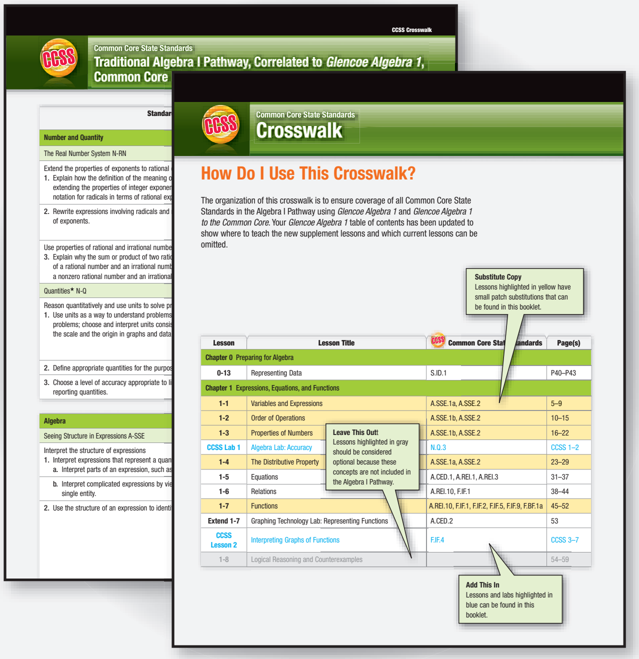
This supplement is your tour guide to understanding how Glencoe Secondary Mathematics programs teach the new Common Core State Standards. Its purpose is to help make a smooth transition from your state standards to the new Common Core State Standards.

Crosswalk

The crosswalk is your guide to understanding how to use your current *Glencoe Algebra 1* program with this supplement to create a Common Core State Standards curriculum. Pages vi–xii show you which lessons in your textbook should be kept, which can be considered optional, which lessons have additional available content, and how the new material fits into the flow of the chapters you already use.

Correlations

Glencoe Algebra 1 and *Glencoe Secondary Mathematics to the Common Core* align your curriculum with the Common Core State Standards and the Traditional Algebra I Pathway. You can use pages xiii–xviii to map each standard to the lesson(s) that address each standard.



CCSS Crosswalk

Common Core State Standards

Traditional Algebra I Pathway, Correlated to *Glencoe Algebra 1*, Common Core

Standar

Number and Quantity

The Real Number System N-RN

Extend the properties of exponents to rational

1. Explain how the definition of the meaning o extending the properties of integer exponen notation for radicals in terms of rational exp

2. Rewrite expressions involving radicals and of exponents.

Use properties of rational and irrational numbe

3. Explain why the sum or product of two ratio of a rational number and an irrational numb a nonzero rational number and an irrational

Quantities★ N-Q

Reason quantitatively and use units to solve pr

1. Use units as a way to understand problems problems; choose and interpret units consis the scale and the origin in graphs and data

2. Define appropriate quantities for the purpos

3. Choose a level of accuracy appropriate to li reporting quantities.

Algebra

Seeing Structure in Expressions A-SSE

Interpret the structure of expressions

1. Interpret expressions that represent a quan
   a. Interpret parts of an expression, such as
   b. Interpret complicated expressions by vie single entity.

2. Use the structure of an expression to identi

Common Core State Standards

Crosswalk

How Do I Use This Crosswalk?

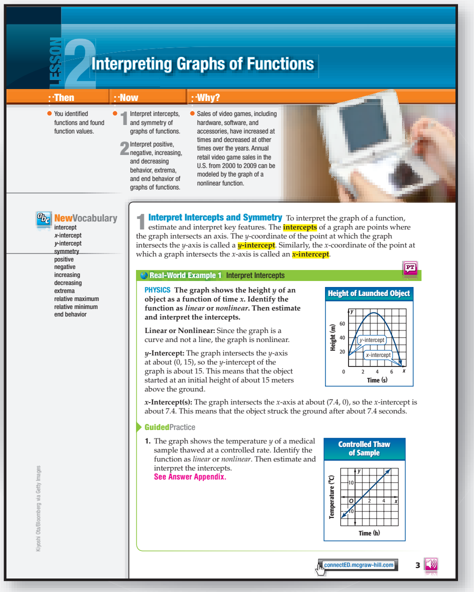
The organization of this crosswalk is to ensure coverage of all Common Core State Standards in the Algebra I Pathway using *Glencoe Algebra 1* and *Glencoe Algebra 1 to the Common Core*. Your *Glencoe Algebra 1* table of contents has been updated to show where to teach the new supplement lessons and which current lessons can be omitted.

Substitute Copy
Lessons highlighted in yellow have small patch substitutions that can be found in this booklet.

Lesson	Lesson Title	Common Core Stat andards	Page(s)
Chapter 0 Preparing for Algebra			
0-13	Representing Data	S.ID.1	P40–P43
Chapter 1 Expressions, Equations, and Functions			
1-1	Variables and Expressions	A.SSE.1a, A.SSE.2	5–9
1-2	Order of Operations	A.SSE.1b, A.SSE.2	10–15
1-3	Properties of Numbers	A.SSE.1b, A.SSE.2	16–22
CCSS Lab 1	Algebra Lab: Accuracy	N.Q.3	CCSS 1–2
1-4	The Distributive Property	A.SSE.1a, A.SSE.2	23–29
1-5	Equations	A.CED.1, A.REI.1, A.REI.3	31–37
1-6	Relations	A.REI.10, F.IF.1	38–44
1-7	Functions	A.REI.10, F.IF.1, F.IF.2, F.IF.5, F.IF.9, F.BF.1a	45–52
Extend 1-7	Graphing Technology Lab: Representing Functions	A.CED.2	53
CCSS Lesson 2	Interpreting Graphs of Functions	F.IF.4	CCSS 3–7
1-8	Logical Reasoning and Counterexamples		54–59

Leave This Outt
Lessons highlighted in gray should be considered optional because these concepts are not included in the Algebra I Pathway.

Add This In
Lessons and labs highlighted in blue can be found in this booklet.



LESSON 2 Interpreting Graphs of Functions

Then
- You identified functions and found function values.

Now
1. Interpret intercepts, and symmetry of graphs of functions.
2. Interpret positive, negative, increasing, and decreasing behavior, extrema, and end behavior of graphs of functions.

Why?
- Sales of video games, including hardware, software, and accessories, have increased at times and decreased at other times over the years. Annual retail video game sales in the U.S. from 2000 to 2009 can be modeled by the graph of a nonlinear function.

NewVocabulary
intercept
x-intercept
y-intercept
symmetry
positive
negative
increasing
decreasing
extrema
relative maximum
relative minimum
end behavior

1 Interpret Intercepts and Symmetry To interpret the graph of a function, estimate and interpret key features. The **intercepts** of a graph are points where the graph intersects an axis. The *y*-coordinate of the point at which the graph intersects the *y*-axis is called a ***y*-intercept**. Similarly, the *x*-coordinate of the point at which a graph intersects the *x*-axis is called an ***x*-intercept**.

Real-World Example 1 Interpret Intercepts

PHYSICS **The graph shows the height *y* of an object as a function of time *x*. Identify the function as *linear* or *nonlinear*. Then estimate and interpret the intercepts.**

Linear or Nonlinear: Since the graph is a curve and not a line, the graph is nonlinear.

***y*-Intercept:** The graph intersects the *y*-axis at about (0, 15), so the *y*-intercept of the graph is about 15. This means that the object started at an initial height of about 15 meters above the ground.

***x*-Intercept(s):** The graph intersects the *x*-axis at about (7.4, 0), so the *x*-intercept is about 7.4. This means that the object struck the ground after about 7.4 seconds.

GuidedPractice

1. The graph shows the temperature *y* of a medical sample thawed at a controlled rate. Identify the function as *linear* or *nonlinear*. Then estimate and interpret the intercepts.
   See Answer Appendix.

Kiyoshi Ota/Bloomberg via Getty Images

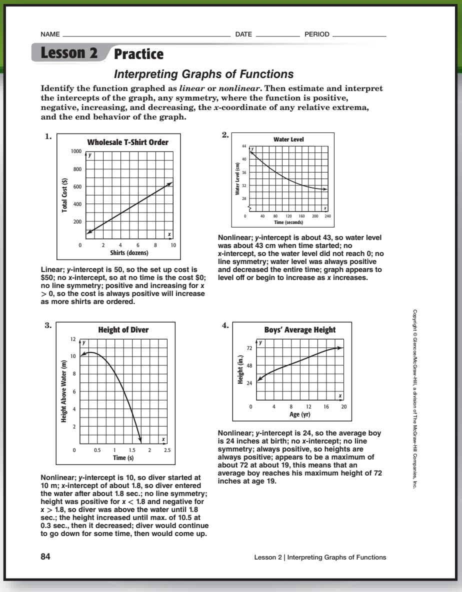
connectED.mcgraw-hill.com 3

In This Booklet

Glencoe Secondary Mathematics to the Common Core contains additional lessons and labs to address the Common Core State Standards and the Traditional Algebra I Pathway. (See pages 1–78.) You can also find copy for patch substitutions that can help you better meet the Common Core State Standards using your existing program. (See pages 79–83.) Refer to the Crosswalk on pages vi-xii for appropriate placement of this content in your *Glencoe Algebra 1* textbook.

Homework Practice

Pages 84–91 of the Student Edition of *Glencoe Secondary Mathematics to the Common Core* contain homework practice pages for the lessons added to meet the Common Core State Standards.



NAME ______________________ DATE ____________ PERIOD ____________

Lesson 2 **Practice**

Interpreting Graphs of Functions

Identify the function graphed as *linear* or *nonlinear*. Then estimate and interpret the intercepts of the graph, any symmetry, where the function is positive, negative, increasing, and decreasing, the x-coordinate of any relative extrema, and the end behavior of the graph.

1.

Linear; y-intercept is 50, so the set up cost is $50; no x-intercept, so at no time is the cost $0; no line symmetry; positive and increasing for $x > 0$, so the cost is always positive will increase as more shirts are ordered.

2.

Nonlinear; y-intercept is about 43, so water level was about 43 cm when time started; no x-intercept, so the water level did not reach 0; no line symmetry; water level was always positive and decreased the entire time; graph appears to level off or begin to increase as x increases.

3.

Nonlinear; y-intercept is 10, so diver started at 10 m; x-intercept of about 1.8, so diver entered the water after about 1.8 sec.; no line symmetry; height was positive for $x < 1.8$ and negative for $x > 1.8$, so diver was above the water until 1.8 sec.; the height increased until max. of 10.5 at 0.3 sec., then it decreased; diver would continue to go down for some time, then would come up.

4.

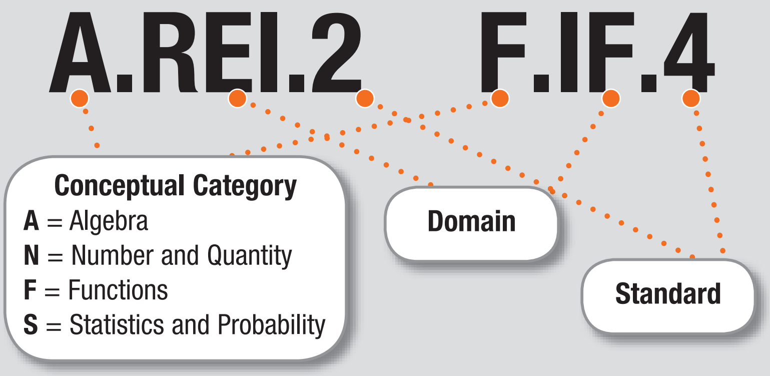
Nonlinear; y-intercept is 24, so the average boy is 24 inches at birth; no x-intercept; no line symmetry; always positive, so heights are always positive; appears to be a maximum of about 72 at about 19, this means that an average boy reaches his maximum height of 72 inches at age 19.

Copyright © Glencoe/McGraw-Hill, a division of The McGraw-Hill Companies, Inc.

84 Lesson 2 | Interpreting Graphs of Functions

Decoding the Common Core State Standards

This diagram provides clarity for decoding the standard identifiers.



Conceptual Category
A = Algebra
N = Number and Quantity
F = Functions
S = Statistics and Probability

Domain Names	Abbreviations
The **R**eal **N**umber System	**RN**
Quantities	**Q**
Seeing **S**tructure in **E**xpressions	**SSE**
Arithmetic with **P**olynomials and **R**ational Expressions	**APR**
Creating **E**quations	**CED**
Reasoning with **E**quations and **I**nequalities	**REI**
Interpreting **F**unctions	**IF**
Building **F**unctions	**BF**
Linear, Quadratic, and **E**xponential Models	**LE**
Interpreting Categorical and Quantitative **D**ata	**ID**

You can choose how to use *Glencoe Secondary Mathematics to the Common Core* in your classroom.

- To print, go to connectED.mcgraw-hill.com.
- To display for whole class instruction, go to connectED.mcgraw-hill.com.
- To order print copies of the Student Edition, contact your local sales representative.

Common Core State Standards

Crosswalk

How Do I Use This Crosswalk?

The organization of this crosswalk is to ensure coverage of all Common Core State Standards in the Algebra I Pathway using *Glencoe Algebra 1* and *Glencoe Algebra 1 to the Common Core*. Your *Glencoe Algebra 1* table of contents has been updated to show where to teach the new supplement lessons and which current lessons can be omitted.

Substitute Copy
Lessons highlighted in yellow have small patch substitutions that can be found in this booklet.

Lesson	Lesson Title	Common Core Stat andards	Page(s)
Chapter 0 Preparing for Algebra			
0-13	Representing Data	S.ID.1	P40–P43
Chapter 1 Expressions, Equations, and Functions			
1-1	Variables and Expressions	A.SSE.1a, A.SSE.2	5–9
1-2	Order of Operations	A.SSE.1b, A.SSE.2	10–15
1-3	Properties of Numbers	A.SSE.1b, A.SSE.2	16–22
CCSS Lab 1	Algebra Lab: Accuracy	N.Q.3	CCSS 1–2
1-4	The Distributive Property	A.SSE.1a, A.SSE.2	23–29
1-5	Equations	A.CED.1, A.REI.1, A.REI.3	31–37
1-6	Relations	A.REI.10, F.IF.1	38–44
1-7	Functions	A.REI.10, F.IF.1, F.IF.2, F.IF.5, F.IF.9, F.BF.1a	45–52
Extend 1-7	Graphing Technology Lab: Representing Functions	A.CED.2	53
CCSS Lesson 2	Interpreting Graphs of Functions	F.IF.4	CCSS 3–7
1-8	Logical Reasoning and Counterexamples		54–59

Leave This Outt
Lessons highlighted in gray should be considered optional because these concepts are not included in the Algebra I Pathway.

Add This In
Lessons and labs highlighted in blue can be found in this booklet.

Lesson	Lesson Title	Common Core State Standards	Page(s)
Chapter 2 Linear Equations			
2-1	Writing Equations	A.CED.1	75–80
Explore 2-2	Algebra Lab: Solving Equations	A.REI.3	81–82
2-2	Solving One-Step Equations	A.CED.1, A.REI.1, A.REI.3	83–89
Explore 2-3	Algebra Lab: Solving Multi-Step Equations	A.REI.3	90
2-3	Solving Multi-Step Equations	A.CED.1, A.REI.1, A.REI.3	91–96
2-4	Solving Equations with the Variable on Each Side	A.CED.1, A.REI.1, A.REI.3	97–102
2-5	Solving Equations Involving Absolute Value	A.CED.1, A.REI.1, A.REI.3	102–109
2-6	Ratios and Proportions	N.Q.1, A.REI.1, A.REI.3	111–117
Extend 2-6	**Spreadsheet Lab:** Financial Ratios	*Use CCSS Lab 3 in place of this lab.*	118
CCSS Lab 3	Spreadsheet Lab: Descriptive Modeling	N.Q.2	CCSS 8
2-7	Percent of Change	N.Q.1, A.REI.3	119–124
2-8	Literal Equations and Dimensional Analysis	N.Q.1, A.CED.4, A.REI.3	126–131
2-9	Weighted Averages	N.Q.1, A.CED.1, A.CED.4, A.REI.1, A.REI.3	132–138

Chapter 3 Linear Functions			
CCSS Lab 4	**Algebra Lab:** Analyzing Linear Graphs	F.IF.4	CCSS 9–10
3-1	Graphing Linear Equations	A.CED.2, A.REI.10, F.IF.4, F.IF.7a, F.BF.1a	153–160
3-2	Solving Linear Equations by Graphing	A.CED.1, A.REI.10, F.IF.7a	161–166
Extend 3-2	Graphing Technology Lab: Graphing Linear Equations	N.Q.1, F.IF.7a	167–168
Explore 3-3	Algebra Lab: Rate of Change of a Linear Function	F.IF.6, F.LE.1a	169
3-3	Rate of Change and Slope	F.IF.6, F.LE.1a	170–178
3-4	Direct Variation	A.CED.2, A.REI.10, F.IF.7a, F.BF.1a, F.LE.5	180–186
3-5	Arithmetic Sequences as Linear Functions	A.CED.2, F.IF.3, F.BF.2, F.LE.1a, F.LE.1b, F.LE.2	187–193
3-6	Proportional and Nonproportional Relationships	A.CED.2, F.IF.2, F.IF.9, F.BF.1a, F.LE.1b, F. LE.2	195–200
Chapter 4 Linear Functions and Relations			
Explore 4-1	Graphing Technology Lab: Investigating Slope-Intercept Form	F.LE.5	213
4-1	Graphing Equations in Slope-Intercept Form	A.CED.2, A.CED.4, F.IF.7a, F.BF.1a, F.LE.5, S.ID.7	214–221

Lesson	Lesson Title	Common Core State Standards	Page(s)
Extend 4-1	Graphing Technology Lab: The Family of Linear Graphs	F.IF.4, F.IF.7a, F.BF.3, F.LE.5, S.ID.7	222–223
4-2	Writing Equations in Slope-Intercept Form	A.CED.2, A.CED.3, F.BF.1, F.LE.2	224–230
4-3	Writing Equations in Point-Slope Form	A.CED.2, F.IF.2, F.IF.9, F.BF.1a, F.LE.2	231–236
4-4	Parallel and Perpendicular Lines	A.CED.2, F.BF.1a, F.LE.2, S.ID.7	237–243
4-5	Scatter Plots and Lines of Fit	N.Q.1, A.CED.2, F.BF.1a, F.LE.2, F.LE.5, S.ID.6a, S.ID.6c, S.ID.7	245–251
Extend 4-5	Algebra Lab: Correlation and Causation	S.ID.9	252
4-6	Regression and Median-Fit Lines	*Use CCSS Lesson 5 in place of this lesson.*	253–260
CCSS Lesson 5	Regression and Median-Fit Lines	A.CED.2, F.BF.1a, F.LE.2, S.ID.6, S.ID.8	CCSS 11–17
4-7	Special Functions	A.SSE.1b, A.REI.11, F.IF.4, F.IF.7b	261–268
Extend 4-7	Graphing Technology Lab: Piecewise-Linear Functions	F.IF.7b	269
CCSS Lesson 6	Inverse Linear Functions	A.CED.2, F.BF.1a, F.BF.4a	CCSS 18–24
CCSS Lab 7	Algebra Lab: Drawing Inverses	F.BF.4a	CCSS 25

Chapter 5 Linear Inequalities

5-1	Solving Inequalities by Addition and Subtraction	A.CED.1, A.REI.3	283–288
Explore 5-2	Algebra Lab: Solving Inequalities	A.REI.3	289
5-2	Solving Inequalities by Multiplication and Division	A.CED.1, A.REI.3	290–295
5-3	Solving Multi-Step Inequalities	A.CED.1, A.REI.3	296–301
Explore 5-4	Algebra Lab: Reading Compound Statements		303
5-4	Solving Compound Inequalities	A.CED.1, A.REI.3	304–309
5-5	Inequalities Involving Absolute Value	A.CED.1, A.REI.3	310–314
5-6	Graphing Inequalities in Two Variables	A.CED.3, A.REI.12	315–320
Extend 5-6	Graphing Technology Lab: Graphing Inequalities	A.REI.12	321

Chapter 6 Systems of Linear Equations and Inequalities

6-1	Graphing Systems of Equations	A.CED.2, A.CED.3, A.REI.6	333–339
Extend 6-1	Graphing Technology Lab: Systems of Equations	A.CED.2, A.REI.6, A.REI.11	340–341
6-2	Substitution	A.CED.2, A.CED.3, A.REI.6	342–347
6-3	Elimination Using Addition and Subtraction	A.CED.2, A.REI.6	348–354

Lesson	Lesson Title	Common Core State Standards	Page(s)
6-4	Elimination Using Multiplication	A.CED.2, A.REI.5, A.REI.6	355–360
6-5	Applying Systems of Linear Equations	A.CED.2, A.REI.6	362–367
6-6	Organizing Data Using Matrices		369–375
6-7	Using Matrices to Solve Systems of Equations	A.REI.6	376–381
6-8	Systems of Inequalities	A.REI.12	382–386
Extend 6-8	Graphing Technology Lab: Systems of Inequalities	A.REI.12	387

Chapter 7 Polynomials			
7-1	Multiplying Monomials	A.SSE.2, F.IF.8b	401–407
7-2	Dividing Monomials	A.SSE.2, F.IF.8b	408–415
CCSS Lesson 8	Rational Exponents	N.RN.1, N.RN.2, A.SSE.2, A.REI.3	CCSS 26–32
7-3	Scientific Notation	A.SSE.2	416–422
7-4	Polynomials		424–429
Explore 7-5	Algebra Lab: Adding and Subtracting Polynomials	A.APR.1	431–432
7-5	Adding and Subtracting Polynomials	A.SSE.1a, A.APR.1	433-438
7-6	Multiplying a Polynomial by a Monomial	A.APR.1	439–444
Explore 7-7	Algebra Lab: Multiplying Polynomials	A.APR.1	445–446
7-7	Multiplying Polynomials	A.APR.1	447–452
7-8	Special Products	A.APR.1	453–458

Chapter 8 Factoring and Quadratic Equations			
8-1	Monomials and Factoring		471–474
Explore 8-2	Algebra	A.SSE.2	475
8-2	Using the Distributive Property	A.SSE.2, A.SSE.3a	476–482
Explore 8-3	Algebra Lab: Factoring Trinomials	A.SSE.2	483–484
8-3	Quadratic Equations: $x^2 + bx + c = 0$	A.SSE.2, A.SSE.3a, A.CED.2, A.REI.1, A.REI.4b	485–491
8-4	Quadratic Equations: $ax^2 + bx + c = 0$	A.SSE.2, A.SSE.3a, A.CED.2, A.REI.1, A.REI.4b	493–498
8-5	Quadratic Equations: Differences of Squares	A.SSE.2, A.SSE.3a, A.CED.2, A.REI.4b	499–504
8-6	Quadratic Equations: Perfect Squares	A.SSE.2, A.SSE.3a, A.REI.1	505–512

Lesson	Lesson Title	Common Core State Standards	Page(s)
Chapter 9 Quadratic and Exponential Functions			
9-1	Graphing Quadratic Functions	A.SSE.1a, A.CED.2, A.REI.10, F.IF.2, F.IF.4, F.IF.5, F.IF.7a, F.IF.9	525–535
Extend 9-1	Algebra Lab: Rate of Change of a Quadratic Function	F.IF.6	536
9-2	Solving Quadratic Equations by Graphing	A.CED.2, A.REI.4b, F.IF.7a, F.IF.8a	537–542
Extend 9-2	Graphing Technology Lab: Quadratic Inequalities	*Use CCSS Lab 9 in place of this lab.*	543
CCSS Lab 9	Graphing Technology Lab: Family of Quadratic Functions	F.IF.7a, F.BF.3	CCSS 33–34
9-3	Transformations of Quadratic Functions	*Use CCSS Lesson 10 in place of this lesson.*	544–549
CCSS Lesson 10	Transformations of Quadratic Functions	A.SSE.3b, F.IF.7a, F.IF.8a, F.IF.9, F.BF.1b, F.BF.3	CCSS 35–41
Extend 9-3	Graphing Technology Lab: Systems of Linear and Quadratic Equations	A.REI.7, A.REI.11, F.IF.7a	550–551
9-4	Solving Quadratic Equations by Completing the Square	A.SSE.3b, A.CED.2, A.REI.4, F.IF.8a	552–557
CCSS Lab 11	Algebra Lab: Finding the Maximum or Minimum Value	A.SSE.3b, A.REI.4b, F.IF.8a	CCSS 42–43
9-5	Solving Quadratic Equations by Using the Quadratic Formula	A.CED.2, A.REI.4	558–564
CCSS Lab 12	Graphing Technology Lab: Family of Exponential Functions	F.IF.7e, F.BF.3	CCSS 44–45
9-6	Exponential Functions	N.Q.1, A.CED.2, A.REI.10, F.IF.2, F.IF.4, F.IF.5, F.IF.7e, F.IF.8b, F.LE.2, F.LE.5	567–572
CCSS Lab 13	Graphing Technology Lab: Solving Exponential Equations and Inequalities	A.REI.11	CCSS 46–47
9-7	Growth and Decay	A.CED.2, F.IF.2, F.IF.5, F.IF.8b, F.BF.1, F.LE.1b, F.LE.1c, F.LE.2, F.LE.5	558–564
CCSS Lab 14	Algebra Lab: Transforming Exponential Expressions	A.SSE.3c, F.IF.8b, F.BF.3	CCSS 48
9-8	Geometric Sequences as Exponential Functions	F.IF.3, F.BF.2, F.LE.1, F.LE.2, F.LE.5	578–583
CCSS Lab 15	Algebra Lab: Average Rate of Change of Exponential Functions	F.IF.6	CCSS 49
CCSS Lesson 16	Recursive Formulas	F.IF.3, F.IF.9, F.BF.1a, F.BF.2	CCSS 50–54
9-9	Analyzing Functions with Successive Differences and Ratios	F.IF.6, F.LE.1, F.LE.2, F.LE.3, F.LE.5	584–589
Extend 9-9	Graphing Technology Lab: Curve Fitting	F.LE.2, F.LE.3, S.ID.6a	590–591

Lesson	Lesson Title	Common Core State Standards	Page(s)
Chapter 10 Radical Functions and Geometry			
CCSS Lab 17	Algebra Lab: Inverse Functions	F.BF.4a	CCSS 55–56
10-1	Square Root Functions	A.CED.2, A.REI.10, F.IF.2, F.IF.4, F.IF.5, F.IF.7b, F.BF.3	605–610
Extend 10-1	Graphing Technology Lab: Graphing Square Root Functions	A.CED.2, F.IF.7b, F.BF.3	611
10-2	Simplifying Radical Expressions	A.REI.4a	612–617
Extend 10-2	Graphing Technology Lab: Rational Exponents	*Use CCSS Lab 18 in place of this lab.*	618
CCSS Lab 18	Algebra Lab: Rational and Irrational Numbers	N.RN.3	CCSS 57
10-3	Operations with Radical Expressions	N.RN.2	619–623
CCSS Lab 19	Algebra Lab: Simplifying nth Root Expressions	N.RN.2	CCSS 58–59
10-4	Radical Equations	N.RN.2, A.CED.2	624–628
10-5	The Pythagorean Theorem		630–635
10-6	The Distance and Midpoint Formulas		636–641
10-7	Similar Triangles		642–647
10-8	Trigonometric Ratios		649–655
Chapter 11 Rational Functions and Equations			
11-1	Inverse Variation		670–676
Explore 11-2	Algebra Lab: Inverse Variation		677
11-2	Rational Functions	A.CED.2	678–683
11-3	Simplifying Rational Expressions		684–690
11-4	Multiplying and Dividing Rational Expressions		692–698
11-5	Dividing Polynomials		700–705
11-6	Adding and Subtracting Rational Expressions		706–713
11-7	Mixed Expressions and Complex Fractions		714–719
11-8	Rational Equations and Functions	A.CED.2	720–726
CCSS Lab 20	Graphing Technology Lab: Solving Rational Equations	A.REI.11	CCSS 60–61

Lesson	Lesson Title	CCSS Common Core State Standards	Page(s)
Chapter 12 Statistics and Probability			
12-1	Designing a Survey		740–745
12-2	Analyzing Survey Results		746–755
12-3	Statistics and Parameters	S.ID.2	756–762
CCSS Lesson 21	Distributions of Data	S.ID.1, S.ID.2, S.ID.3	CCSS 62–67
CCSS Lesson 22	Comparing Sets of Data	S.ID.1, S.ID.2, S.ID.3	CCSS 68–74
12-4	Permutations and Combinations		764–770
12-5	Probability of Compound Events		771–778
12-6	Probability Distributions		779–784
Extend 12-6	Graphing Technology Lab: The Normal Curve	*Use CCSS Lab 23 in place of this lab.*	785–786
CCSS Lab 23	Graphing Technology Lab: The Normal Curve	S.ID.2	CCSS 75–76
12-7	Probability Simulations		787–792
CCSS Lab 24	Algebra Lab: Two-Way Frequency Tables	S.ID.5	CCSS 77–78

Common Core State Standards

Traditional Algebra I Pathway, Correlated to *Glencoe Algebra 1*, Common Core Edition

Standards	Student Edition Lesson(s)	Student Edition Page(s)
Number and Quantity		
The Real Number System N-RN		
Extend the properties of exponents to rational exponents. **1.** Explain how the definition of the meaning of rational exponents follows from extending the properties of integer exponents to those values, allowing for a notation for radicals in terms of rational exponents.	CCSS Lesson 8	CCSS 26–32
2. Rewrite expressions involving radicals and rational exponents using the properties of exponents.	10-3, 10-4 CCSS Lesson 8, CCSS Lab 19	619–623, 624–628 CCSS 26–32, CCSS 58–59
Use properties of rational and irrational numbers. **3.** Explain why the sum or product of two rational numbers is rational; that the sum of a rational number and an irrational number is irrational; and that the product of a nonzero rational number and an irrational number is irrational.	CCSS Lab 18	CCSS 57
Quantities★ N-Q		
Reason quantitatively and use units to solve problems. **1.** Use units as a way to understand problems and to guide the solution of multi-step problems; choose and interpret units consistently in formulas; choose and interpret the scale and the origin in graphs and data displays.	Throughout the text; for example, 2-6, 2-7, 2-8, 2-9, Extend 3-2, 4-5, 9-6	Throughout the text; for example, 111–117, 119–124, 126–131, 132–138, 167–168, 245–251, 567–572
2. Define appropriate quantities for the purpose of descriptive modeling.	CCSS Lab 3	CCSS 8
3. Choose a level of accuracy appropriate to limitations on measurement when reporting quantities.	CCSS Lab 1	CCSS 1–2
Algebra		
Seeing Structure in Expressions A-SSE		
Interpret the structure of expressions **1.** Interpret expressions that represent a quantity in terms of its context.★ **a.** Interpret parts of an expression, such as terms, factors, and coefficients.	1-1, 1-4, 7-5, 9-1	5–9, 25–31, 433–438, 525–535
b. Interpret complicated expressions by viewing one or more of their parts as a single entity.	1-2, 1-3, 4-7	10–15, 16–22, 261–268
2. Use the structure of an expression to identify ways to rewrite it.	1-1, 1-2, 1-3, 1-4, 7-1, 7-2, 7-3, Explore 8-2, 8-2, Explore 8-3, 8-3, 8-4, 8-5, 8-6 CCSS Lesson 8	5–9, 10–15, 16–22, 25–31, 401–407, 408–415, 416–422, 475, 476–482, 483–484, 485–491, 493–498, 499–504, 505–512 CCSS 26–32

Standards	Student Edition Lesson(s)	Student Edition Page(s)
Write expressions in equivalent forms to solve problems. **3.** Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression.★ **a.** Factor a quadratic expression to reveal the zeros of the function it defines.	8-2, 8-3, 8-4, 8-5, 8-6	476–482, 485–491, 493–498, 499–504, 505–512
b. Complete the square in a quadratic expression to reveal the maximum or minimum value of the function it defines.	9-4 CCSS Lesson 10, CCSS Lab 11	552–557 CCSS 35–41, CCSS 42–43
c. Use the properties of exponents to transform expressions for exponential functions.	CCSS Lab 14	CCSS 48
Arithmetic with Polynomials and Rational Expressions A-APR		
Perform arithmetic operations on polynomials. **1.** Understand that polynomials form a system analogous to the integers, namely, they are closed under the operations of addition, subtraction, and multiplication; add, subtract, and multiply polynomials.	Explore 7-5, 7-5, 7-6, Explore 7-7, 7-7, 7-8	431–432, 433–438, 439–444, 445–446, 447–452, 453–458
Creating Equations★ A-CED		
Create equations that describe numbers or relationships. **1.** Create equations and inequalities in one variable and use them to solve problems.	1-5, 2-1, 2-2, 2-3, 2-4, 2-5, 2-9, 3-2, 5-1, 5-2, 5-3, 5-4, 5-5	33–39, 75–80, 83–89, 91–96, 97–102, 103–109, 132–138, 161–166, 283–288, 290–295, 296–301, 304–309, 310–314
2. Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.	Extend 1-7, 3-1, 3-4, 3-5, 3-6, 4-1, 4-2, 4-3, 4-4, 4-5, 6-1, Extend 6-1, 6-2, 6-3, 6-4, 6-5, 8-3, 8-4, 8-5, 9-1, 9-2, 9-4, 9-5, 9-6, 9-7, 10-1, Extend 10-1, 10-4, 11-2, 11-8 CCSS Lesson 5, CCSS Lesson 6	55, 153–160, 180–186, 187–193, 195–200, 214–221, 224–230, 231–236, 237–243, 245–251, 333–339, 340–341, 342–347, 348–354, 355–360, 362–367, 485–491, 493–498, 499–504, 525–535, 537–542, 552–557, 567–572, 558–564, 605–610, 611, 624–628, 678–683, 720–726 CCSS 11–17, CCSS 18–24
3. Represent constraints by equations or inequalities, and by systems of equations and/or inequalities, and interpret solutions as viable or nonviable options in a modeling context.	4-2, 5-6, 6-1, 6-2	224–230, 315–319, 333–339, 342–347

Standards	Student Edition Lesson(s)	Student Edition Page(s)
4. Rearrange formulas to highlight a quantity of interest, using the same reasoning as in solving equations.	2-8, 2-9, 4-1	126–131, 132–138, 214–221
Reasoning with Equations and Inequalities A-REI		
Understand solving equations as a process of reasoning and explain the reasoning. **1.** Explain each step in solving a simple equation as following from the equality of numbers asserted at the previous step, starting from the assumption that the original equation has a solution. Construct a viable argument to justify a solution method.	1-5, 2-2, 2-3, 2-4, 2-5, 2-6, 2-9, 8-3, 8-4, 8-6	33–39, 83–89, 91–96, 97–102, 102–109, 111–117, 132–138, 485–491, 493–498, 505–512
Solve equations and inequalities in one variable. **3.** Solve linear equations and inequalities in one variable, including equations with coefficients represented by letters.	1-5, Explore 2-2, 2-2, Explore 2-3, 2-3, 2-4, 2-5, 2-6, 2-7, 2-8, 2-9, 5-1, Explore 5-2, 5-2, 5-3, 5-4, 5-5 CCSS Lesson 8	33–39, 81–82, 83–89, 90, 91–96, 97–102, 103–109, 111–117, 119–124, 126–131, 132–138, 283–288, 289, 290–295, 296–301, 304–309, 310–314 CCSS 26–32
4. Solve quadratic equations in one variable. **a.** Use the method of completing the square to transform any quadratic equation in x into an equation of the form $(x - p)^2 = q$ that has the same solutions. Derive the quadratic formula from this form.	9-4, 9-5, 10-2	552–557, 558–564, 612–617
b. Solve quadratic equations by inspection (e.g., for $x^2 = 49$), taking square roots, completing the square, the quadratic formula and factoring, as appropriate to the initial form of the equation. Recognize when the quadratic formula gives complex solutions and write them as a ± bi for real numbers a and b.	8-3, 8-4, 8-5, 9-2, 9-4, 9-5 CCSS Lab 11	485–491, 493–498, 499–504, 537–542, 552–557, 558–564 CCSS 42–43
Solve systems of equations. **5.** Prove that, given a system of two equations in two variables, replacing one equation by the sum of that equation and a multiple of the other produces a system with the same solutions.	6-4	355–0360
6. Solve systems of linear equations exactly and approximately (e.g., with graphs), focusing on pairs of linear equations in two variables.	6-1, Extend 6-1, 6-2, 6-3, 6-4, 6-5, 6-7	333–339, 340–341, 342–347, 348–354, 355–360, 362–367, 376–381
7. Solve a simple system consisting of a linear equation and a quadratic equation in two variables algebraically and graphically.	Extend 9-3	550–551
Represent and solve equations and inequalities graphically. **10.** Understand that the graph of an equation in two variables is the set of all its solutions plotted in the coordinate plane, often forming a curve (which could be a line).	1-6, 1-7, 3-1, 3-2, 3-4, 9-1, 9-6, 10-1	40–46, 47–54, 153–160, 161–166, 180–186, 525–535, 567–572, 605–610

Standards	Student Edition Lesson(s)	Student Edition Page(s)
11. Explain why the x-coordinates of the points where the graphs of the equations $y = f(x)$ and $y = g(x)$ intersect are the solutions of the equation $f(x) = g(x)$; find the solutions approximately, e.g., using technology to graph the functions, make tables of values, or find successive approximations. Include cases where $f(x)$ and/or $g(x)$ are linear, polynomial, rational, absolute value, exponential, and logarithmic functions.★	4-7, Extend 6-1, Extend 9-3 CCSS Lab 13, CCSS Lab 20	261–268, 340–341, 572–573 CCSS 46–47, CCSS 60–61
12. Graph the solutions to a linear inequality in two variables as a halfplane (excluding the boundary in the case of a strict inequality), and graph the solution set to a system of linear inequalities in two variables as the intersection of the corresponding half-planes.	5-6, Extend 5-6, 6-8, Extend 6-8	315–319, 321, 382–386, 387
Functions		
Interpreting Functions F-IF		
Understand the concept of a function and use function notation. **1.** Understand that a function from one set (called the domain) to another set (called the range) assigns to each element of the domain exactly one element of the range. If f is a function and x is an element of its domain, then $f(x)$ denotes the output of f corresponding to the input x. The graph of f is the graph of the equation $y = f(x)$.	1-6, 1-7	40–46, 47–54
2. Use function notation, evaluate functions for inputs in their domains, and interpret statements that use function notation in terms of a context.	1-7, 3-6, 4-3, 9-1, 9-6, 9-7, 10-1	47–54, 195–200, 231–236, 261–268, 525–535, 567–572, 573–577, 605–610
3. Recognize that sequences are functions, sometimes defined recursively, whose domain is a subset of the integers.	3-5, 9-8 CCSS Lesson 16	187–193, 578–583 CCSS 50–54
Interpret functions that arise in applications in terms of the context. **4.** For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship.★	3-1, Extend 4-1, 4-7, 9-1, 9-6, 10-1 CCSS Lesson 2, CCSS Lab 4	153–160, 222–223, 261–268, 525–535, 567–572, 605–610 CCSS 3–7, CCSS 9–10
5. Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes.	1-7, 9-1, 9-6, 9-7, 10-1	47–54, 525–535, 567–572, 573–577, 605–610
6. Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph.★	Explore 3-3, 3-3, Extend 9-1, 9-9 CCSS Lab 15	169, 170–178, 536, 584–589 CCSS 49

Standards	Student Edition Lesson(s)	Student Edition Page(s)
Analyze functions using different representations. **7.** Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.★ **a.** Graph linear and quadratic functions and show intercepts, maxima, and minima.	3-1, 3-2, Extend 3-2, 3-4, 4-1, Extend 4-1, 9-1, 9-2, Explore 9-3, Extend 9-3 CCSS Lab 9, CCSS Lesson 10	153–160, 161–166, 167–168, 180–186, 214–221, 222–223, 525–535, 537–542, 550–551 CCSS 33–34, CCSS 35–41
b. Graph square root, cube root, and piecewise-defined functions, including step functions and absolute value functions.	4-7, 10-1, Extend 10-1 CCSS Lab 14	261–268, 605–610, 611 CCSS 48
e. Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude.	9-6 CCSS Lab 12	567–572 CCSS 44–45
8. Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function. **a.** Use the process of factoring and completing the square in a quadratic function to show zeros, extreme values, and symmetry of the graph, and interpret these in terms of a context.	9-2, 9-4 CCSS Lesson 10, CCSS Lab 11	537–542, 552–557 CCSS 35–41, CCSS 42–43
b. Use the properties of exponents to interpret expressions for exponential functions.	7-1, 7-2, 9-6, 9-7 CCSS Lab 14	401–407, 408–415, 567–572, 573–577 CCSS 48
9. Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions).	1-7, 3-6, 4-3, 9-1 CCSS Lesson 10, CCSS Lesson 16	47–54, 195–200, 231–236, 525–535 CCSS 35–41, CCSS 50–54
Building Functions F-BF		
Build a function that models a relationship between two quantities. **1.** Write a function that describes a relationship between two quantities.★ **a.** Determine an explicit expression, a recursive process, or steps for calculation from a context.	1-7, 3-1, 3-4, 3-6, 4-1, 4-2, 4-3, 4-4, 4-5, 9-7 CCSS Lesson 5, CCSS Lesson 6, CCSS Lesson 16	47–54, 153–160, 180–186, 195–200, 214–221, 224–230, 231–236, 237–243, 245-251, 573–577 CCSS 11–17, CCSS 18–24, CCSS 50–54
b. Combine standard function types using arithmetic operations.	4-2, 9-7 CCSS Lesson 10	224–230, 573–577 CCSS 35–41
2. Write arithmetic and geometric sequences both recursively and with an explicit formula, use them to model situations, and translate between the two forms.★	3-5, 9-8 CCSS Lesson 16	187–193, 578–583 CCSS 50–54

Standards	Student Edition Lesson(s)	Student Edition Page(s)
Build new functions from existing functions. **3.** Identify the effect on the graph of replacing $f(x)$ by $f(x) + k$, $k\,f(x)$, $f(kx)$, and $f(x + k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology.	Extend 4-1, 10-1, Extend 10-1 CCSS Lab 9, CCSS Lesson 10, CCSS Lab 12, CCSS Lab 14	222–223, 605–610, 611 CCSS 33–34, CCSS 35–41, CCSS 44–45, CCSS 48
4. Find inverse functions. **a.** Solve an equation of the form $f(x) = c$ for a simple function f that has an inverse and write an expression for the inverse.	CCSS Lesson 6, CCSS Lab 7, CCSS Lab 17	CCSS 18–24, CCSS 25, CCSS 55–56
Linear, Quadratic, and Exponential Models F-LE		
Construct and compare linear, quadratic, and exponential models and solve problems. **1.** Distinguish between situations that can be modeled with linear functions and with exponential functions. **a.** Prove that linear functions grow by equal differences over equal intervals, and that exponential functions grow by equal factors over equal intervals.	Explore 3-3, 3-3, 3-5, 9-8, 9-9	169, 170–178, 187–193, 578–583, 584–589
b. Recognize situations in which one quantity changes at a constant rate per unit interval relative to another.	3-5, 3-6, 9-7, 9-8, 9-9	187–193, 195–200, 573–577, 578–583, 584–589
c. Recognize situations in which a quantity grows or decays by a constant percent rate per unit interval relative to another.	9-7, 9-8, 9-9	573–577, 578–583, 584–589
2. Construct linear and exponential functions, including arithmetic and geometric sequences, given a graph, a description of a relationship, or two input-output pairs (include reading these from a table).	3-5, 3-6, 4-2, 4-3, 4-4, 4-5, 9-6, 9-7, 9-8, 9-9, Extend 9-9 CCSS Lesson 5	187–193, 195–200, 224–230, 231–236, 237–243, 245–251, 567–572, 573–577, 578–583, 584–589, 590–591 CCSS 11–17
3. Observe using graphs and tables that a quantity increasing exponentially eventually exceeds a quantity increasing linearly, quadratically, or (more generally) as a polynomial function.	9-9, Extend 9-9	590–591, 584–589
Interpret expressions for functions in terms of the situation they model. **5.** Interpret the parameters in a linear or exponential function in terms of a context.	3-4, Explore 4-1, 4-1, Extend 4-1, 4-5, 9-6, 9-7, 9-8, 9-9	180–186, 213, 214–221, 222–223, 245–251, 567–572, 573–577, 578–583, 584–589

Standards	Student Edition Lesson(s)	Student Edition Page(s)
Statistics and Probability		
Interpreting Categorical and Quantitative Data S-ID		
Summarize, represent, and interpret data on a single count or measurement variable. **1.** Represent data with plots on the real number line (dot plots, histograms, and box plots).	0-13 CCSS Lesson 21, CCSS Lesson 22	P40–P46 CCSS 62–67, CCSS 68–74
2. Use statistics appropriate to the shape of the data distribution to compare center (median, mean) and spread (interquartile range, standard deviation) of two or more different data sets.	12-3 CCSS Lesson 21, CCSS Lesson 22, CCSS Lab 23	756–762 CCSS 62–67, CCSS 68–74, CCSS 75–76
3. Interpret differences in shape, center, and spread in the context of the data sets, accounting for possible effects of extreme data points (outliers).	CCSS Lesson 21, CCSS Lesson 22	CCSS 62–67, CCSS 68–74
Summarize, represent, and interpret data on two categorical and quantitative variables. **5.** Summarize categorical data for two categories in two-way frequency tables. Interpret relative frequencies in the context of the data (including joint, marginal, and conditional relative frequencies). Recognize possible associations and trends in the data.	CCSS Lab 24	CCSS 77–78
6. Represent data on two quantitative variables on a scatter plot, and describe how the variables are related. **a.** Fit a function to the data; use functions fitted to data to solve problems in the context of the data. Use given functions or choose a function suggested by the context. Emphasize linear, quadratic, and exponential models.	4-5, Extend 9-9 CCSS Lesson 5	245–251, 590–591 CCSS 11–17
b. Informally assess the fit of a function by plotting and analyzing residuals.	4-6	253–260
c. Fit a linear function for a scatter plot that suggests a linear association.	4-5, 4-6	245–251, 253–260
Interpret linear models. **7.** Interpret the slope (rate of change) and the intercept (constant term) of a linear model in the context of the data.	4-1, Extend 4-1, 4-4, 4-5	214–221, 222–223, 237–243, 245–251
8. Compute (using technology) and interpret the correlation coefficient of a linear fit.	CCSS Lesson 5	CCSS 11–17
9. Distinguish between correlation and causation.	Extend 4-5	252

LAB 1

Algebra Lab
Accuracy

All measurements taken in the real world are approximations. The greater the care with which a measurement is taken, the more accurate it will be. **Accuracy** refers to how close a measured value comes to the actual or desired value. For example, a fraction is more accurate than a rounded decimal.

Common Core State Standards
N.Q.3

Activity 1 When Is Close Good Enough?

Measure the length of your desktop. Record your results in centimeters, in meters, and in millimeters.

Analyze the Results

2. Sample answer: Yes; in centimeters, the measurement was rounded to the half; in meters, the measurement was rounded to the tenth.

1. Did you round to the nearest whole measure? If so, when? **Sample answer: yes; in millimeters**
2. Did you round to the nearest half, tenth, or smaller? If so, when?
3. Which unit of measure was the most appropriate for this task? **centimeters**
4. Which unit of measure was the most accurate? **millimeters**

Deciding where to round a measurement depends on how the measurement will be used. But calculations should not be carried out to greater accuracy than that of the original data.

Activity 2 Decide Where to Round

a. Elan has \$13 that he wants to divide among his 6 nephews. When he types 13 ÷ 6 into his calculator, the number that appears is 2.166666667. Where should Elan round?

Since Elan is rounding money, the smallest increment is a penny, so round to the hundredths place. This will give him 2.17, and \$2.17 × 6 = \$13.02. Elan will be two pennies short, so round to \$2.16. Since \$2.16 × 6 = \$12.96, Elan can give each of his nephews \$2.16.

b. Dante's mother brings him a dozen cookies, but before she leaves she eats one and tells Dante he has to share with his two sisters. Dante types 11 ÷ 3 into his calculator and gets 3.666666667. Where should Dante round?

After each sibling receives 3 cookies, there are two cookies left. In this case, it is more accurate to convert the decimal portion to a fraction and give each sibling $\frac{2}{3}$ of a cookie.

c. Eva measures the dimensions of a box as 8.7, 9.52, and 3.16 inches. She multiplies these three numbers to find the measure of the volume. The result shown on her calculator is 261.72384. Where should Eva round?

Eva should round to the tenths place, 261.7, because she was only accurate to the tenths place with one of her measures.

Exercises

5. Jessica wants to divide \$23 six ways. Her calculator shows 3.833333333. Where should she round? **the hundredths place; \$3.83**
6. Ms. Harris wants to share 2 pizzas among 6 people. Her calculator shows 0.3333333333. Where should she round? **She should convert to a fraction and give each person $\frac{1}{3}$.**
7. The measurements of an aquarium are 12.9, 7.67, and 4.11 inches. The measure of the volume is given by the product 406.65573. Where should the number be rounded? **the tenths place; 406.7**

(continued on the next page)

1 Focus

Objective Choose a level of accuracy appropriate to limitations on measurement when reporting quantities.

Materials for Each Student

- rulers that measure in centimeters, millimeters, and meters
- calculators

2 Teach

Working in Cooperative Groups

Have students work in groups of three or four, mixing abilities, to complete Activities 1–3, Analyze the Results 1–4, and Exercises 5–7.

Teaching Tip

It is important that students understand that a solution to a problem cannot be more accurate then the level of accuracy in the given measurements, or values.

3 Assess

Formative Assessment

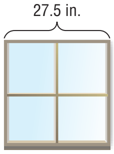
Use Exercises 8–15 to assess whether students understand how to choose a level of accuracy appropriate to limitations on measurement when reporting quantities.

Algebra Lab

Accuracy *Continued*

For most real-world measurements, a decision must be made on the level of accuracy needed or desired.

Activity 3 Find an Appropriate Level of Accuracy



a. Jon needs to buy a shade for the window opening shown, but the shades are only available in whole inch increments. What size shade should he buy?

He should buy the 27-inch shade because it will be enough to cover the glass.

b. Tom is buying flea medicine for his dog. The amount of medicine depends on the dog's weight. The medicine is available in packages that vary by 10 dog pounds. How accurate does Tom need to be to buy the correct medicine?

He needs to be accurate to within 10 pounds.

c. Tyrone is building a jet engine. How accurate do you think he needs to be with his measurements?

He needs to be very accurate, perhaps to the thousandth of an inch.

Exercises

8. Matt's table is missing a leg. He wants to cut a piece of wood to replace the leg. How accurate do you think he needs to be with his measurements? **Accurate to the nearest centimeter, or the table will wobble.**

For each situation, determine where the rounding should occur and give the rounded answer.

9. Sam wants to divide $111 seven ways. His calculator shows 15.85714286. **the hundredths place; $15.85**

10. Kiri wants to share 3 pies among 11 people. Her calculator shows 0.2727272727.
She should convert to a fraction and give each person about a quarter of a pie.

11. Evan's calculator gives him the volume of his soccer ball as 137.2582774. Evan measured the radius of the ball to be 3.2 inches. **the tenths place; 137.3**

For each situation, determine the level of accuracy needed. Explain.

12. You are estimating the length of your school's basketball court. Which unit of measure should you use: 1 foot, 1 inch, or $\frac{1}{16}$ inch?

12. 1 ft; this is a large surface, using the inch measure or smaller would most likely be more precise than necessary, so it stands to use the largest measure available.

13. You are estimating the height of a small child. Which unit of measure should you use: 1 foot, 1 inch, or $\frac{1}{16}$ inch?

13. 1 in.; this is a small height, if a foot measure was used it would not be very accurate, so using an inch would be the most acceptable measure.

14. **TRAVEL** Curt is measuring the driving distance from one city to another. How accurate do you think he needs to be with his measurement? **Sample answer: He needs to be accurate to within a few miles.**

15. **MEDICINE** A nurse is administering medicine to a patient based on his weight. How accurate do you think she needs to be with her measurements?
Sample answer: She needs to be very accurate, perhaps to the thousandths of a milliliter.

LESSON 2 Interpreting Graphs of Functions

Then

- You identified functions and found function values.

Now

1 Interpret intercepts, and symmetry of graphs of functions.

2 Interpret positive, negative, increasing, and decreasing behavior, extrema, and end behavior of graphs of functions.

Why?

- Sales of video games, including hardware, software, and accessories, have increased at times and decreased at other times over the years. Annual retail video game sales in the U.S. from 2000 to 2009 can be modeled by the graph of a nonlinear function.

NewVocabulary

intercept
x-intercept
y-intercept
symmetry
positive
negative
increasing
decreasing
extrema
relative maximum
relative minimum
end behavior

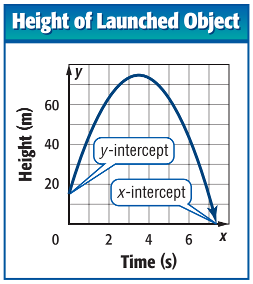
Kiyoshi Ota/Bloomberg via Getty Images

Common Core State Standards
F.IF.4

1 Interpret Intercepts and Symmetry

To interpret the graph of a function, estimate and interpret key features. The **intercepts** of a graph are points where the graph intersects an axis. The y-coordinate of the point at which the graph intersects the y-axis is called a ***y*-intercept**. Similarly, the x-coordinate of the point at which a graph intersects the x-axis is called an ***x*-intercept**.

Real-World Example 1 Interpret Intercepts


Height of Launched Object

PHYSICS The graph shows the height y of an object as a function of time x. Identify the function as *linear* or *nonlinear*. Then estimate and interpret the intercepts.

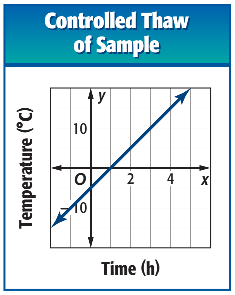
Linear or Nonlinear: Since the graph is a curve and not a line, the graph is nonlinear.

***y*-Intercept:** The graph intersects the y-axis at about (0, 15), so the y-intercept of the graph is about 15. This means that the object started at an initial height of about 15 meters above the ground.

***x*-Intercept(s):** The graph intersects the x-axis at about (7.4, 0), so the x-intercept is about 7.4. This means that the object struck the ground after about 7.4 seconds.

GuidedPractice

1. The graph shows the temperature y of a medical sample thawed at a controlled rate. Identify the function as *linear* or *nonlinear*. Then estimate and interpret the intercepts.
See Answer Appendix.


Controlled Thaw of Sample

1 Focus

VerticalAlignment

Before Lesson 2 Represent relations. Interpret graphs as relations.

Lesson 2 Interpret key features of graphs of functions.

After Lesson 2 Interpret key features of graphs of quadratic, exponential, radical, and rational functions.

2 Teach

Scaffolding Questions

Have students read the **Why?** section of the lesson.

Ask:

- Why would a linear function not model the sale of video games well? Linear functions can only increase over time or decrease over time. Linear functions cannot increase and then decrease, or vice versa.
- Describe some points or areas on a graph of video game sales that might be of more interest to someone in the video game industry than other points. The high and low points on the graph and the far right side of the graph.

1 Interpret Intercepts and Symmetry

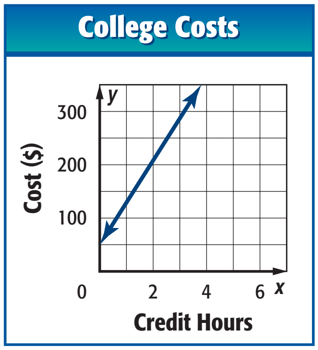
Example 1 shows how to identify the graph of a function as linear or nonlinear and interpret the intercepts. **Example 2** shows how to identify and interpret any symmetry exhibited by the graph of a function.

Formative Assessment

Use the Guided Practice exercises after each Example to determine students' understanding of concepts.

Additional Example

1 **COLLEGE** The graph shows the cost at a community college *y* as a function of the number of credit hours taken *x*. Identify the function as *linear* or *nonlinear.* Then estimate and interpret the intercepts of the graph of the function



Linear; no *x*-intercept, the cost of college is never 0; *y*-intercept ≈ 50, there is an additional fee of $50 added to the cost charged per credit hour taken.

Additional Examples also in Interactive Classroom PowerPoint® Presentations

IWB Interactive White Board READY

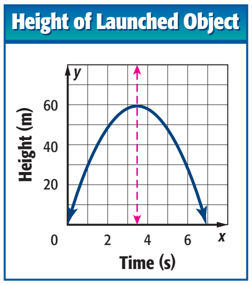
The graphs of some functions exhibit another key feature: symmetry. A graph possesses **line symmetry** in the *y*-axis or some other vertical line if each half of the graph on either side of the line matches exactly.

StudyTip

Symmetry The graphs of most real-world functions do not exhibit symmetry over the entire domain. However, many have symmetry over smaller portions of the domain that are worth analyzing.

Real-World Example 2 Interpret Symmetry

PHYSICS An object is launched. The graph shows the height *y* of the object as a function of time *x*. Describe and interpret any symmetry.



The right half of the graph is the mirror image of the left half in approximately the line $x = 3.5$ between approximately $x = 0$ and $x = 7$.

In the context of the situation, the symmetry of the graph tells you that the time it took the object to go up is equal to the time it took to come down.

GuidedPractice

2. Describe and interpret any symmetry exhibited by the graph in Guided Practice 1. There is no vertical line symmetry.

2 Interpret Extrema and End Behavior

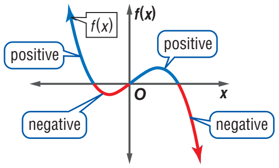
Interpreting a graph also involves estimating and interpreting where the function is increasing, decreasing, positive, or negative, and where the function has any extreme values, either high or low.

StudyTip

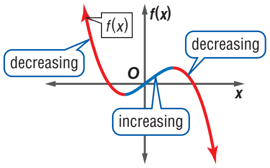
End Behavior The end behavior of some graphs can be described as approaching a specific *y*-value. In this case, a portion of the graph looks like a horizontal line.

KeyConcepts Positive, Negative, Increasing, Decreasing, Extrema, and End Behavior

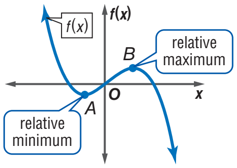
A function is **positive** where its graph lies *above* the *x*-axis, and **negative** where its graph lies *below* the *x*-axis.



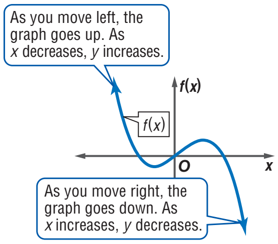
A function is **increasing** where the graph goes *up* and **decreasing** where the graph goes *down* when viewed from left to right.



The points shown are the locations of relatively high or low function values called **extrema**. Point *A* is a **relative minimum**, since no other nearby points have a lesser *y*-coordinate. Point *B* is a **relative maximum**, since no other nearby points have a greater *y*-coordinate.



End behavior describes the values of a function at the positive and negative extremes in its domain.



DifferentiatedInstruction AL OL ELL

If students are having trouble interpreting key features of graphs,

Then have students work together discussing examples in this lesson. You may also want them to complete some exercises cooperatively.

Real-WorldLink

The first successful commercially sold portable video game system was released in 1989 and sold for $120.

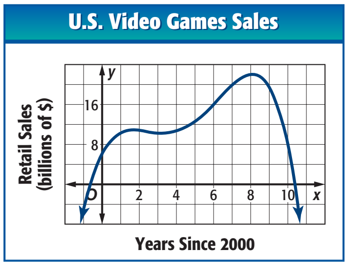
Source: *PCWorld*

StudyTip

Constant A function is *constant* where the graph does not go up or down as the graph is viewed from left to right.

Real-World Example 3 Interpret Extrema and End Behavior

VIDEO GAMES U.S. retail sales of video games from 2000 to 2009 can be modeled by the function graphed at the right. Estimate and interpret where the function is positive, negative, increasing, and decreasing, the x-coordinates of any relative extrema, and the end behavior of the graph.



Positive: between about $x = -0.6$ and $x = 10.4$

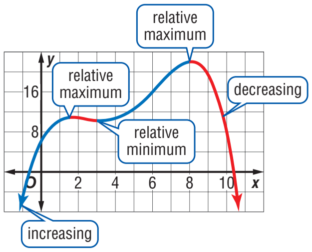
Negative: for about $x < -0.6$ and $x > 10.4$

This means that there were positive sales between about 2000 and 2010, but the model predicts negative sales after about 2010, indicating the unlikely collapse of the industry.

Increasing: for about $x < 1.5$ and between about $x = 3$ and $x = 8$

Decreasing: between about $x = 2$ and $x = 3$ and for about $x > 8$

This means that sales increased from about 2000 to 2002, decreased from 2002 to 2003, increased from 2003 to 2008, and have been decreasing since 2008.



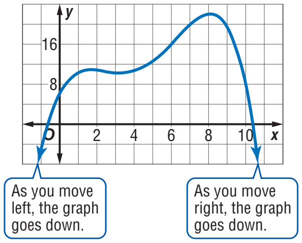
Relative Maxima: at about $x = 1.5$ and $x = 8$

Relative Minima: at about $x = 3$

The extrema of the graph indicate that the industry experienced two relative peaks in sales during this period: one around 2002 of approximately $10.5 billion and another around 2008 of approximately $22 billion. A relative low of $10 billion in sales came in about 2003.

End Behavior:
As x increases or decreases, the value of y decreases.

The end behavior of the graph indicates negative sales several years prior to 2000 and several years after 2009, which is unlikely. This graph appears to only model sales well between 2000 and 2009 and can only be used to predict sales in 2010.



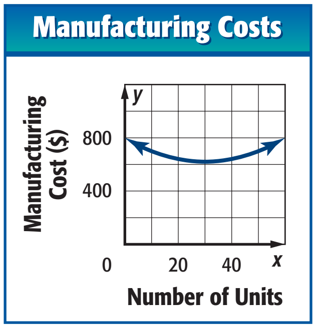
GuidedPractice

3. Estimate and interpret where the function graphed in Guided Practice 1 is positive, negative, increasing, or decreasing, the x-coordinate of any relative extrema, and the end behavior of the graph. **See Answer Appendix.**

Science & Society Picture Library/SSPL/Getty Images

Additional Example

2 **MANUFACTURING** The graph shows the cost y to manufacture x units of a product. Describe and interpret any symmetry.



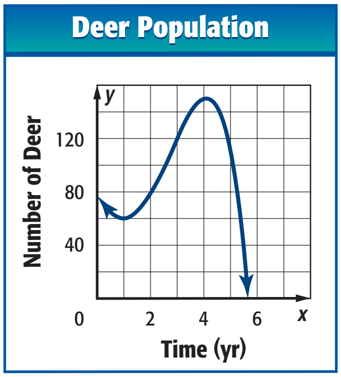
The right half of the graph is the mirror image of the left half in approximately the line $x = 30$. The symmetry of the graph tells you that the cost to produce n more or n less than 30 units will be the same.

2 Interpret Extrema and End Behavior

Example 3 shows how to estimate and interpret where a function is increasing, decreasing, positive, and negative, the x-coordinates of any relative extrema, and end behavior.

Additional Example

3 **DEER** The graph shows the population y of deer x years after the animals are introduced on an island. Estimate and interpret where the function is positive, negative, increasing, and decreasing, the x-coordinates of any relative extrema, and the end behavior of the graph.



See bottom margin.

Additional Answer (Additional Example)

3. Nonlinear; the y-intercept is about 75, indicating an initial population of 75 deer. The x-intercept of about 5.6 indicates that the population died out about 5.6 years after the deer were introduced to the island. The graph has no line symmetry. The population decreased the first year, increased between year 1 and year 4, and then decreased to 0 after year 4. The population experienced a relative low at the end of year 1 and a relative high at then end of year 4. As the years passed, the population decreased to 0.

3 Practice

Formative Assessment

Use Exercises 1–3 to check for understanding.

Use the chart at the bottom of this page to customize assignments for your students.

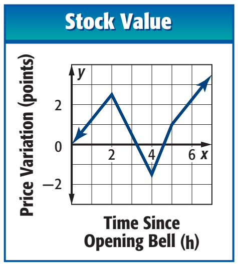
Teach with Tech

Interactive Whiteboard Display graphs of linear, quadratic, and polynomial functions on the board. Use the highlighter tool to show intercepts, symmetry, where the graph is positive, negative, increasing, and decreasing, and end behavior.

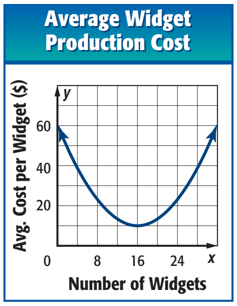
Check Your Understanding

Examples 1–3 Identify the function graphed as *linear* or *nonlinear*. Then estimate and interpret the intercepts of the graph, any symmetry, where the function is positive, negative, increasing, and decreasing, the x-coordinate of any relative extrema, and the end behavior of the graph. **1–3. See Answer Appendix.**

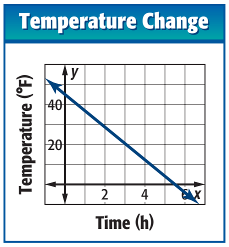
1.



2.



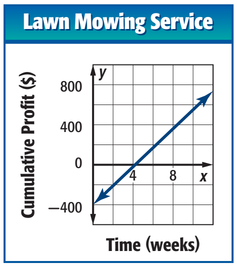
3.



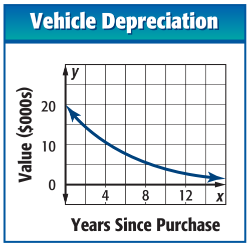
Practice and Problem Solving

Examples 1–3 Identify the function graphed as *linear* or *nonlinear*. Then estimate and interpret the intercepts of the graph, any symmetry, where the function is positive, negative, increasing, and decreasing, the x-coordinate of any relative extrema, and the end behavior of the graph. **4–9. See Answer Appendix.**

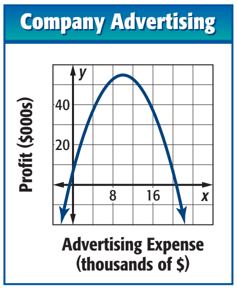
4.



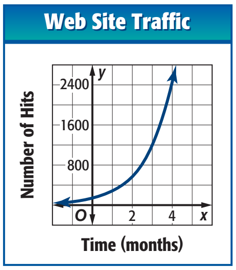
5.



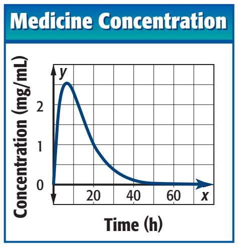
6.



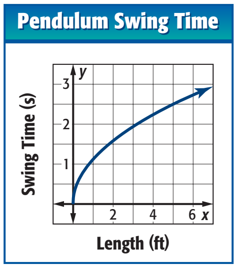
7.



8.



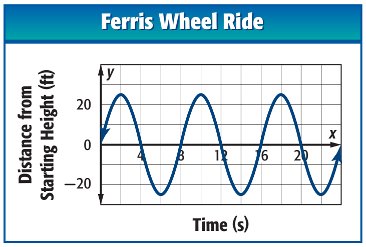
9.



Differentiated Homework Options

Level	Assignment	Two-Day Option	
AL Basic	4–9, 18, 20–22	5–9 odd	4–8 even, 18, 20–22
OL Core	5–9 odd, 10, 11–17 odd, 18, 20–22	4–9	10, 11–17 odd, 18, 20–22
BL Advanced	10–22		

B **10.** **FERRIS WHEEL** At the beginning of a Ferris wheel ride, a passenger cart is located at the same height as the center of the wheel. The position y in feet of this cart relative to the center t seconds after the ride starts is given by the function graphed at the right. Identify and interpret the key features of the graph. (*Hint:* Look for a pattern in the graph to help you describe its end behavior.) **See Answer Appendix.**



Sketch a graph of a function that could represent each situation. Identify and interpret the intercepts of the graph, where the graph is increasing and decreasing, and any relative extrema. **11–13. See Answer Appendix.**

11. the height of a corn plant from the time the seed is planted until it reaches maturity 120 days later

12. the height of a football from the time it is punted until it reaches the ground 2.8 seconds later

13. the balance due on a car loan from the date the car was purchased until it was sold 4 years later

Sketch graphs of functions with the following characteristics. **14–17. See Answer Appendix.**

14. The graph is linear with an x-intercept at -2. The graph is positive for $x < -2$, and negative for $x > -2$.

C **15.** A nonlinear graph has x-intercepts at -2 and 2 and a y-intercept at -4. The graph has a relative minimum of 4 at $x = 0$. The graph is decreasing for $x < 0$ and increasing for $x > 0$.

16. A nonlinear graph has a y-intercept at 2, but no x-intercepts. The graph is positive and increasing for all values of x.

17. A nonlinear graph has x-intercepts at -8 and -2 and a y-intercept at 3. The graph has relative minimums at $x = -3$ and $x = 6$ and a relative maximum at $x = 2$. The graph is positive for $x < -8$ and $x > -1$ and negative between $x = -8$ and $x = -1$. As x decreases, y increases and as x increases, y increases.

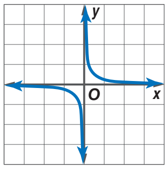
20. True; a function can have no more than one y-intercept. If a graph has more than one y-intercept, then it is not the graph of a function. A function can also have no y-intercept if it is not defined for $x = 0$.

H.O.T. Problems Use Higher-Order Thinking Skills

18. **ERROR ANALYSIS** Katara thinks that all linear functions have exactly one x-intercept. Desmond thinks that a linear function can have at most one x-intercept. Is either of them correct? Explain your reasoning. **See margin.**

19. As x increases or decreases, y approaches 0.

19. **CHALLENGE** Describe the end behavior of the graph shown.



20. **REASONING** Determine whether the following statement is *true* or *false*. Explain. **See margin.**

Functions have at most one y-intercept.

21. **OPEN ENDED** Sketch the graph of a function with one relative maximum and one relative minimum that could represent a real-world function. Label each axis and include appropriate units. Then identify and interpret the relative extrema of your graph. **21, 22. See Answer Appendix.**

22. **WRITING IN MATH** Describe how you would identify the key features of a graph described in this lesson using a table of values for a function.

DifferentiatedInstruction OL BL

Extension Have pairs of students challenge each other to draw graphs with given key features. One student draws a graph without showing it to the other and describes its key features. The second student should draw a graph that fits the description. Discuss similarities and differences in the graphs and whether both graphs fit the description. Then switch roles and repeat.

WatchOut!

Error Anlaysis In Exercise 18, neither person is correct. Linear functions can have 0, 1, or infinitely many x-intercepts. Remind students that linear functions can also be horizontal. These functions are called constant functions. The function $f(x) = 0$, called the zero function, lies entirely on the x-axis. Therefore, it has infinitely many x-intercepts. All other constant functions lie entirely above or below the x-axis and parallel to it. These functions have no x-intercepts.

4 Assess

Ticket Out the Door Make several copies each of graphs for five real-world functions. Give one graph to each student. As the students leave the room, ask them to describe and interpret one or more of the following characteristics of the graph: intercepts, its symmetry, where the function is positive, negative, increasing, or decreasing, the location of any relative extrema, or end behavior.

Additional Answers

18. Neither; the line $y = 2$ has no x-intercept while the line $y = 0$ has infinitely many x-intercepts.

20. True; a function can have no more than one y-intercept. If a graph has more than one y-intercept, then it is not the graph of a function. A function can also have no y-intercept if it is not defined for $x = 0$.

1 Focus

Objective Use a spreadsheet to investigate different metrics.

Materials for Each Student

- computer
- spreadsheet software

Teaching Tip

You may want to explain that lending institutions use the debt-to-income ratio to determine their risk in lending money to an individual. Debt-to-income ratios are also used for car loans and other major purchases.

2 Teach

Working in Cooperative Groups

Have students work in pairs to complete the Activity.

Ask:

- What would happen to the debt-to-income ratio if more items were added to the expenses column? It would increase.
- What would happen to the debt-to-income ratio if more items were added to the salary column? It would decrease.
- What would happen if Dorrie's mortgage is higher or lower than her rent? A higher mortgage means a greater debt-to-income ratio; a lower mortgage means a lower debt-to-income ratio.

Practice Have students complete Exercises 1–4.

LAB 3 Spreadsheet Lab
Descriptive Modeling

When using numbers to model a real-world situation, it is often helpful to have a metric. A **metric** is a rule for assigning a number to some characteristic or attribute. For example, teachers use metrics to determine grades. Each teacher determines an appropriate metric for assessing a student's performance and assigning a grade.

Common Core State Standards
N.Q.2

You can use a spreadsheet to calculate different metrics.

Activity

Dorrie wants to buy a house. She has the following expenses: rent of $650, credit card monthly bills of $320, a car payment of $410, and a student loan payment of $115. Dorrie has a yearly salary of $46,500. Use a spreadsheet to find Dorrie's debt-to-income ratio.

Image Source/CORBIS

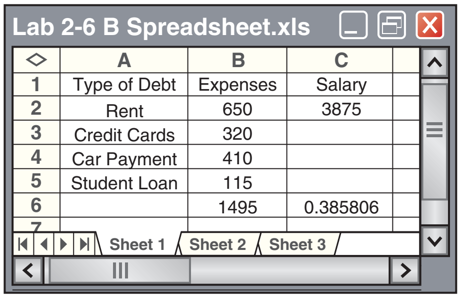
Step 1 Enter Dorrie's debts in column B.

Step 2 Add her debts using a function in cell B6. Go to Insert and then Function. Then choose Sum. The sum of 1495 appears in B6.

Step 3 Now insert Dorrie's salary in column C. Remember to find her monthly salary by dividing the yearly salary by 12.

A mortgage company will use the debt-to-income ratio as a metric to determine if Dorrie qualifies for a loan. The **debt-to-income ratio** is calculated as *how much she owes per month* divided by *how much she earns each month*.

Step 4 Enter a formula to find the debt-to-income ratio in cell C6. In the formula bar, enter =B6/C2.



Lab 2-6 B Spreadsheet.xls

◇	A	B	C
1	Type of Debt	Expenses	Salary
2	Rent	650	3875
3	Credit Cards	320	
4	Car Payment	410	
5	Student Loan	115	
6		1495	0.385806
7			

The ratio of about 0.39 appears. An ideal ratio would be 0.36 or less. A ratio higher than 0.36 would cause an increased interest rate or may require a higher down payment.

The spreadsheet shows a debt-to-income ratio of about 0.39. Dorrie should try to eliminate or reduce some debts or try to earn more money in order to lower her debt-to-income ratio.

1. Sample answer: Reduce or eliminate credit card debt, reduce or eliminate car payments, or earn a higher salary.

Exercises

1. How could Dorrie improve her debt-to-income ratio?
2. Another metric mortgage companies use is the ratio of monthly mortgage to total monthly income. An ideal ratio is 0.28. Using this metric, how much could Dorrie afford to pay for a mortgage each month? **$1085**
3. How effective are each of these metrics as measures of whether Dorrie can afford to buy a house? Explain your reasoning. **See students' work.**
4. **RESEARCH** Metrics are used to compare athletes. For example, ERAs are used to compare pitchers. Find a metric and evaluate its effectiveness for modeling. Compare it to other metrics, and then define your own metric. **See students' work.**

3 Assess

Formative Assessment

Use Exercise 2 to assess whether students understand how to use mortgage metrics.

From Concrete to Abstract

Use Exercise 4 to assess whether students understand how to set up and use a spreadsheet to use metrics.

LAB 4

Algebra Lab
Analyzing Linear Graphs

Analyzing a graph can help you learn about the relationship between two quantities. A linear function is a function for which the graph is a line. There are four types of linear graphs. Let's analyze each type.

Common Core State Standards
F.IF.4

Activity 1 Line that Slants Up

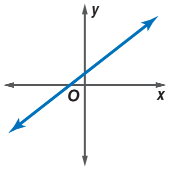
Analyze the function graphed at the right.

a. Describe the domain, range, and end behavior.

b. Describe the intercepts and any maximum or minimum points.

c. Identify where the function is positive, negative, increasing, and decreasing.

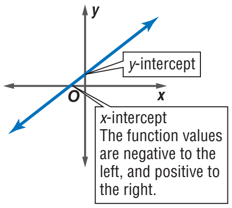
d. Describe any symmetry.



a. The domain and range are all real numbers. As you move left, the graph goes down. So as x decreases, y decreases. As you move right, the graph goes up. So as x increases, y increases.

b. There is one x-intercept and one y-intercept. There are no maximum or minimum points.

c. The function value is 0 at the x-intercept. The function values are negative to the left of the x-intercept and positive to the right. The function goes up from left to right, so it is increasing on the entire domain.



d. The graph has no symmetry.

Lines that slant down from left to right have some different key features.

Activity 2 Line that Slants Down

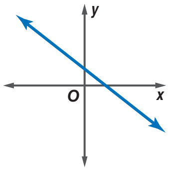
Analyze the function graphed at the right.

a. Describe the domain, range, and end behavior.

b. Describe the intercepts and any maximum or minimum points.

c. Identify where the function is positive, negative, increasing, and decreasing.

d. Describe any symmetry.



a. The domain and range are all real numbers. As you move left, the graph goes up. So as x decreases, y increases. As you move right, the graph goes down. So as x increases, y decreases.

b. There is one x-intercept and one y-intercept. There are no maximum or minimum points.

c. The function values are positive to the left of the x-intercept and negative to the right.
The function goes down from left to right, so it is decreasing on the entire domain.

d. The graph has no symmetry.

(continued on the next page)

1 Focus

Objective
Analyze the key features of linear graphs.

Teaching Tips

- Students may find that placing a pencil on a coordinate plane to represent a line helps them visualize the key features of the graph. Suggest that students move the pencil to represent different lines and evaluate each feature.
- Have students experiment with their pencils on coordinate planes to verify that there are four general types of lines—slant up, slant down, horizontal, and vertical.

2 Teach

Working in Cooperative Groups

Divide the class into pairs. Work through Activities 1–4. Then ask students to work with their partners to complete Exercises 1–4.

Practice Have students complete Exercises 5–8.

3 Assess

Formative Assessment

Use Exercise 5 to assess each student's ability to analyze the key features of a linear graph.

From Concrete to Abstract

Ask students to summarize the key features of each type of linear graph.

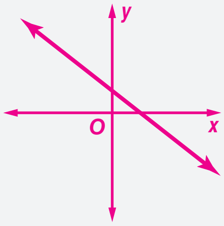
Additional Answers

3a. The key features are the same except that the x- and y-intercepts are the same point, (0, 0).

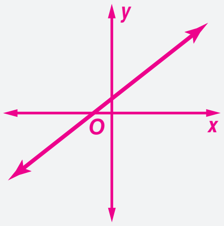
3b. The key features are the same except that the x- and y-intercepts are the same point, (0, 0).

3c. The horizontal line that passes through the origin is the x-axis. The key features of the graph are the same as those of the line in Activity 3 except that instead of no x-intercept, it has infinitely many x-intercepts. The vertical line that passes through the origin is the y-axis. The key features of the graph are the same as those of the line in Activity 4 except that instead of no y-intercept, it has infinitely many y-intercepts.

5. Sample answer: any line that slants down



6. Sample answer: any nonvertical, nonhorizontal line



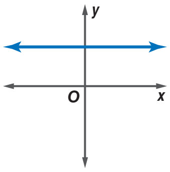
Algebra Lab

Analyzing Linear Graphs *Continued*

Horizontal lines represent special functions called constant functions.

Activity 3 Horizontal Line

Analyze the function graphed at the right.



a. The domain is all real numbers, and the range is one value. As you move left or right, the graph stays constant. So as x decreases or increases, y is constant.

b. The graph does not intersect the x-axis, so there is no x-intercept. The graph has one y-intercept. There are no maximum or minimum points.

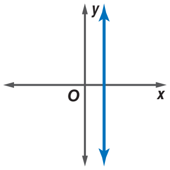
c. The function values are all positive. The function is constant on the entire domain.

d. The graph is symmetric about any vertical line.

Vertical lines represent linear relations that are *not* functions.

Activity 4 Vertical Line

Analyze the relation graphed at the right.



a. The domain is one value, and the range is all real numbers. This relation is not a function. Because you cannot move left or right on the graph, there is no end behavior.

b. There is one x-intercept and no y-intercept. There are no maximum or minimum points.

c. The y-values are positive above the x-axis and negative below. Because you cannot move left or right on the graph, the relation is neither increasing nor decreasing.

d. The graph is symmetric about itself.

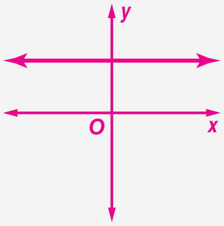
Analyze the Results **1, 2. See Answer Appendix.**

1. Compare and contrast the key features of lines that slant up and lines that slant down.
2. How would the key features of a horizontal line below the x-axis differ from the features of a line above the x-axis?
3. Consider lines that pass through the origin. **a–c. See margin.**
   - **a.** How do the key features of a line that slants up and passes through the origin compare to the key features of the line in Activity 1?
   - **b.** Compare the key features of a line that slants down and passes through the origin to the key features of the line in Activity 2.
   - **c.** Describe a horizontal line that passes through the origin and a vertical line that passes through the origin. Compare their key features to those of the lines in Activities 3 and 4.
4. Place a pencil on a coordinate plane to represent a line. Move the pencil to represent different lines and evaluate each conjecture.
   - **a.** *True* or *false*: A line can have more than one x-intercept. **false**
   - **b.** *True* or *false*: If the end behavior of a line is that as x increases, y increases, then the function values are increasing over the entire domain. **true**
   - **c.** *True* or *false*: Two different lines can have the same x- and y-intercepts. **false**

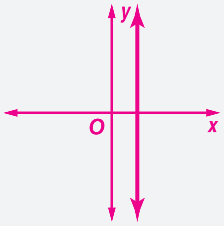
Sketch a linear graph that fits each description. **5–8. See margin.**

5. as x increases, y decreases
6. one x-intercept and one y-intercept
7. has symmetry
8. is not a function

7. Sample answer: any vertical or horizontal line



8. Sample answer: any vertical line



LESSON 5 Regression and Median-Fit Lines

Then

- You used lines of fit and scatter plots to evaluate trends and make predictions.

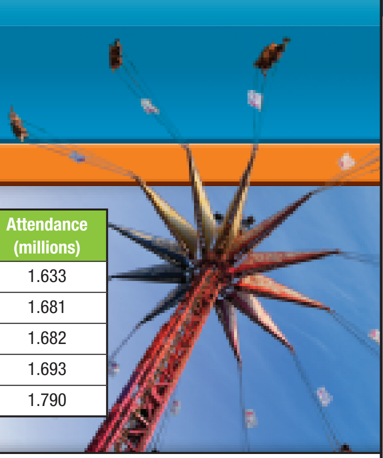
Now

1. Write equations of best-fit lines using linear regression.
2. Write equations of median-fit lines.

Why?

- The table shows the total attendance, in millions of people, at the Minnesota State Fair from 2005 to 2009. You can use a graphing calculator to find the equation of a *best-fit line* and use it to make predictions about future attendance at the fair.

Year	Attendance (millions)
2005	1.633
2006	1.681
2007	1.682
2008	1.693
2009	1.790



Jonathan Larsen/age fotostock

NewVocabulary
best-fit line
linear regression
correlation coefficient
residual
median-fit line

Common Core State Standards
A.CED.2, F.BF.1a, F.LE.2, S.ID.6, S.ID.8

1 Best-Fit Lines

You have learned how to find and write equations for lines of fit by hand. Many calculators use complex algorithms that find a more precise line of fit called the **best-fit line**. One algorithm is called **linear regression**.

Your calculator may also compute a number called the **correlation coefficient**. This number will tell you if your correlation is positive or negative and how closely the equation is modeling the data. The closer the correlation coefficient is to 1 or −1, the more closely the equation models the data.

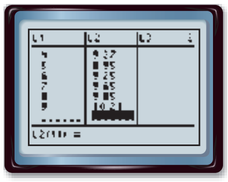
Real-World Example 1 Best-Fit Line

MOVIES **The table shows the amount of money made by movies in the United States. Use a graphing calculator to write an equation for the best-fit line for that data.**

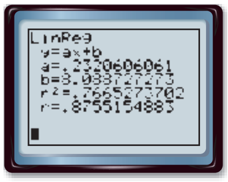
Year	2000	2001	2002	2003	2004	2005	2006	2007	2008	2009
Income ($ billion)	7.48	8.13	9.19	9.35	9.27	8.95	9.25	9.65	9.85	10.21

Before you begin, make sure that your Diagnostic setting is on. You can find this under the **CATALOG** menu. Press **D** and then scroll down and click **DiagnosticOn**. Then press ENTER.

Step 1 Enter the data by pressing STAT and selecting the **Edit** option. Let the year 2000 be represented by 0. Enter the years since 2000 into List 1 (L1). These will represent the x-values. Enter the income ($ billion) into List 2 (L2). These will represent the y-values.



Step 2 Perform the regression by pressing STAT and selecting the **CALC** option. Scroll down to **LinReg (ax+b)** and press ENTER twice.



```
LinReg
 y=ax+b
 a=.2320606061
 b=8.033727273
 r²=.7665273702
 r=.8755154883
```

1 Focus

VerticalAlignment

Before Lesson 5 Use lines of fit and scatter plots to evaluate trends and make predictions.

Lesson 5 Write equations of best-fit lines using linear regression. Write equations of median-fit lines.

After Lesson 5 Find a curve of best-fit in the form of a polynomial function for data.

2 Teach

Scaffolding Questions

Have students read the **Why?** section of the lesson.

Ask:

- What type of correlation does the data in the table have? positive
- How could you estimate the Minnesota State Fair Attendance in 2012? Make a scatter plot of the data in the table. Draw a line that seems to be close to all the points, then use two points on that line to write an equation. Finally, substitute 2012 into the equation for x and solve for y.
- Is this the only line of fit for the data? No, other lines could be drawn that fit.

1 Equations of Best-Fit Lines

Examples 1–2 show how to use a calculator to write an equation for a best-fit line and how to analyze the applicable residual plot for a set of data.

Formative Assessment

Use the Guided Practice exercises after each Example to determine students' understanding of concepts.

Additional Examples

1 **EARNINGS** The table below shows Ariana's hourly earnings for 2001–2007. Use a graphing calculator to write an equation for the best-fit line for that data. Name the correlation coefficient. Let x be the number of years since 2000.

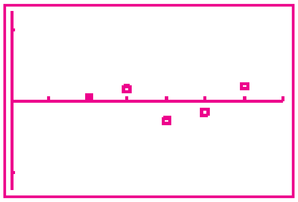
Year	Cost	Year	Cost
2001	\$10	2005	\$15
2002	\$10.50	2006	\$15.75
2003	\$11	2007	\$16.50
2004	\$13		

$y = 1.21x + 8.25$; ≈ 0.9801

2 Graph and analyze the residual plot for the data comparing the years since the 2000 and the cost of repairs.

Years	Cost
2	1236
3	1560
4	1423
5	1740
6	2230

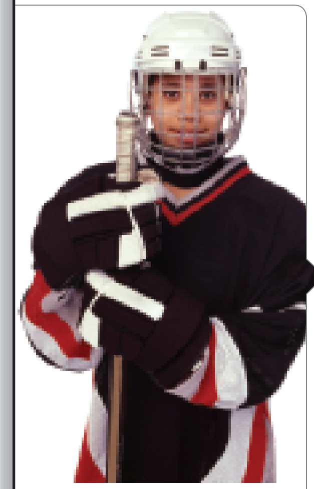
The plot appears to have a curved pattern, so the regression line may not fit the data well.



[0, 7] scl: 1 by [−1000, 1000] scl: 800

Additional Examples also in Interactive Classroom PowerPoint® Presentations

IWB Interactive White Board READY



Real-WorldLink

In 1994, Minnesota became the first state to sanction girls' ice hockey as a high school varsity sport.

Source: ESPNET SportsZone

Jim Mone/AP Images

Step 3 Write the equation of the regression line by rounding the a and b values on the screen. The form that we chose for the regression was $ax + b$, so the equation is $y = 0.23x + 8.09$. The correlation coefficient is about 0.8755, which means that the equation models the data fairly well.

GuidedPractice

Write an equation of the best-fit line for the data in each table. Name the correlation coefficient. Round to the nearest ten-thousandth. Let x be the number of years since 2003.

1A. **HOCKEY** The table shows the number of goals of leading scorers for the Mustang Girls Hockey Team.

Year	2003	2004	2005	2006	2007	2008	2009	2010
Goals	30	23	41	35	31	43	33	45

1B. **HOCKEY** The table gives the number of goals scored by the team each season.

Year	2003	2004	2005	2006	2007	2008	2009	2010
Goals	63	44	55	63	81	85	93	84

1A. For x as years since 2003, $y = 1.87x + 28.58$; 0.6142

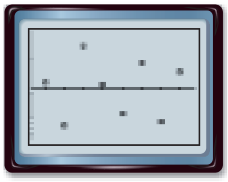
1B. For x as years since 2003, $y = 5.95x + 50.17$; 0.8495

We know that not all of the points will lie on the best-fit line. The difference between an observed y-value and its predicted y-value (found on the best-fit line) is called a **residual**. Residuals measure how much the data deviate from the regression line. When residuals are plotted on a scatter plot they can help to assess how well the best-fit line describes the data. If the best-fit line is a good fit, there is no pattern in the residual plot.

Real-World Example 2 Graph and Analyze a Residual Plot

HOCKEY **Graph and analyze the residual plot for the data for Guided Practice 1A. Determine if the best-fit line models the data well.**

After calculating the least-squares regression line in Guided Practice 1A, you can obtain the residual plot of the data. Turn on **Plot2** under the **STAT PLOT** menu and choose ⁖. Use **L1** for the **Xlist** and **RESID** for the **Ylist**. You can obtain **RESID** by pressing [2nd] **[STAT]** and selecting **RESID** from the list of names. Graph the scatter plot of the residuals by pressing [ZOOM] and choosing **ZoomStat**.



[0, 8] scl: 1 by [−10, 10] scl: 2

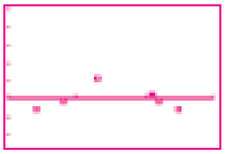
The residuals appear to be randomly scattered and centered about the line $y = 0$. Thus, the best-fit line seems to model the data well.

GuidedPractice

2. **UNEMPLOYMENT** Graph and analyze the residual plot for the following data comparing graduation rates and unemployment rates.

Graduation Rate	73	85	64	81	68	82
Unemployment Rate	6.9	4.1	3.2	5.5	4.3	5.1

2. The plot appears to have a curved pattern, so the regression line may not fit the data well.



[60, 90] scl: 10 by [−5, 10] scl: 2

Focus on Mathematical Content

Equation of a Best-Fit Line The TI-83/84 Plus graphing calculator has two methods to compute the equation of a best-fit line:

- LinReg($ax + b$) for linear regression and
- Med-Med for a median-fit line.

The linear regression method uses a least-squares fit method to determine the values for a and b. This utilizes calculus involving the distance each point is from the best-fit line. The median-fit method calculates the medians of the coordinates of the data points.

A residual is positive when the observed value is above the line, negative when the observed value is below the line, and zero when it is on the line. One common measure of goodness of fit is the sum of squared vertical distances from the points to the line. The best-fit line, which is also called the *least-squares regression line*, minimizes the sum of the squares of those distances.

We can use points on the best-fit line to estimate values that are not in the data. Recall that when we estimate values that are between known values, this is called *linear interpolation*. When we estimate a number outside of the range of the data, it is called *linear extrapolation*.

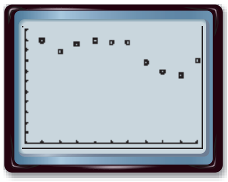
Real-World Example 3 Use Interpolation and Extrapolation

PAINTBALL **The table shows the points received by the top ten paintball teams at a tournament. Estimate how many points the 20th-ranked team received.**

Rank	1	2	3	4	5	6	7	8	9	10
Score	100	89	96	99	97	98	78	70	64	80

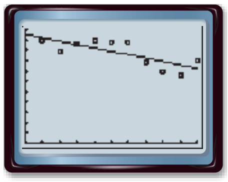
Write an equation of the best-fit line for the data. Then extrapolate to find the missing value.

Step 1 Enter the data from the table into the lists. Let the ranks be the x-values and the scores be the y-values. Then graph the scatter plot.


[0, 10] scl: 1 by [0, 110] scl: 10

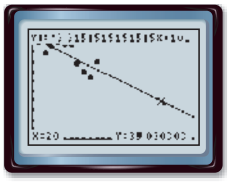
Step 2 Perform the linear regression using the data in the lists. Find the equation of the best-fit line.

The equation is about $y = -3.32x + 105.3$.


[0, 10] scl: 1 by [0, 110] scl: 10

Step 3 Graph the best-fit line. Press Y= VARS and choose **Statistics**. From the **EQ** menu, choose **RegEQ**. Then press GRAPH.

Step 4 Use the graph to predict the points that the 20th-ranked team received. Change the viewing window to include the x-value to be evaluated. Press 2nd **[CALC]** ENTER 20 ENTER to find that when $x = 20$, $y \approx 39$. It is estimated that the 20th ranked team received 39 points.



[0, 25] scl: 1 by [0, 110] scl: 1

StudyTip

Median-Fit Line The median-fit line is computed using a different algorithm than linear regression.

DifferentiatedInstruction AL OL

If students have trouble remembering certain keystrokes on their graphing calculators,

Then have students make “recipe” cards for each calculator activity they study. For example, have them title a card “Best-fit Line” and then write the keystrokes for using their graphing calculator to find the best-fit line for data and correlation coefficient. Students can refer to these cards as needed until they commit the steps to memory. Be sure students include a card for how to clear all previous data lists using the following keystrokes: 2nd [MEM] 4 ENTER.

Tips for New Teachers

Graphing Calculator Make sure that students understand the keystrokes for the STAT menu. Make sure students have cleared the L1 and L2 lists before entering new data. If students enter their data into L3 and L4 instead of L1 and L2, they must type in “L3, L4” after “LinReg(ax+b)” and then press ENTER.

Correlation Coefficient With DiagnosticOn, the calculator also displays values for r^2 and r. The closer the value for $|r|$ is to 1, the better the equation fits the data.

Example 3 shows how to use a calculator and an equation of the best-fit line for data for estimating a number outside of the range of the data.

Additional Example

3 **BOWLING** The table below shows the points earned by the top ten bowlers in a tournament. Estimate how many points the 15th-ranked bowler earned.

Rank	Score	Rank	Score
1	210	6	147
2	197	7	144
3	164	8	142
4	158	9	134
5	151	10	132

about 83

Teach with Tech

Video Recording Have students work in pairs to create a video showing how to find a regression equation or median-fit line. Share each group’s videos with the class.

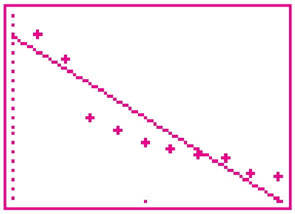
2 Equations of Median-Fit Lines

Example 4 shows how to calculate the equation of a median-fit line using a graphing calculator and then use the equation to predict a value.

Additional Example

4 Find and graph the equation of a median-fit line for the data on the bowling tournament in Additional Example 2. Then predict the score of the 20th ranked bowler.

$y = -9x + 209.5$; ≈ 30



[0, 22] scl: 1 by [0, 225] scl: 10

3 Practice

Formative Assessment

Use Exercises 1–3 to check for understanding.

Use the chart at the bottom of this page to customize assignments for your students.

GuidedPractice

ONLINE GAMES **Use linear interpolation to estimate the percent of Americans that play online games for the following ages.**

Age	15	20	30	40	50
Percent	81	54	37	29	25

Source: Pew Internet & American Life Survey

3A. 35 years ≈39%

3B. 18 years ≈64%

Real-WorldLink

Paintball is more popular with 12- to 17-year-olds than any other age group. In a recent year, 3,649,000 teenagers participated in paintball while 2,195,000 18- to 24-year-olds participated.

Source: *Statistical Abstract* of the *United States*

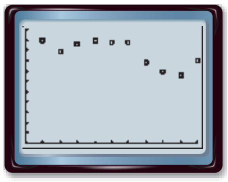
Alan Thornton/Stone/Getty Images

2 Median-Fit Lines A second type of fit line that can be found using a graphing calculator is a **median-fit line**. The equation of a median-fit line is calculated using the medians of the coordinates of the data points.

Example 4 Median-Fit Line

PAINTBALL **Find and graph the equation of a median-fit line for the data in Example 3. Then predict the score of the 15th ranked team.**

Step 1 Reenter the data if it is not in the lists. Clear the **Y=** list and graph the scatter plot.



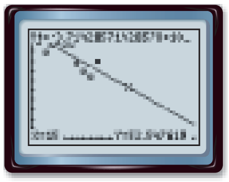
[0, 10] scl: 1 by [0, 110] scl: 10

Step 2 To find the median-fit equation, press the STAT key and select the **CALC** option. Scroll down to the **Med-Med** option and press ENTER. The value of a is the slope, and the value of b is the y-intercept.

The equation for the median-fit line is about $y = -3.71x + 108.26$.

Step 3 Copy the equation to the **Y=** list and graph. Use the **value** option to find the value of y when $x = 15$.

The 15th place team scored about 53 points.



[0, 25] scl: 1 by [0, 110] scl: 1

Notice that the equations for the regression line and the median-fit line are very similar.

GuidedPractice

4. ≈63%, ≈38%; These values are relatively close to those in Guided Practice 3A and 3B.

4. Use the data from Guided Practice 3 and a median-fit line to estimate the numbers of 18- and 35-year-olds who play online games. Compare these values with the answers from the regression line.

Differentiated Homework Options

Level	Assignment	Two-Day Option	
AL Basic	4–7, 16–18	5, 7	4, 6, 16–18
OL Core	5, 7, 9–14, 16–18	4–7	8–14, 16–18
BL Advanced	8–18		

Check Your Understanding

Examples 1, 2 **1. POTTERY** A local university is keeping track of the number of art students who use the pottery studio each day.

Day	1	2	3	4	5	6	7
Students	10	15	18	15	13	19	20

a. Write an equation of the regression line and find the correlation coefficient. $y = 1.18x + 11$; 0.7181

b. Graph the residual plot and determine if the regression line models the data well. See margin.

Example 3 **2. COMPUTERS** The table below shows the percent of Americans with a broadband connection at home in a recent year. Use linear extrapolation and a regression equation to estimate the percentage of 60-year-olds with broadband at home. 29%

Age	25	30	35	40	45	50
Percent	40	42	36	35	36	32

Example 4 **3. VACATION** The Smiths want to rent a house on the lake that sleeps eight people. The cost of the house per night is based on how close it is to the water.

Distance from Lake (mi)	0.0 (houseboat)	0.3	0.5	1.0	1.25	1.5	2.0
Price/Night ($)	785	325	250	200	150	140	100

a. Find and graph an equation for the median-fit line.

b. What would you estimate is the cost of a rental 1.75 miles from the lake? $78.69

a. $y = -271.88x + 554.48$; See margin for graph.

Practice and Problem Solving

Example 1 **Write an equation of the regression line for the data in each table. Then find the correlation coefficient.**

$y = -2.75x + 102.53$; -0.6071

4. SKYSCRAPERS The table ranks the ten tallest buildings in the world.

Rank	1	2	3	4	5	6	7	8	9	10
Stories	101	88	110	88	88	80	69	102	78	70

5. MUSIC The table gives the number of annual violin auditions held by a youth symphony each year since 2004. Let x be the number of years since 2004.

$y = 3.54x + 19.68$; 0.9007

Year	2004	2005	2006	2007	2008	2009	2010
Auditions	22	19	25	37	32	35	42

Example 2 **6. RETAIL** The table gives the sales at a clothing chain since 2004. Let x be the number of years since 2004.

Year	2004	2005	2006	2007	2008	2009
Sales (Millions of Dollars)	6.84	7.6	10.9	15.4	17.6	21.2

a. Write an equation of the regression line. $y = 3.32x + 5.20$

b. Graph and analyze the residual plot. See margin.

Follow-up

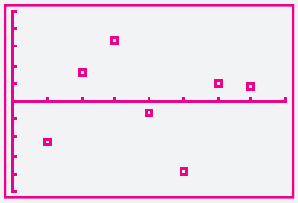
Students have explored lines of best fit and linear regression.

Ask:

- How can you use a set of data to make predictions? Sample answer: You can create a scatter plot of the data and determine the line of best fit. You could also enter the data into a calculator and find the equation of the regression line. Then, you could use the equation to estimate values that are between known values or outside the range of the data.

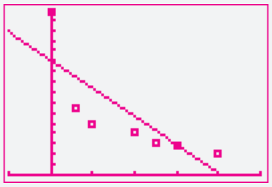
Additional Answers

1b. The residuals appear to be randomly scattered, so the regression line fits the data reasonably well.



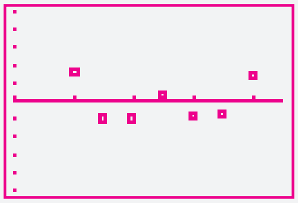
[0, 8] scl: 1.5 by [−5, 5] scl: 1.5

3a.



[−0.5, 2.5] scl: 1 by [0, 785] scl: 10

6b. residuals appear to be randomly scattered, so the regression line fits the data well.

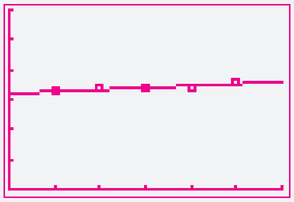


[2, 11] scl: 2 by [−5, 5]

Additional Answers

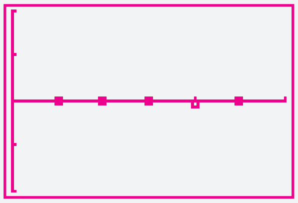
9b.

11a. $y = 0.0326x + 1.598$



[0, 6] scl: 1 by [0, 3] scl: 0.5

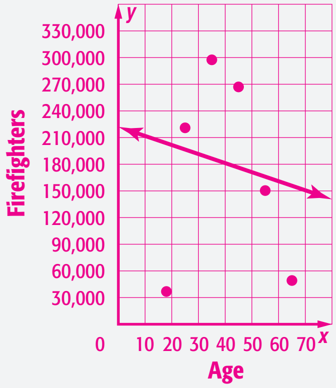
11b. The regression line is a good fit as the residuals appear to almost be on the line.



[0, 6] scl: 1 by [−2, 2] scl: 1

12a. $y = -841.42x + 223{,}288$

12b.



Examples 3, 4

7. **MARATHON** The number of entrants in the Boston Marathon every five years since 1975 is shown. Let x be the number of years since 1975.

Year	1975	1980	1985	1990	1995	2000	2005	2010
Entrants	2395	5417	5594	9412	9416	17,813	20,453	26,735

a. Find an equation for the median-fit line. $y = 601.44x + 1236.13$

b. According to the equation, how many entrants were there in 2003? about 18,076

B

8. **CAMPING** A campground keeps a record of the number of campsites rented the week of July 4 for several years. Let x be the number of years since 2000.

Year	2002	2003	2004	2005	2006	2007	2008	2009	2010
Sites Rented	34	45	42	53	58	47	57	65	59

a. Find an equation for the regression line. $y = 3.07x + 32.71$

b. Predict the number of campsites that will be rented in 2012. about 70 campsites

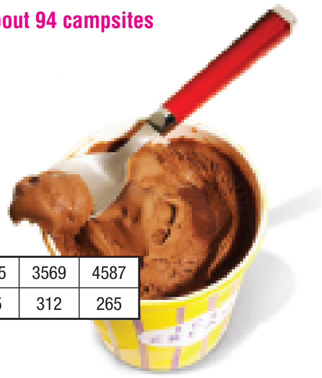
c. Predict the number of campsites that will be rented in 2020. about 94 campsites

9. **ICE CREAM** An ice cream company keeps a count of the tubs of chocolate ice cream delivered to each of their stores in a particular area.

a. Find an equation for the median-fit line. $y = 0.095x - 94.58$

b. Graph the points and the median-fit line. See margin.

c. How many tubs would be delivered to a 1500-square-foot store? a 5000-square-foot store? about 48 tubs; about 380 tubs



Store Size (ft^2)	2100	2225	3135	3569	4587
Tubs (hundreds)	110	102	215	312	265

10. **FINANCIAL LITERACY** The prices of the eight top-selling brands of jeans at Jeanie's Jeans are given in the table below.

Sales Rank	1	2	3	4	5	6	7	8
Price ($)	43	44	50	61	64	135	108	78

a. Find the equation for the regression line. $y = 9.8x + 28.79$

b. According to the equation, what would be the price of a pair of the 12th best-selling brand? $146.39

c. Is this a reasonable prediction? Explain.

10c. Sample answer: Yes; the number is within a reasonable range of the other pairs of jeans.

11. **STATE FAIRS** Refer to the beginning of the lesson. a, b. See margin.

a. Graph a scatter plot of the data, where $x = 1$ represents 2005. Then find and graph the equation for the best-fit line.

b. Graph and analyze the residual plot.

c. Predict the total attendance in 2020. about 2.12 million people

Alex Cao/Photodisc/Getty Images

Differentiated Instruction OL BL

Extension Remind students that outliers are points that are significantly distant from the other data points. Ask students to determine if removing the outlier in Exercise 3 affects the estimated cost of a rental 1.75 miles from the lake. Yes, the outlier (0.0, 785) causes the estimate to be lower than it currently is. Removing the outlier and using the median-fit line for the remaining data points results in a more realistic estimated cost of $112 per night for renting a house 1.75 miles from the lake.

12. **FIREFIGHTERS** The table shows statistics from the U.S. Fire Administration.

Age	Number of Firefighters
18	40,919
25	245,516
35	330,516
45	296,665
55	167,087
65	54,559

a. Find an equation for the median-fit line. **See margin.**

b. Graph the points and the median-fit line. **See margin.**

c. Does the median-fit line give you an accurate picture of the number of firefighters? Explain. **No; sample answer: The points show no linear correlation; therefore a line cannot accurately portray the data.**

13. **ATHLETICS** The table shows the number of participants in high school athletics.

Year Since 1970	1	10	20	30	35
Athletes	3,960,932	5,356,913	5,298,671	6,705,223	7,159,904

a. Find an equation for the regression line. **$y = 87{,}390.5x + 4{,}018{,}431$**

b. According to the equation, how many participated in 1988? **about 5,591,460**

14. **ART** A count was kept on the number of paintings sold at an auction by the year in which they were painted. Let x be the number of years since 1950.

Year Painted	1950	1955	1960	1965	1970	1975
Paintings Solds	8	5	25	21	9	22

a. Find the equation for the linear regression line. **$y = 0.446x + 9.43$**

b. How many paintings were sold that were painted in 1961? **about 14 paintings**

c. Is the linear regression equation an accurate model of the data? Explain why or why not. **No, the correlation coefficient is 0.48, so the linear model is not a good fit for the data.**

H.O.T. Problems Use Higher-Order Thinking Skills

15. **CHALLENGE** Below are the results of the World Superpipe Championships in 2008.

Men	Score	Rank	Women	Score
Shaun White	93.00	1	Torah Bright	96.67
Mason Aguirre	90.33	2	Kelly Clark	93.00
Janne Korpi	85.33	3	Soko Yamaoka	85.00
Luke Mitrani	85.00	4	Ellery Hollingsworth	79.33
Keir Dillion	81.33	5	Sophie Rodriguez	71.00

Find an equation of the regression line for each, and graph them on the same coordinate plane. Compare and contrast the men's and women's graphs. **See Answer Appendix.**

16. **REASONING** For a class project, the scores that 10 randomly selected students earned on the first 8 tests of the school year are given. Explain how to find a line of best fit. Could it be used to predict the scores of other students? Explain your reasoning. **See margin.**

17. **OPEN ENDED** For 10 different people, measure their heights and the lengths of their heads from chin to top. Use these data to generate a linear regression equation and a median-fit equation. Make a prediction using both of the equations. **See students' work.**

18. **WRITING IN MATH** How are lines of fit and linear regression similar? different? **See margin.**

4 Assess

Yesterday's News Ask students to write how investigating relationships using scatter plots and lines of fit in yesterday's lesson helped them in today's new material.

Additional Answers

16. Apply a linear regression model to the data. Use the number of each test as the independent variable and the score on each test as the dependent variable. If there is no correlation, the r value will not be close to 1 or -1. If this is the case, the line of fit could not be used to predict the scores of the other students.

18. Sample answer: Both lines of fit and linear regression are used to model data. However, you could have numerous lines of fit, while linear regression results in one line of best fit. If linear regression is used, you can also use the correlation coefficient to see how closely the model fits the data.

1 Focus

Vertical Alignment

Before Lesson 6 You represented relations as tables, graphs, and mappings.

Lesson 6 Find the inverse of a relation. Find the inverse of a linear function.

After Lesson 6 Find the inverse of a quadratic function.

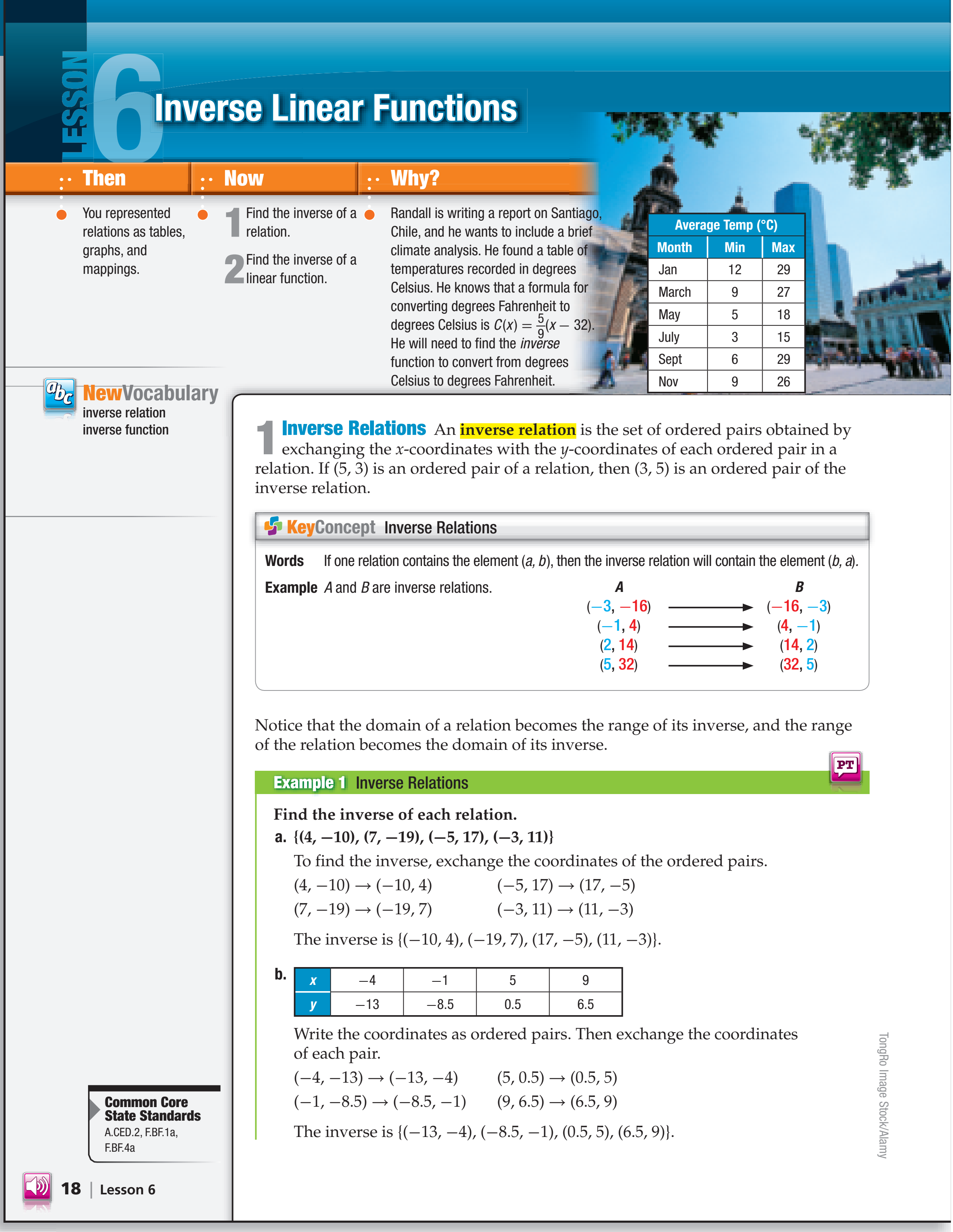
2 Teach

Scaffolding Questions

Have students read the **Why?** section of the lesson.

Ask:

- What do $C(x)$ and x represent in the context of the function? $C(x)$ is degrees Celsius, and x is degrees Fahrenheit.
- How can the equation be used to convert 12° C to degree Fahrenheit? Sample answer: Substitute 12 for $C(x)$ in the given equation and then solve for x.
- Convert 12°C to degrees Fahrenheit. 53.6° F



LESSON 6 Inverse Linear Functions

Then

- You represented relations as tables, graphs, and mappings.

Now

- 1 Find the inverse of a relation.
- 2 Find the inverse of a linear function.

Why?

- Randall is writing a report on Santiago, Chile, and he wants to include a brief climate analysis. He found a table of temperatures recorded in degrees Celsius. He knows that a formula for converting degrees Fahrenheit to degrees Celsius is $C(x) = \frac{5}{9}(x - 32)$. He will need to find the *inverse* function to convert from degrees Celsius to degrees Fahrenheit.

Average Temp (°C)		
Month	Min	Max
Jan	12	29
March	9	27
May	5	18
July	3	15
Sept	6	29
Nov	9	26

New Vocabulary
inverse relation
inverse function

Common Core State Standards
A.CED.2, F.BF.1a, F.BF.4a

1 Inverse Relations

An **inverse relation** is the set of ordered pairs obtained by exchanging the x-coordinates with the y-coordinates of each ordered pair in a relation. If (5, 3) is an ordered pair of a relation, then (3, 5) is an ordered pair of the inverse relation.

Key Concept Inverse Relations

Words If one relation contains the element (a, b), then the inverse relation will contain the element (b, a).

Example A and B are inverse relations.

A		B
(−3, −16)	⟶	(−16, −3)
(−1, 4)	⟶	(4, −1)
(2, 14)	⟶	(14, 2)
(5, 32)	⟶	(32, 5)

Notice that the domain of a relation becomes the range of its inverse, and the range of the relation becomes the domain of its inverse.

Example 1 Inverse Relations

Find the inverse of each relation.

a. **{(4, −10), (7, −19), (−5, 17), (−3, 11)}**

To find the inverse, exchange the coordinates of the ordered pairs.

(4, −10) → (−10, 4)　　(−5, 17) → (17, −5)
(7, −19) → (−19, 7)　　(−3, 11) → (11, −3)

The inverse is {(−10, 4), (−19, 7), (17, −5), (11, −3)}.

b.

x	−4	−1	5	9
y	−13	−8.5	0.5	6.5

Write the coordinates as ordered pairs. Then exchange the coordinates of each pair.

(−4, −13) → (−13, −4)　　(5, 0.5) → (0.5, 5)
(−1, −8.5) → (−8.5, −1)　　(9, 6.5) → (6.5, 9)

The inverse is {(−13, −4), (−8.5, −1), (0.5, 5), (6.5, 9)}.

TongRo Image Stock/Alamy

18 | Lesson 6

GuidedPractice

1A. {(−6, 8), (−15, 11), (9, 3), (0, 6)}
{(8, −6), (11, −15), (3, 9), (6, 0)}

1B.

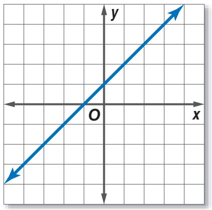
x	−10	−4	−3	0
y	5	11	12	15

1B. {(5, −10), (11, −4), (12, −3), (15, 0)}

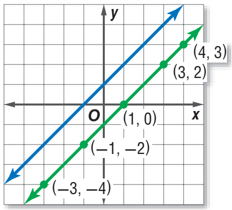
The graphs of relations can be used to find and graph inverse relations.

Example 2 Graph Inverse Relations

Graph the inverse of the relation.



The graph of the relation passes through the points at (−4, −3), (−2, −1), (0, 1), (2, 3), and (3, 4). To find points through which the graph of the inverse passes, exchange the coordinates of the ordered pairs. The graph of the inverse passes through the points at (−3, −4), (−1, −2), (1, 0), (3, 2), and (4, 3). Graph these points and then draw the line that passes through them.

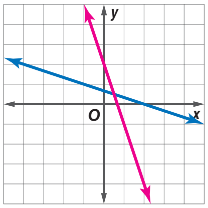


StudyTip
Graphing Inverses Only two points are necessary to graph the inverse of a line, but several should be used to avoid possible error.

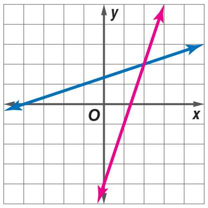
GuidedPractice

Graph the inverse of each relation.

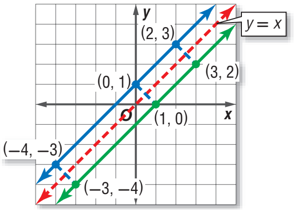
2A.



2B.



The graphs from Example 2 are graphed on the right with the line $y = x$. Notice that the graph of an inverse is the graph of the original relation reflected in the line $y = x$. For every point (x, y) on the graph of the original relation, the graph of the inverse will include the point (y, x).



2 Inverse Functions

A linear relation that is described by a function has an inverse function that can generate ordered pairs of the inverse relation The inverse of the linear function $f(x)$ can be written as $f^{-1}(x)$ and is read *f of x inverse* or *the inverse of f of x*.

1 Inverse Relations

Example 1 shows how to find the inverses of relations. **Example 2** shows how to graph the inverse of a relation.

Formative Assessment

Use the Guided Practice exercises after each example to determine students' understanding of concepts.

Additional Examples

1 Find the inverse of each relation.

a. {(−3, 26), (2, 11), (6, −1), (−1, 20)} {(26, −3), (11, 2), (−1, 6), (20, −1)}

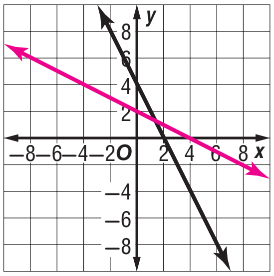
b.

x	y
−4	−3
−2	0
1	4.5
5	10.5

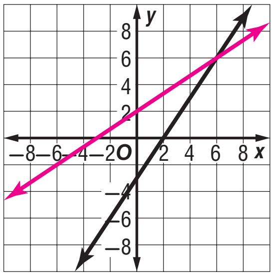
{(−3, −4), (0, −2), (4.5, 1), (10.5, 5)}

2 Graph the inverse of each relation.

a.



b.



Additional Examples also in Interactive Classroom PowerPoint® Presentations

IWB Interactive White Board READY

2 Inverse Functions

Example 3 shows how to find inverses of functions. **Example 4** shows how to find and use an inverse function.

Additional Example

3 Find the inverse of each function.

a. $f(x) = -3x + 27$

$f^{-1}(x) = -\frac{1}{3}x + 9$

b. $f(x) = \frac{5}{4}x - 8$

$f^{-1}(x) = \frac{4}{5}x + \frac{32}{5}$

Tips for New Teachers

Inverse Functions If $f(x)$ and $g(x)$ are inverse functions, then $f(a) = b$ and $g(b) = a$. However, the converse of this statement is not necessarily true. Let $f(x) = 3x + 9$ and $g(x) = x - 13$. $f(2) = 15$ and $g(15) = 2$, but $f(x)$ and $g(x)$ are not inverse functions. Students will explore this in Exercises 40 and 41.

KeyConcept Finding Inverse Functions

To find the inverse function $f^{-1}(x)$ of the linear function $f(x)$, complete the following steps.

Step 1 Replace $f(x)$ with y in the equation for $f(x)$.

Step 2 Interchange y and x in the equation.

Step 3 Solve the equation for y.

Step 4 Replace y with $f^{-1}(x)$ in the new equation.

Example 3 Find Inverse Linear Functions

Find the inverse of each function.

a. $f(x) = 4x - 8$

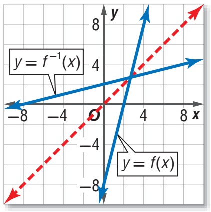
Step 1	$f(x) = 4x - 8$	Original equation
	$y = 4x - 8$	Replace $f(x)$ with y.
Step 2	$x = 4y - 8$	Interchange y and x.
Step 3	$x + 8 = 4y$	Add 8 to each side.
	$\frac{x+8}{4} = y$	Divide each side by 4.
Step 4	$\frac{x+8}{4} = f^{-1}(x)$	Replace y with $f^{-1}(x)$.

The inverse of $f(x) = 4x - 8$ is $f^{-1}(x) = \frac{x+8}{4}$ or $f^{-1}(x) = \frac{1}{4}x + 2$.

WatchOut!

Notation The -1 in $f^{-1}(x)$ is *not* an exponent.

CHECK Graph both functions and the line $y = x$ on the same coordinate plane. $f^{-1}(x)$ appears to be the reflection of $f(x)$ in the line $y = x$. ✓



b. $f(x) = -\frac{1}{2}x + 11$

Step 1	$f(x) = -\frac{1}{2}x + 11$	Original equation
	$y = -\frac{1}{2}x + 11$	Replace $f(x)$ with y.
Step 2	$x = -\frac{1}{2}y + 11$	Interchange y and x.
Step 3	$x - 11 = -\frac{1}{2}y$	Subtract 11 from each side.
	$-2(x - 11) = y$	Multiply each side by -2.
	$-2x + 22 = y$	Distributive Property
Step 4	$-2x + 22 = f^{-1}(x)$	Replace y with $f^{-1}(x)$.

The inverse of $f(x) = -\frac{1}{2}x + 11$ is $f^{-1}(x) = -2x + 22$.

GuidedPractice

3A. $f(x) = 4x - 12$ $f^{-1}(x) = \frac{1}{4}x + 3$

3B. $f(x) = \frac{1}{3}x + 7$ $f^{-1}(x) = 3x - 21$

DifferentiatedInstruction AL OL

Interpersonal Learners Write the equation of a linear function on an index card. Sketch the graph of the function on a second card. On a third card, display a table of points for the function. Repeat this process for the inverse of the function. Create similar cards for several functions and their inverses. Have students work in groups of two or three. Give each group a copy of all of the cards. Have the groups divide the cards by matching each equation with its graph and table, and then with the equation, graph, and table of its inverse.

Real-WorldLink

The winter months in Chile occur during the summer months in the U.S. due to Chile's location in the southern hemisphere. The average daily high temperature of Santiago during its winter months is about 60° F.

Source: World Weather Information Service

Real-World Example 4 Use an Inverse Function

TEMPERATURE **Refer to the beginning of the lesson. Randall wants to convert the temperatures from degrees Celsius to degrees Fahrenheit.**

a. Find the inverse function $C^{-1}(x)$.

Step 1	$C(x) = \frac{5}{9}(x - 32)$	Original equation
	$y = \frac{5}{9}(x - 32)$	Replace $C(x)$ with y.
Step 2	$x = \frac{5}{9}(y - 32)$	Interchange y and x.
Step 3	$\frac{9}{5}x = y - 32$	Multiply each side by $\frac{9}{5}$.
	$\frac{9}{5}x + 32 = y$	Add 32 to each side.
Step 4	$\frac{9}{5}x + 32 = C^{-1}(x)$	Replace y with $C^{-1}(x)$.

The inverse function of $C(x)$ is $C^{-1}(x) = \frac{9}{5}x + 32$.

b. What do x and $C^{-1}(x)$ represent in the context of the inverse function?

x represents the temperature in degrees Celsius. $C^{-1}(x)$ represents the temperature in degrees Fahrenheit.

c. Find the average temperatures for July in degrees Fahrenheit.

The average minimum and maximum temperatures for July are 3° C and 15° C, respectively. To find the average minimum temperature, find $C^{-1}(3)$.

$C^{-1}(x) = \frac{9}{5}x + 32$	Original equation
$C^{-1}(3) = \frac{9}{5}(3) + 32$	Subsitute 3 for x.
$= 37.4$	Simplify.

To find the average maximum temperature, find $C^{-1}(15)$.

$C^{-1}(x) = \frac{9}{5}x + 32$	Original equation
$C^{-1}(15) = \frac{9}{5}(15) + 32$	Subsitute 15 for x.
$= 59$	Simplify.

The average minimum and maximum temperatures for July are 37.4° F and 59° F, respectively.

GuidedPractice

4. **RENTAL CAR** Peggy rents a car for the day. The total cost $C(x)$ in dollars is given by $C(x) = 19.99 + 0.3x$, where x is the number of miles she drives.

   A. Find the inverse function $C^{-1}(x)$. $C^{-1}(x) = \frac{x - 19.99}{0.3}$

   B. What do x and $C^{-1}(x)$ represent in the context of the inverse function? x is the total cost, and $C^{-1}(x)$ is the total number of miles driven.

   C. How many miles did Peggy drive if her total cost was $34.99? 50

Felipe Dupouy/Lifesize/Getty Images

Additional Example

4 **SALES** Carter sells paper supplies and makes a base salary of $2200 each month. He also earns 5% commission on his total sales. His total earnings $f(x)$ for a month in which he compiled x dollars in total sales is $f(x) = 2200 + 0.05x$.

a. Find the inverse function. $f^{-1}(x) = 20x - 44{,}000$

b. What do x and $f^{-1}(x)$ represent in the context of the inverse function? x is Carter's total earnings for the month, and $f^{-1}(x)$ is his total sales for the month.

c. Find Carter's total sales for last month if his earnings for that month were $3450. $25,000

3 Practice

Formative Assessment

Use Exercises 1–7 to check for understanding.

Use the chart at the bottom of this page to customize assignments for your students.

Check Your Understanding

Example 1 Find the inverse of each relation.

1. {(4, −15), (−8, −18), (−2, −16.5), (3, −15.25)} **{(−15, 4), (−18, −8), (−16.5, −2), (−15.25, 3)}**

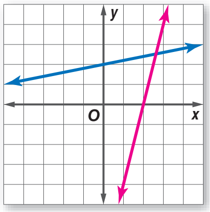
2.

x	−3	0	1	6
y	11.8	3.7	1	−12.5

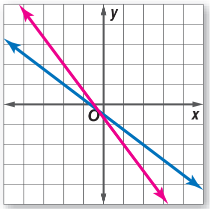
{(11.8, −3), (3.7, 0), (1, 1), (−12.5, 6)}

Example 2 Graph the inverse of each relation.

3.



4.



Example 3 Find the inverse of each function.

5. $f(x) = -2x + 7$ $f^{-1}(x) = -\frac{1}{2}x + \frac{7}{2}$

6. $f(x) = \frac{2}{3}x + 6$ $f^{-1}(x) = \frac{3}{2}x - 9$

Example 4

7. **TICKETS** Dwayne and his brother purchase season tickets to the Cleveland Crusaders games. The ticket package requires a one-time purchase of a personal seat license costing $1200 for two seats. A ticket to each game costs $70. The cost $C(x)$ in dollars for Dwayne for the first season is $C(x) = 600 + 70x$, where x is the number of games Dwayne attends.

a. Find the inverse function. $C^{-1}(x) = \frac{1}{70}x - \frac{60}{7}$

b. What do x and $C^{-1}(x)$ represent in the context of the inverse function?

b. **x is Dwayne's total cost, and $C^{-1}(x)$ is the number of games Dwayne attended.**

c. How many games did Dwayne attend if his total cost for the season was $950? **5**

Practice and Problem Solving

Example 1 Find the inverse of each relation.

8. {(−5, 13), (6, 10.8), (3, 11.4), (−10, 14)} **{(13, −5), (10.8, 6), (11.4, 3), (14, −10)}**

9. {(−4, −49), (8, 35), (−1, −28), (4, 7)} **{(−49, −4), (35, 8), (−28, −1), (7, 4)}**

10.

x	y
−8	−36.4
−2	−15.4
1	−4.9
5	9.1
11	30.1

{(−36.4, −8), (−15.4, −2), (−4.9, 1), (9.1, 5), (30.1, 11)}

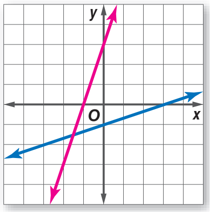
11.

x	y
−3	7.4
−1	4
1	0.6
3	−2.8
5	−6.2

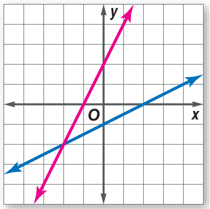
{(7.4, −3), (4, −1), (0.6, 1), (−2.8, 3), (−6.2, 5)}

Example 2 Graph the inverse of each relation.

12.



13.



Differentiated Homework Options

Level	Assignment	Two-Day Option	
AL Basic	8–21, 40–43	9–21 odd	8–20 even, 40–43
OL Core	9–35 odd, 36, 37, 40–43	8–21	22–37, 40–43
BL Advanced	22–43		

Example 3 **Find the inverse of each function.**

14. $f(x) = 25 + 4x$ $f^{-1}(x) = \frac{1}{4}x - \frac{25}{4}$

15. $f(x) = 17 - \frac{1}{3}x$ $f^{-1}(x) = -3x + 51$

16. $f(x) = 4(x + 17)$ $f^{-1}(x) = \frac{1}{4}x - 17$

17. $f(x) = 12 - 6x$ $f^{-1}(x) = -\frac{1}{6}x + 2$

18. $f(x) = \frac{2}{5}x + 10$ $f^{-1}(x) = \frac{5}{2}x - 25$

19. $f(x) = -16 - \frac{4}{3}x$ $f^{-1}(x) = -\frac{3}{4}x - 12$

Example 4 **20.** **DOWNLOADS** An online music subscription service allows members to download songs for \$0.99 each after paying a monthly service charge of \$3.99. The total monthly cost $C(x)$ of the service in dollars is $C(x) = 3.99 + 0.99x$, where x is the number of songs downloaded.

a. Find the inverse function. $C^{-1}(x) = \frac{x - 3.99}{0.99}$

b. What do x and $C^{-1}(x)$ represent in the context of the inverse function?

b. x **is the total monthly cost of the service, and** $C^{-1}(x)$ **is the number of songs downloaded.**

c. How many songs were downloaded if a member's monthly bill is \$27.75? **24**

21. **LANDSCAPING** At the start of the mowing season, Chuck collects a one-time maintenance fee of \$10 from his customers. He charges the Fosters \$35 for each cut. The total amount collected from the Fosters in dollars for the season is $C(x) = 10 + 35x$, where x is the number of times Chuck mows the Fosters' lawn.

a. Find the inverse function. $C^{-1}(x) = \frac{1}{35}x - \frac{2}{7}$

b. What do x and $C^{-1}(x)$ represent in the context of the inverse function?

c. How many times did Chuck mow the Fosters' lawn if he collected a total of \$780 from them? **22**

21b. x **is the total amount collected from the Fosters, and** $C^{-1}(x)$ **is the number of times Chuck mowed the Fosters' lawn.**

B **Write the inverse of each equation in** $f^{-1}(x)$ **notation.**

22. $3y - 12x = -72$ $f^{-1}(x) = \frac{1}{4}x + 6$

23. $x + 5y = 15$ $f^{-1}(x) = 15 - 5x$

24. $-42 + 6y = x$ $f^{-1}(x) = 6x - 42$

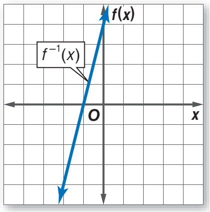
25. $3y + 24 = 2x$ $f^{-1}(x) = \frac{3}{2}x + 12$

26. $-7y + 2x = -28$ $f^{-1}(x) = \frac{7}{2}x - 14$

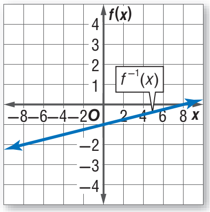
27. $3y - x = 3$ $f^{-1}(x) = 3x - 3$

Match each function with the graph of its inverse.

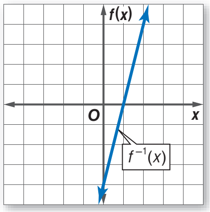
A.



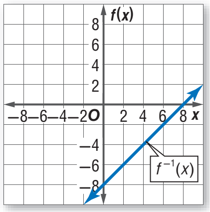
B.



C.



D.



28. $f(x) = x + 4$ **D**

29. $f(x) = 4x + 4$ **B**

30. $f(x) = \frac{1}{4}x + 1$ **C**

31. $f(x) = \frac{1}{4}x - 1$ **A**

4 Assess

Ticket Out the Door Have students identify a linear function by writing its equation and graphing it. Then have them find the equation and graph for the inverse of the function.

Multiple Representations

In Exercise 37, students use graphs to investigate the domains and ranges of inverse functions.

Write an equation for the inverse function $f^{-1}(x)$ that satisfies the given conditions.

32. slope of $f(x)$ is 7; graph of $f^{-1}(x)$ contains the point (13, 1) **$f^{-1}(x) = \frac{1}{7}x - \frac{6}{7}$**

33. graph of $f(x)$ contains the points $(-3, 6)$ and (6, 12) **$f^{-1}(x) = \frac{3}{2}x - 12$**

34. graph of $f(x)$ contains the point (10, 16); graph of $f^{-1}(x)$ contains the point $(3, -16)$ **$f^{-1}(x) = 2x - 22$**

35. slope of $f(x)$ is 4; $f^{-1}(5) = 2$ **$f^{-1}(x) = \frac{1}{4}x + \frac{3}{4}$**

C **36.** **CELL PHONES** Mary Ann pays a monthly fee for her cell phone package which includes 700 minutes. She gets billed an additional charge for every minute she uses the phone past the 700 minutes. During her first month, Mary Ann used 26 additional minutes and her bill was \$37.79. During her second month, Mary Ann used 38 additional minutes and her bill was \$41.39.

a. Write a function that represents the total monthly cost $C(x)$ of Mary Ann's cell phone package, where x is the number of additional minutes used. **$C(x) = 0.3x + 29.99$**

b. Find the inverse function. **$C^{-1}(x) = \frac{x - 29.99}{0.3}$**

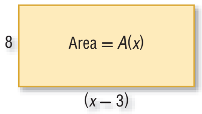
c. What do x and $C^{-1}(x)$ represent in the context of the inverse function?

d. How many additional minutes did Mary Ann use if her bill for her third month was \$48.89? **63**

36c. x is Mary Ann's total monthly cost, and $C^{-1}(x)$ is the number of additional minutes used.

37. **MULTIPLE REPRESENTATIONS** In this problem, you will explore the domain and range of inverse functions. **b, d. See Answer Appendix.**

a. Algebraic Write a function for the area $A(x)$ of the rectangle shown. **$A(x) = 8(x - 3)$ or $A(x) = 8x - 24$**



b. Graphical Graph $A(x)$. Describe the domain and range of $A(x)$ in the context of the situation.

c. Algebraic Write the inverse of $A(x)$. What do x and $A^{-1}(x)$ represent in the context of the inverse function?

d. Graphical Graph $A^{-1}(x)$. Describe the domain and range of $A^{-1}(x)$ in the context of the situation.

e. Logical Determine the relationship between the domains and ranges of $A(x)$ and $A^{-1}(x)$. **Sample answer: The domain of $A(x)$ is the range of $A^{-1}(x)$, and the range of $A(x)$ is the domain of $A^{-1}(x)$.**

37c. $A^{-1}(x) = \frac{1}{8}x + 3$; x is the area of the rectangle and $A^{-1}(x)$ is the value of x in the expression for the length of the side of the rectangle $x - 3$.

40. True; sample answer: If $f(a) = b$, then the graph of $f(x)$ includes the point (a, b). If $f(x)$ and $g(x)$ are inverses, then the graph of $g(x)$ includes the point (b, a). If (b, a) is included on the graph of $g(x)$, then $g(b) = a$.

H.O.T. Problems Use Higher-Order Thinking Skills

38. **CHALLENGE** If $f(x) = 5x + a$ and $f^{-1}(10) = -1$, find a. **15**

39. **CHALLENGE** If $f(x) = \frac{1}{a}x + 7$ and $f^{-1}(x) = 2x - b$, find a and b. **$a = 2$; $b = 14$**

REASONING **Determine whether the following statements are *sometimes*, *always*, or *never* true. Explain your reasoning.** **41–43. See Answer Appendix.**

40. If $f(x)$ and $g(x)$ are inverse functions, then $f(a) = b$ and $g(b) = a$.

41. If $f(a) = b$ and $g(b) = a$, then $f(x)$ and $g(x)$ are inverse functions.

42. **OPEN ENDED** Give an example of a function and its inverse. Verify that the two functions are inverses by graphing the functions and the line $y = x$ on the same coordinate plane.

43. **WRITING IN MATH** Explain why it may be helpful to find the inverse of a function.

Differentiated Instruction BL

Extension Have students graph the inverse of $y = x^2$. First, have students graph $y = x^2$ on a coordinate plane by creating a table of points. Then, have them find points on the graph of the inverse by exchanging the x-coordinates with the y-coordinates of each ordered pair in the table. Have students plot these new points on the same coordinate plane. Finally, have students connect the new points with a smooth curve, using the graph of $y = x^2$ as a guide. Remind students that the graphs of inverse relations are reflections of each other in the line $y = x$.

LAB 7

Algebra Lab
Drawing Inverses

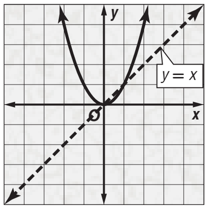
You can use patty paper to draw the graph of an inverse relation by reflecting the original graph in the line $y = x$.

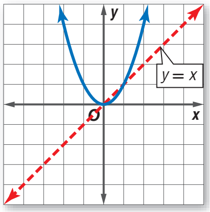
Common Core State Standards
F.BF.4a

Activity Draw an Inverse

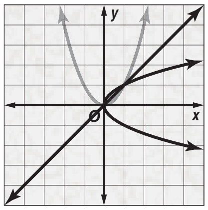
Consider the graphs shown.

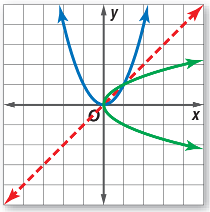
Step 1 Trace the graphs onto a square of patty paper, waxed paper, or tracing paper.





Step 2 Flip the patty paper over and lay it on the original graph so that the traced $y = x$ is on the original $y = x$.





Notice that the result is the reflection of the graph in the line $y = x$ or the inverse of the graph.

6. Sample answer: Draw a horizontal line to see if the inverse represents a function. If a horizontal line intersects the graph more than once, then the inverse is not a function. If a horizontal line intersects the graph at only one point, the inverse is a function.

Analyze The Results

1. Is the graph of the original relation a function? Explain. **Yes; it passes the vertical line test.**
2. Is the graph of the inverse relation a function? Explain. **No; it does not pass the vertical line test.**
3. What are the domain and range of the original relation? of the inverse relation? **D = {all real numbers}, R = $\{y \mid y \geq 0\}$; D = $\{x \mid x \geq 0\}$; R = {all real numbers}**
4. If the domain of the original relation is restricted to $D = \{x \mid x \geq 0\}$, is the inverse relation a function? Explain. **Yes; none of the domain values would have more than one corresponding range value.**
5. If the graph of a relation is a function, what can you conclude about the graph of its inverse? **Nothing; its inverse may or may not be a function. However, if the domain of the original is restricted, the inverse will likely be a function.**
6. **CHALLENGE** The vertical line test can be used to determine whether a relation is a function. Write a rule that can be used to determine whether a function has an inverse that is also a function.

1 Focus

Objective Draw the inverse of a relation and determine whether the inverse is a function.

Materials for Each Student

- patty paper (1 sheet)
- colored pencils

Teach with Tech

Graphing Calculator Students may also be interested to know that most of their calculators have a draw inverse feature. They can access it by entering their original equation into the **Y=** list. Press 2nd **[DRAW]** 8. **DrawInv** will appear on the screen. Select the equation entered into **Y1** by pressing VARS ▶ 1 ENTER. Press ENTER once more to see the graph of the function and its inverse.

2 Teach

Working in Cooperative Groups

Divide the class into pairs to work through the Activity. Then ask students to work with their partners to complete Exercises 1–3.

Practice Have students complete Exercises 4–6.

3 Assess

Formative Assessment

Use Exercises 1–3 to assess each student's ability to determine the domain and range of a graph and its inverse and to determine if a relation and its inverse are functions.

From Concrete to Abstract

Exercise 4 asks about the specific case shown, Exercise 5 asks students to expand that knowledge to a general case, and Exercise 6 extends the concept to the horizontal line test.

1 Focus

VerticalAlignment

Before Lesson 8 Used the laws of exponents to find products and quotients of monomials.

Lesson 8 Evaluate and rewrite expressions involving rational exponents. Solve equations involving expressions with rational exponents.

After Lesson 8 Express numbers in scientific notation. Find products and quotients of numbers expressed in scientific notation.

2 Teach

Scaffolding Questions

Have students read the **Why?** section of the lesson.

Ask:

- How does $50f^{0.2}$ differ from other exponential expressions students have seen before? Sample answer: The exponent is not an integer.
- Does a higher SPF offer more or less protection from sun damage? more
- What SPF numbers have you seen on sunscreens in stores? Sample answer: 2, 4, 15, 30

LESSON 8 Rational Exponents

Then

- You used the laws of exponents to find products and quotients of monomials.

Now

1. Evaluate and rewrite expressions involving rational exponents.
2. Solve equations involving expressions with rational exponents.

Why?

- It's important to protect your skin with sunscreen to prevent damage. The sun protection factor (SPF) of a sunscreen indicates how well it protects you. Sunscreen with an SPF of f absorbs about p percent of the UV-B rays, where $p = 50f^{0.2}$.

NewVocabulary

rational exponent
cube root
*n*th root
exponential equation

1 Rational Exponents You know that an exponent represents the number of times that the base is used as a factor. But how do you evaluate an expression with an exponent that is not an integer like the one above? Let's investigate **rational exponents** by assuming that they behave like integer exponents.

$$\left(b^{\frac{1}{2}}\right)^2 = b^{\frac{1}{2}} \cdot b^{\frac{1}{2}} \qquad \text{Write as a multiplication expression.}$$
$$= b^{\frac{1}{2}+\frac{1}{2}} \qquad \text{Product of Powers}$$
$$= b^1 \text{ or } b \qquad \text{Simplify.}$$

Thus, $b^{\frac{1}{2}}$ is a number with a square equal to b. So $b^{\frac{1}{2}} = \sqrt{b}$.

KeyConcept $b^{\frac{1}{2}}$

Words	For any nonnegative real number b, $b^{\frac{1}{2}} = \sqrt{b}$.	
Examples	$16^{\frac{1}{2}} = \sqrt{16}$ or 4	$38^{\frac{1}{2}} = \sqrt{38}$

PT

Example 1 Radical and Exponential Forms

Write each expression in radical form, or write each radical in exponential form.

a. $25^{\frac{1}{2}}$

$25^{\frac{1}{2}} = \sqrt{25}$ Definition of $b^{\frac{1}{2}}$

$= 5$ Simplify.

b. $\sqrt{18}$

$\sqrt{18} = 18^{\frac{1}{2}}$ Definition of $b^{\frac{1}{2}}$

c. $5x^{\frac{1}{2}}$

$5x^{\frac{1}{2}} = 5\sqrt{x}$ Definition of $b^{\frac{1}{2}}$

d. $\sqrt{8p}$

$\sqrt{8p} = (8p)^{\frac{1}{2}}$ Definition of $b^{\frac{1}{2}}$

GuidedPractice

1A. $a^{\frac{1}{2}}$ $\sqrt{a}$

1B. $\sqrt{22}$ $22^{\frac{1}{2}}$

1C. $(7w)^{\frac{1}{2}}$ $\sqrt{7w}$

1D. $2\sqrt{x}$ $2x^{\frac{1}{2}}$

Ariel Skelley/Blend Images/Getty Images

Common Core State Standards
N.RN.1, N.RN.2, A.SSE.2, A.REI.3

You know that to find the square root of a number a you find a number with a square of a. In the same way, you can find other roots of numbers. If $a^3 = b$, then a is the **cube root** of b, and if $a^n = b$ for a positive integer n, then a is an **nth root** of b.

KeyConcept *n*th Root

Words	For any real numbers a and b and any positive integer n, if $a^n = b$, then a is an nth root of b.
Symbols	If $a^n = b$, then $\sqrt[n]{b} = a$.
Example	Because $2^4 = 16$, 2 is a fourth root of 16; $\sqrt[4]{16} = 2$.

StudyTip

Graphing Calculator You can use a graphing calculator to find nth roots. Enter n, then press MATH and choose $\sqrt[x]{\ }$.

Since $3^2 = 9$ and $(-3)^2 = 9$, both 3 and -3 are square roots of 9. Similarly, since $2^4 = 16$ and $(-2)^4 = 16$, both 2 and -2 are fourth roots of 16. The positive roots are called *principal roots*. Radical symbols indicate principal roots, so $\sqrt[4]{16} = 2$.

Example 2 *n*th roots

Simplify.

a. $\sqrt[3]{27}$

$$\sqrt[3]{27} = \sqrt[3]{3 \cdot 3 \cdot 3}$$
$$= 3$$

b. $\sqrt[5]{32}$

$$\sqrt[5]{32} = \sqrt[5]{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2}$$
$$= 2$$

GuidedPractice

2A. $\sqrt[3]{64}$ **4**

2B. $\sqrt[4]{10{,}000}$ **10**

Like square roots, nth roots can be represented by rational exponents.

$$\left(b^{\frac{1}{n}}\right)^n = \underbrace{b^{\frac{1}{n}} \cdot b^{\frac{1}{n}} \cdot \ldots \cdot b^{\frac{1}{n}}}_{n \text{ factors}} \qquad \text{Write as a multiplication expression.}$$

$$= b^{\frac{1}{n} + \frac{1}{n} + \cdots + \frac{1}{n}} \qquad \text{Product of Powers}$$

$$= b^1 \text{ or } b \qquad \text{Simplify.}$$

Thus, $b^{\frac{1}{n}}$ is a number with an nth power equal to b. So $b^{\frac{1}{n}} = \sqrt[n]{b}$.

KeyConcept $b^{\frac{1}{n}}$

Words	For any positive real number b and any integer $n > 1$, $b^{\frac{1}{n}} = \sqrt[n]{b}$.
Example	$8^{\frac{1}{3}} = \sqrt[3]{8} = \sqrt[3]{2 \cdot 2 \cdot 2}$ or 2

1 Rational Exponents

Example 1 shows how to write an expression with a rational exponent in radical form and how to write an expression in radical form as an expression with a rational exponent. **Example 2** shows how to evaluate nth roots. **Example 3** shows how to evaluate expressions of the form $b^{\frac{1}{n}}$. **Example 4** shows how to evaluate expressions of the form $b^{\frac{m}{n}}$.

Formative Assessment

Use the Guided Practice exercises after each example to determine students' understanding of concepts.

Additional Examples

1 Write each expression in radical form, or write each radical in exponential form.

a. $81^{\frac{1}{2}}$ 9

b. $\sqrt{38}$ $38^{\frac{1}{2}}$

c. $12m^{\frac{1}{2}}$ $12\sqrt{m}$

d. $\sqrt{32w}$ $(32w)^{\frac{1}{2}}$

2 Simplify.

a. $\sqrt[4]{256}$ 4

b. $\sqrt[6]{15{,}625}$ 5

Additional Examples also in Interactive Classroom PowerPoint® Presentations

Interactive White Board READY

Tips for New Teachers

Order of Operations Remind students to observe the order of operations with radical and exponential expressions. Emphasize that expressions like $5x^{\frac{1}{2}}$ and $(5x)^{\frac{1}{2}}$ are different.

Notation Emphasize that if there is no index on a radical symbol, it denotes a square root.

Additional Examples

3 Simplify.

a. $1331^{\frac{1}{3}}$ 11

b. $2401^{\frac{1}{4}}$ 7

4 Simplify.

a. $32^{\frac{2}{5}}$ 4

b. $81^{\frac{5}{2}}$ 59,049

Focus on Mathematical Content

Rational Exponents The meaning of a rational exponent follows from extending the properties of integer exponents to those values. This allows for a notation for radicals in terms of rational exponents. For example, we define $2^{\frac{1}{2}}$ to be the square root of 2 because we want $\left(2^{\frac{1}{2}}\right)^2 = 2^{\frac{1}{2}\cdot 2} = 2^1$ to hold, so $\left(2^{\frac{1}{2}}\right)^2$ must equal 2.

StudyTip

Rational Exponents on a Calculator Use parentheses to evaluate expressions involving rational exponents on a graphing calculator. For example to find $125^{\frac{1}{3}}$, press 125 [^] [(] 1 [÷] 3 [)] [ENTER].

Example 3 Evaluate $b^{\frac{1}{n}}$ Expressions

Simplify.

a. $125^{\frac{1}{3}}$

$$\begin{aligned} 125^{\frac{1}{3}} &= \sqrt[3]{125} && b^{\frac{1}{n}} = \sqrt[n]{b} \\ &= \sqrt[3]{5 \cdot 5 \cdot 5} && 125 = 5^3 \\ &= 5 && \text{Simplify.} \end{aligned}$$

b. $1296^{\frac{1}{4}}$

$$\begin{aligned} 1296^{\frac{1}{4}} &= \sqrt[4]{1296} && b^{\frac{1}{n}} = \sqrt[n]{b} \\ &= \sqrt[4]{6 \cdot 6 \cdot 6 \cdot 6} && 1296 = 6^4 \\ &= 6 && \text{Simplify.} \end{aligned}$$

GuidedPractice

3A. $27^{\frac{1}{3}}$ **3**

3B. $256^{\frac{1}{4}}$ **4**

The Power of a Power property allows us to extend the definition of $b^{\frac{1}{n}}$ to $b^{\frac{m}{n}}$.

$$\begin{aligned} b^{\frac{m}{n}} &= \left(b^{\frac{1}{n}}\right)^m && \text{Power of a Power} \\ &= \left(\sqrt[n]{b}\right)^m \text{ or } \sqrt[n]{b^m} && b^{\frac{1}{n}} = \sqrt[n]{b} \end{aligned}$$

KeyConcept $b^{\frac{m}{n}}$

Words	For any positive real number b and any integers m and $n > 1$, $b^{\frac{m}{n}} = \left(\sqrt[n]{b}\right)^m$ or $\sqrt[n]{b^m}$.
Example	$8^{\frac{2}{3}} = \left(\sqrt[3]{8}\right)^2 = 2^2$ or 4

Example 4 Evaluate $b^{\frac{m}{n}}$ Expressions

Simplify.

a. $64^{\frac{2}{3}}$

$$\begin{aligned} 64^{\frac{2}{3}} &= \left(\sqrt[3]{64}\right)^2 && b^{\frac{m}{n}} = \left(\sqrt[n]{b}\right)^m \\ &= \left(\sqrt[3]{4 \cdot 4 \cdot 4}\right)^2 && 64 = 4^3 \\ &= 4^2 \text{ or } 16 && \text{Simplify.} \end{aligned}$$

b. $36^{\frac{3}{2}}$

$$\begin{aligned} 36^{\frac{3}{2}} &= \left(\sqrt[2]{36}\right)^3 && b^{\frac{m}{n}} = \left(\sqrt[n]{b}\right)^m \\ &= 6^3 && \sqrt{36} = 6 \\ &= 216 && \text{Simplify.} \end{aligned}$$

GuidedPractice

4A. $27^{\frac{2}{3}}$ **9**

4B. $256^{\frac{5}{4}}$ **1024**

DifferentiatedInstruction AL OL BL ELL

Interpersonal Learners Divide the class into groups of two or three students. Have students discuss what they knew about exponents before starting the lesson and how it relates to rational exponents. For example, if an exponent tells you how many times to use the base as a factor, what does an exponent of $\frac{3}{2}$ represent?

2 Solve Exponential Equations

In an **exponential equation**, variables occur as exponents. The Power Property of Equality and the other properties of exponents can be used to solve exponential equations.

KeyConcept Power Property of Equality

Words For any real number $b > 0$ and $b \neq 1$, $b^x = b^y$ if and only if $x = y$.

Examples If $5^x = 5^3$, then $x = 3$. If $n = \frac{1}{2}$, then $4^n = 4^{\frac{1}{2}}$.

Example 5 Solve Exponential Equations

Solve each equation.

a. $6^x = 216$

$6^x = 216$ Original equation

$6^x = 6^3$ Rewrite 216 as 6^3.

$x = 3$ Property of Equality

CHECK $6^x = 216$

$6^3 \overset{?}{=} 216$

$216 = 216$ ✓

b. $25^{x-1} = 5$

$25^{x-1} = 5$ Original equation

$(5^2)^{x-1} = 5$ Rewrite 9 as 3^2.

$5^{2x-2} = 5^1$ Power of a Power, Distributive Property

$2x - 2 = 1$ Power Property of Equality

$2x = 3$ Add 2 to each side.

$x = \frac{3}{2}$ Divide each side by 2.

CHECK $25^{x-1} = 5$

$25^{\frac{3}{2}-1} \overset{?}{=} 5$

$25^{\frac{1}{2}} = 5$ ✓

GuidedPractice

5A. $5^x = 125$ **3**

5B. $12^{2x+3} = 144$ $-\frac{1}{2}$

Real-World Example 6 Solve Exponential Equations

SUNSCREEN **Refer to the beginning of the lesson. Find the SPF that absorbs 100% of UV-B rays.**

$p = 50f^{0.2}$ Original equation

$100 = 50f^{0.2}$ $p = 100$

$2 = f^{0.2}$ Divide each side by 50.

$2 = f^{\frac{1}{5}}$ $0.2 = \frac{1}{5}$

$(2^5)^{\frac{1}{5}} = f^{\frac{1}{5}}$ $2 = 2^1 = (2^5)^{\frac{1}{5}}$

$2^5 = f$ Power Property of Equality

$32 = f$ Simplify.

GuidedPractice

6. **CHEMISTRY** The radius r of the nucleus of an atom of mass number A is $r = 1.2A^{\frac{1}{3}}$ femtometers. Find A if $r = 3.6$ femtometers. **27**

Real-WorldLink

Use extra caution near snow, water, and sand because they reflect the damaging rays of the Sun, which can increase your chance of sunburn.

Source: American Academy of Dermatology

Ryan McVay/Digital Vision/Getty Images

2 Exponential Equations

Example 5 shows how to solve an exponential equation by applying the Power Property of Equality. **Example 6** shows how to solve a real-world exponential equation.

Additional Examples

5 Solve each equation.

a. $9^x = 729$ 3

b. $16^{2x-1} = 8$ $\frac{7}{8}$

6 **BIOLOGY** The population p of a culture that begins with 40 bacteria and doubles every 8 hours can be modeled by $p = 40(2)^{\frac{t}{8}}$, where t is time in hours. Find t if $p = 20{,}480$. 72 hours

3 Practice

Formative Assessment

Use Exercises 1–16 to check for understanding.

Use the chart at the bottom of this page to customize assignments for your students.

Check Your Understanding

Example 1 Write each expression in radical form, or write each radical in exponential form.

1. $12^{\frac{1}{2}}$ $\sqrt{12}$
2. $3x^{\frac{1}{2}}$ $3\sqrt{x}$
3. $\sqrt{33}$ $33^{\frac{1}{2}}$
4. $\sqrt{8n}$ $(8n)^{\frac{1}{2}}$

Examples 2–4 Simplify.

5. $\sqrt[3]{512}$ 8
6. $\sqrt[5]{243}$ 3
7. $343^{\frac{1}{3}}$ 7
8. $\left(\frac{1}{16}\right)^{\frac{1}{4}}$ $\frac{1}{2}$
9. $343^{\frac{2}{3}}$ 49
10. $81^{\frac{3}{4}}$ 27
11. $216^{\frac{4}{3}}$ 1296
12. $\left(\frac{1}{49}\right)^{\frac{3}{2}}$ $\frac{1}{343}$

Example 5 Solve each equation.

13. $8^x = 4096$ 4
14. $3^{3x+1} = 81$ 1
15. $4^{x-3} = 32$ 5.5

Example 6

16. **ECOLOGY** A weir is used to measure water flow in a channel. For a rectangular broad crested weir, the flow Q in cubic feet per second is related to the weir length L in feet and height H of the water by $Q = 1.6LH^{\frac{3}{2}}$. Find the water height for a weir that is 3 feet long and has flow of 38.4 cubic feet per second. **4 ft**

Practice and Problem Solving

Example 1 Write each expression in radical form, or write each radical in exponential form.

17. $15^{\frac{1}{2}}$ $\sqrt{15}$
18. $24^{\frac{1}{2}}$ $\sqrt{24}$
19. $4k^{\frac{1}{2}}$ $4\sqrt{k}$
20. $(12y)^{\frac{1}{2}}$ $\sqrt{12y}$
21. $\sqrt{26}$ $26^{\frac{1}{2}}$
22. $\sqrt{44}$ $44^{\frac{1}{2}}$
23. $2\sqrt{ab}$ $2(ab)^{\frac{1}{2}}$
24. $\sqrt{3xyz}$ $(3xyz)^{\frac{1}{2}}$

Examples 2–4 Simplify.

25. $\sqrt[3]{8}$ 2
26. $\sqrt[5]{1024}$ 4
27. $\sqrt[3]{216}$ 6
28. $\sqrt[4]{10{,}000}$ 10
29. $\sqrt[3]{0.001}$ 0.1
30. $\sqrt[4]{\frac{16}{81}}$ $\frac{2}{3}$
31. $1331^{\frac{1}{3}}$ 11
32. $64^{\frac{1}{6}}$ 2
33. $3375^{\frac{1}{3}}$ 15
34. $512^{\frac{1}{9}}$ 2
35. $\left(\frac{1}{81}\right)^{\frac{1}{4}}$ $\frac{1}{3}$
36. $\left(\frac{3125}{32}\right)^{\frac{1}{5}}$ $\frac{5}{2}$
37. $8^{\frac{2}{3}}$ 4
38. $625^{\frac{3}{4}}$ 125
39. $729^{\frac{5}{6}}$ 243
40. $256^{\frac{3}{8}}$ 8
41. $125^{\frac{4}{3}}$ 625
42. $49^{\frac{5}{2}}$ $16{,}807$
43. $\left(\frac{9}{100}\right)^{\frac{3}{2}}$ $\frac{27}{1000}$
44. $\left(\frac{8}{125}\right)^{\frac{4}{3}}$ $\frac{16}{625}$

Design Pics/The Irish Image Collection/Getty Images

Differentiated Homework Options

Level	Assignment	Two-Day Option	
AL Basic	17–58, 89, 90, 92, 93	17–57 odd	18-58 even, 89, 90, 92, 93
OL Core	17–83 odd, 85–90, 92, 93	17–58	59–90, 92, 93
BL Advanced	59–93		

Example 5 **Solve each equation.**

45. $3^x = 243$ **5**
46. $12^x = 144$ **2**
47. $16^x = 4$ $\frac{1}{2}$
48. $27^x = 3$ $\frac{1}{3}$
49. $9^x = 27$ $\frac{3}{2}$
50. $32^x = 4$ $\frac{2}{5}$
51. $2^{x-1} = 128$ **8**
52. $4^{2x+1} = 1024$ **2**
53. $6^{x-4} = 1296$ **8**
54. $9^{2x+3} = 2187$ $\frac{1}{4}$
55. $4^{3x} = 512$ $\frac{3}{2}$
56. $128^{3x} = 8$ $\frac{1}{7}$

Example 6 **57. CONSERVATION** Water collected in a rain barrel can be used to water plants and reduce city water use. Water flowing from an open rain barrel has velocity $v = 8h^{\frac{1}{2}}$, where v is in feet per second and h is the height of the water in feet. Find the height of the water if it is flowing at 16 feet per second. **4 ft**

58. ELECTRICITY The radius r in millimeters of a platinum wire L centimeters long with resistance 0.1 ohm is $r = 0.059L^{\frac{1}{2}}$. How long is a wire with radius 0.236 millimeters? **16 cm**

B **Write each expression in radical form, or write each radical in exponential form.**

59. $17^{\frac{1}{3}}$ $\sqrt[3]{17}$
60. $q^{\frac{1}{4}}$ $\sqrt[4]{q}$
61. $7b^{\frac{1}{3}}$ $7\sqrt[3]{b}$
62. $m^{\frac{2}{3}}$ $\sqrt[3]{m^2}$
63. $\sqrt[3]{29}$ $29^{\frac{1}{3}}$
64. $\sqrt[5]{h}$ $h^{\frac{1}{5}}$
65. $2\sqrt[3]{a}$ $2a^{\frac{1}{3}}$
66. $\sqrt[3]{xy^2}$ $x^{\frac{1}{3}}y^{\frac{2}{3}}$

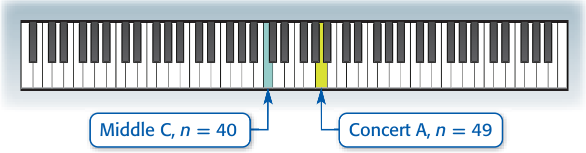
Simplify.

67. $\sqrt[3]{0.027}$ **0.3**
68. $\sqrt[4]{\frac{n^4}{16}}$ $\frac{n}{2}$
69. $a^{\frac{1}{3}} \cdot a^{\frac{2}{3}}$ a
70. $c^{\frac{1}{2}} \cdot c^{\frac{3}{2}}$ c^2
71. $(8^2)^{\frac{2}{3}}$ **16**
72. $\left(y^{\frac{3}{4}}\right)^{\frac{1}{2}}$ $y^{\frac{3}{8}}$
73. $9^{-\frac{1}{2}}$ $\frac{1}{3}$
74. $16^{-\frac{3}{2}}$ $\frac{1}{64}$
75. $(3^2)^{-\frac{3}{2}}$ $\frac{1}{27}$
76. $\left(81^{\frac{1}{4}}\right)^{-2}$ $\frac{1}{9}$
77. $k^{-\frac{1}{2}}$ $\frac{1}{\sqrt{k}}$
78. $\left(d^{\frac{4}{3}}\right)^0$ **1**

Solve each equation.

79. $2^{5x} = 8^{2x-4}$ **12**
80. $81^{2x-3} = 9^{x+3}$ **3**
81. $2^{4x} = 32^{x+1}$ -5
82. $16^x = \frac{1}{2}$ $-\frac{1}{4}$
83. $25^x = \frac{1}{125}$ $-\frac{3}{2}$
84. $6^{8-x} = \frac{1}{216}$ **11**

85. MUSIC The frequency f in hertz of the nth key on a piano is $f = 440\left(2^{\frac{1}{12}}\right)^{n-49}$.



a. What is the frequency of Concert A? **440 Hz**
b. Which note has a frequency of 220 Hz? **the A below middle C, the 37th note**

Alan Buckingham/Dorling Kindersley/Getty Images

Multiple Representations

In Exercise 88, students use a table and a graph to explore the graph of an exponential function.

WatchOut!

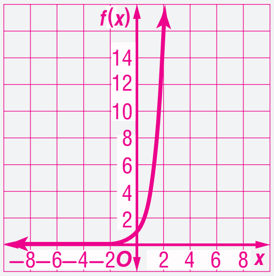
Error Analysis In Exercise 92, students should recognize that the bases of the expressions must have equal bases to apply the Power Property of Equality.

4 Assess

Ticket Out the Door Have each student an exponential expression and an equivalent radical expression.

Additional Answers

88b.



88c. The graph of $f(x) = 4^x$ is a curve. It has no x-intercept, a y-intercept of 1, the domain is all reals, the range is all positive reals, it is increasing over the entire domain, as x approaches infinity $f(x)$ approaches infinity, as x approaches negative infinity $f(x)$ approaches 0.

89. Sample answer: $2^{\frac{1}{2}}$ and $4^{\frac{1}{4}}$

90a. sometimes; true only when $x = 1$

90b. sometimes; true only when $x = 1$

90c sometimes; true only when $x = 1$

90d. always; by definition of $x^{\frac{1}{2}}$

90e. always; $\left(x^{\frac{1}{2}}\right)^2 = x^{\frac{1}{2} \cdot 2} = x^1$ or x

90f. sometimes; true only when $x = 1$

86. RANDOM WALKS Suppose you go on a walk where you choose the direction of each step at random. The path of a molecule in a liquid or a gas, the path of a foraging animal, and a fluctuating stock price are all modeled as random walks. The number of possible random walks w of n steps where you choose one of d directions at each step is $w = d^n$.

a. How many steps have been taken in a 2-direction random walk if there are 4096 possible walks? **12**

b. How many steps have been taken in a 4-direction random walk if there are 65,536 possible walks? **8**

c. If a walk of 7 steps has 2187 possible walks, how many directions could be taken at each step? **3**

87. SOCCER The radius r of a ball that holds V cubic units of air is modeled by $r = 0.62V^{\frac{1}{3}}$. What are the possible volumes of each size soccer ball? **Size 3, 204.0 to 230.2 in^3; Size 4, 268.5 to 299.9 in^3; Size 5, 333.6 to 382.4 in^3**

Soccer Ball Dimensions

Size	Diameter (in.)
3	7.3–7.6
4	8.0–8.3
5	8.6–9.0

88. MULTIPLE REPRESENTATIONS In this problem, you will explore the graph of an exponential function.

a. TABULAR Copy and complete the table below.

x	-2	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2
$f(x) = 4^x$	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8	16

b. GRAPHICAL Graph $f(x)$ by plotting the points and connecting them with a smooth curve. **b–c. See margin.**

c. VERBAL Describe the shape of the graph of $f(x)$. What are its key features? Is it linear?

H.O.T. Problems Use Higher-Order Thinking Skills

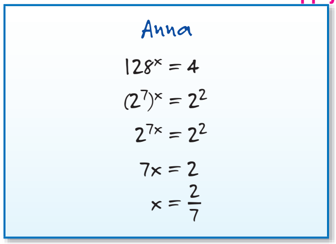
89. OPEN ENDED Write two different expressions with rational exponents equal to $\sqrt{2}$. **See margin.**

90. REASONING Determine whether each statement is *always*, *sometimes*, or *never* true. Assume that x is a nonnegative real number. Explain your reasoning. **a–f. See margin.**

a. $x^2 = x^{\frac{1}{2}}$
b. $x^{-2} = x^{\frac{1}{2}}$
c. $x^{\frac{1}{3}} = x^{\frac{1}{2}}$
d. $\sqrt{x} = x^{\frac{1}{2}}$
e. $\left(x^{\frac{1}{2}}\right)^2 = x$
f. $x^{\frac{1}{2}} \cdot x^2 = x$

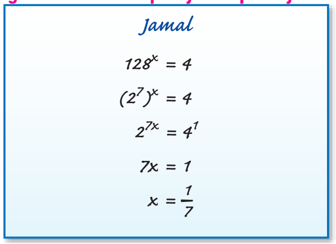
91. CHALLENGE For what values of x is $x = x^{\frac{1}{3}}$? **−1, 0, 1**

92. ERROR ANALYSIS Anna and Jamal are solving $128^x = 4$. Is either of them correct? Explain your reasoning. **Anna; Jamal did not write the expressions with equal bases before applying the Power Property of Equality.**



Anna

$128^x = 4$
$(2^7)^x = 2^2$
$2^{7x} = 2^2$
$7x = 2$
$x = \frac{2}{7}$



Jamal

$128^x = 4$
$(2^7)^x = 4$
$2^{7x} = 4^1$
$7x = 1$
$x = \frac{1}{7}$

93. WRITING IN MATH Explain why 2 is the principal fourth root of 16. **Sample answer: 2 is the principal fourth root of 16 because 2 is positive and $2^4 = 16$.**

DifferentiatedInstruction BL

Extension For Exercise 86, students found the numbers of random walks of a given length of steps. Have students investigate the applications of random walks.

LAB 9

Graphing Technology Lab

Family of Quadratic Functions

You have studied the effects of changing parameters in the equations of linear and exponential functions. You can use a graphing calculator to analyze how changing the parameters of the equation of a quadratic function affects the graphs in the family of quadratic functions.

Common Core State Standards
F.IF.7a, F.BF.3

Activity 1 Change k in $y = a(x - h)^2 + k$

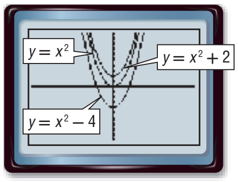
Graph the set of equations on the same screen in the standard viewing window. Describe any similarities and differences among the graphs.

$y = x^2$, $y = x^2 + 2$, $y = x^2 - 4$

Enter the equations in the **Y =** list and graph in the standard viewing window. Use the **ZOOM** feature to investigate the key features of the graphs.

The graphs have the same shape, and all open up. The vertex of each graph is on the y-axis, which is the axis of symmetry.

However, the graphs have different vertical positions. The graph of $y = x^2 + 2$ is shifted up 2 units. The graph of $y = x^2 - 4$ is shifted down 4 units.



[−10, 10] scl: 1 by [−10, 10] scl: 1

Changing the value of h in $y = a(x - h)^2 + k$ affects the graphs in a different way than changing k.

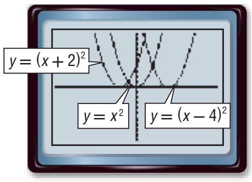
Activity 2 Change h in $y = a(x - h)^2 + k$

Graph the set of equations on the same screen in the standard viewing window. Describe any similarities and differences among the graphs.

$y = x^2$, $y = (x + 2)^2$, $y = (x - 4)^2$

The graphs have the same shape, and all open up. The vertex of each graph is on the x-axis.

However, the graphs have different horizontal positions. Each has a different axis of symmetry. The graph of $y = (x + 2)^2$ is shifted to the left 2 units. The graph of $y = (x - 4)^2$ is shifted to the right 4 units.



[−10, 10] scl: 1 by [−10, 10] scl: 1

It appears that changing the values of h and k in $y = a(x - h)^2 + k$ moves the graph vertically or horizontally. How does changing the value of a affect the graphs?

(continued on the next page)

1 Focus

Objective Use a graphing calculator to investigate families of quadratic functions.

Materials

- TI-83/84 Plus or other graphing calculator

Teaching Tips

- Encourage students to compare the key characteristics of quadratic functions to those of linear and exponential functions.
- Remind students to use ZOOM 6 to show graphs in the standard viewing window.
- In Activities 2 and 3, students will need to clear the Y= lists. To do this, they can use CLEAR.
- Use TRACE and ▲ or ▼ to display the equation for a graph.

2 Teach

Working in Cooperative Groups

Divide the class into pairs. Work through Activities 1–3. Then ask students to work with their partners to complete Exercises 1–3.

Practice Have students complete Exercises 4–11.

3 Assess

Formative Assessment

Use Exercise 11 to assess each student's ability to predict similarities and differences in quadratic functions.

From Concrete to Abstract

Ask students to summarize using technology to investigate families of quadratic functions.

Graphing Technology Lab
Family of Quadratic Functions *Continued*

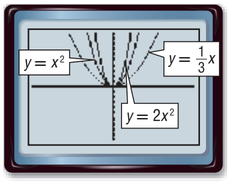
Activity 3 Change a in $y = a(x - h)^2 + k$

Graph each set of equations on the same screen in the standard viewing window. Describe any similarities and differences among the graphs.

a. $y = x^2, y = 2x^2, y = \frac{1}{3}x^2$

The graphs have the same vertex, they have the same axis of symmetry, and all open up.

However, the graphs have different widths. The graph of $y = 2x^2$ is narrower than the graph of $y = x^2$. The graph of $y = \frac{1}{3}x^2$ is wider than the graph of $y = x^2$.

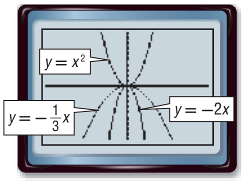


[−10, 10] scl: 1 by [−10, 10] scl: 1

b. $y = x^2, y = -\frac{1}{3}x^2, y = -2x^2$

The graphs have the same vertex and the same axis of symmetry.

However, the graphs of $y = -\frac{1}{3}x^2$ and $y = -2x^2$ open down. Also the graph of $y = -2x^2$ is narrower than the graph of $y = x^2$. The graph $y = -\frac{1}{3}x^2$ is wider than the graph of $y = x^2$.



[−10, 10] scl: 1 by [−10, 10] scl: 1

Model and Analyze

How does each parameter affect the graph of $y = a(x - h)^2 + k$? Give examples. **1–3. See margin.**

1. k

2. h

3. a

Examine each pair of equations and predict the similarities and differences in their graphs. Use a graphing calculator to confirm your predictions. Write a sentence or two comparing the two graphs. **4–11. See Answer Appendix.**

4. $y = x^2, y = x^2 + 3$

5. $y = \frac{1}{2}x^2, y = 3x^2$

6. $y = x^2, y = (x - 5)^2$

7. $y = 3x^2, y = -3x^2$

8. $y = x^2, y = -4x^2$

9. $y = x^2 - 1, y = x^2 + 2$

10. $y = \frac{1}{2}x^2 + 3, y = -2x^2$

11. $y = x^2 - 4, y = (x - 4)^2$

Additional Answers

1. The value of k affects the vertical position of the graph. Sample answer: The graph of $y = x^2 + 5$ is 5 units above the x-axis.
2. The value of h affects the horizontal position of the graph. Sample answer: The graph of $y = (x + 5)^2$ is 5 units to the left of the y-axis.
3. The value of a affects the width and direction of opening of the graph. If a is positive the graph opens up and if a is negative the graph opens down. Sample answer: The graph of $y = -2x^2$ is narrower than the graph of $y = x^2$; the graph of $y = -\frac{1}{2}x^2$ is wider than the graph of $y = x^2$; the graph of $y = x^2$ opens up and the graphs of $y = -x^2$, $y = -2x^2$, and $y = -\frac{1}{2}x^2$ open down.

LESSON 10 Transformations of Quadratic Functions

Then

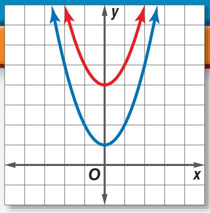
- You graphed quadratic functions by using the vertex and axis of symmetry.

Now

- 1 Apply translations to quadratic functions.
- 2 Apply dilations and reflections to quadratic functions.

Why?

- The graphs of the parabolas shown at the right are the same size and shape, but notice that the vertex of the red parabola is higher on the *y*-axis than the vertex of the blue parabola. Shifting a parabola up and down is an example of a transformation.



NewVocabulary

transformation
translation
dilation
reflection
vertex form

1 Translations

A **transformation** changes the position or size of a figure. One transformation, a **translation**, moves a figure up, down, left, or right. When a constant k is added to or subtracted from the parent function, the graph of the resulting function $f(x) \pm k$ is the graph of the parent function translated up or down.

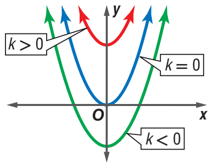
The parent function of the family of quadratics is $f(x) = x^2$. All other quadratic functions have graphs that are transformations of the graph of $f(x) = x^2$.

KeyConcept Vertical Translations

The graph of $f(x) = x^2 + k$ is the graph of $f(x) = x^2$ translated vertically.

If $k > 0$, the graph of $f(x) = x^2$ is translated $|k|$ units **up**.

If $k < 0$, the graph of $f(x) = x^2$ is translated $|k|$ units **down**.



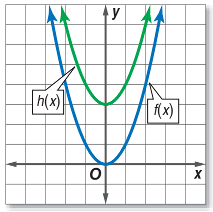
Example 1 Describe and Graph Translations

Describe how the graph of each function is related to the graph of $f(x) = x^2$.

a. $h(x) = x^2 + 3$

$k = 3$ and $3 > 0$

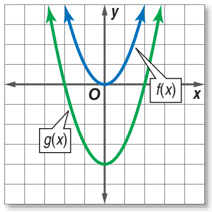
$h(x)$ is a translation of the graph of $f(x) = x^2$ up 3 units.



b. $g(x) = x^2 - 4$

$k = -4$ and $-4 < 0$

$g(x)$ is a translation of the graph of $f(x) = x^2$ down 4 units.



GuidedPractice

1A. $f(x) = x^2 - 7$ translated down 7
1B. $g(x) = 5 + x^2$ translated up 5
1C. $h(x) = -5 + x^2$ translated down 5
1D. $f(x) = x^2 + 1$ translated up 1

Common Core State Standards
A.SSE.3b, F.IF.7a, F.IF.8a, F.IF.9, F.BF.1b, F.BF.3

1 Focus

VerticalAlignment

Before Lesson 10 Graph quadratic functions by using the vertex and axis of symmetry.

Lesson 10 Apply translations to quadratic functions. Apply dilations and reflections to quadratic functions.

After Lesson 10 Write equations that model data.

2 Teach

Scaffolding Questions

Have students read the **Why?** section of the lesson.

Ask:

- What is the vertex of each parabola? blue: (0, 1); red: (0, 4)
- What do we know about the coefficient of the x^2 term for the equations of these graphs? They are both positive.
- What is the axis of symmetry for these two graphs? the *y*-axis
- What do we know about the *b* value in the equations for these graphs? Since $-\frac{b}{2a} = 0$, we know then that *b* must equal 0 for both.

1 Translations

Example 1 shows how to describe the graph of a quadratic function of the form $f(x) = x^2 + c$ based on its relationship to the parent function $f(x) = x^2$. **Example 2** shows how to describe the graph of a quadratic function of the form $f(x) = (x - h)^2$ based on its relationship to the parent function of $f(x) = x^2$. **Example 3** shows combined vertical and horizontal translations.

Formative Assessment

Use the Guided Practice exercises after each example to determine students' understanding of concepts.

Additional Examples

1 Describe how the graph of each function is related to the graph of $f(x) = x^2$.

a. $g(x) = 10 + x^2$ translated up 10 units

b. $g(x) = x^2 - 8$ translated down 8 units

2 Describe how the graph of each function is related to the graph of $f(x) = x^2$.

a. $g(x) = (x + 1)^2$ translated to left 1 unit

b. $g(x) = (x - 4)^2$ translated to right 4 units

3 Describe how the graph of each function is related to the graph of $f(x) = x^2$.

a. $g(x) = (x + 1)^2 + 1$ translated to left 1 unit and up 1 unit

b. $g(x) = (x - 2)^2 + 6$ translated right 2 units and up 6 units

Additional Examples also in Interactive Classroom PowerPoint® Presentations

Interactive White Board READY

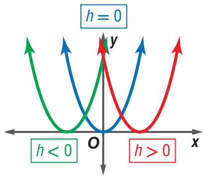
A quadratic graph can be translated horizontally by subtracting an h term from x.

KeyConcept Horizontal Translations

The graph of $g(x) = (x - h)^2$ is the graph of $f(x) = x^2$ translated horizontally.

If $h > 0$, the graph of $f(x) = x^2$ is translated h units to the **right**.

If $h < 0$, the graph of $f(x) = x^2$ is translated $|h|$ units to the **left**.



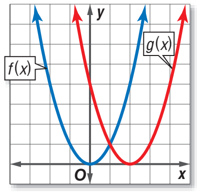
Example 2 Horizontal Translations

Describe how the graph of each function is related to the graph of $f(x) = x^2$.

a. $g(x) = (x - 2)^2$

$k = 0$, $h = 2$ and $2 > 0$

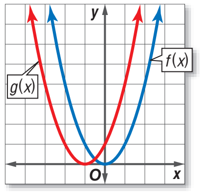
$g(x)$ is a translation of the graph of $f(x) = x^2$ to the right 2 units.



b. $g(x) = (x + 1)^2$

$k = 0$, $h = -1$ and $-1 < 0$

$g(x)$ is a translation of the graph of $f(x) = x^2$ to the left 1 unit.



GuidedPractice

2A. $g(x) = (x - 3)^2$ translated right 3

2B. $g(x) = (x + 2)^2$ translated left 2

A quadratic graph can be translated both horizontally and vertically.

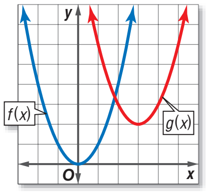
Example 3 Horizontal and Vertical Translations

Describe how the graph of each function is related to the graph of $f(x) = x^2$.

a. $g(x) = (x - 3)^2 + 2$

$k = 2$, $h = 3$ and $3 > 0$

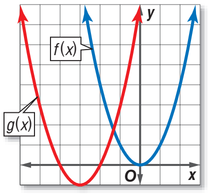
$g(x)$ is a translation of the graph of $f(x) = x^2$ to the right 3 units and up 2 units.



b. $g(x) = (x + 3)^2 - 1$

$k = -1$, $h = -3$ and $-3 < 0$

$g(x)$ is a translation of the graph of $f(x) = x^2$ to the left 3 units and down 1 unit.



GuidedPractice

3A. $g(x) = (x + 2)^2 + 3$

3B. $g(x) = (x - 4)^2 - 4$

3A. translated left 2 and up 3

3B. translated down 4 and left 4

2 Dilations and Reflections

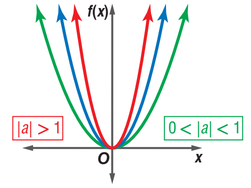
Another type of transformation is a dilation. A **dilation** makes the graph narrower than the parent graph or wider than the parent graph. When the parent function $f(x) = x^2$ is multiplied by a constant a, the graph of the resulting function $f(x) = ax^2$ is either stretched or compressed vertically.

KeyConcept Dilations

The graph of $g(x) = ax^2$ is the graph of $f(x) = x^2$ stretched or compressed vertically.

If $|a| > 1$, the graph of $f(x) = x^2$ is stretched vertically.

If $0 < |a| < 1$, the graph of $f(x) = x^2$ is compressed vertically.



StudyTip

Compress or Stretch When the graph of a quadratic function is stretched vertically, the shape of the graph is narrower than that of the parent function. When it is compressed vertically, the graph is wider than the parent function.

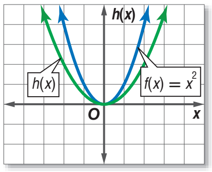
Example 4 Describe and Graph Dilations

Describe how the graph of each function is related to the graph of $f(x) = x^2$.

a. $h(x) = \frac{1}{2}x^2$

$a = \frac{1}{2}$ and $0 < \frac{1}{2} < 1$

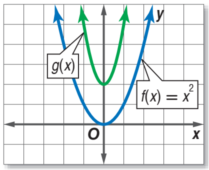
$h(x)$ is a dilation of the graph of $f(x) = x^2$ that is compressed vertically.



b. $g(x) = 3x^2 + 2$

$a = 3$ and $3 > 1$, $k = 2$ and $2 > 0$

$g(x)$ is a dilation of the graph of $f(x) = x^2$ that is stretched vertically and translated up 2 units.



GuidedPractice

4A. $j(x) = 2x^2$

4B. $h(x) = 5x^2 - 2$

4C. $g(x) = \frac{1}{3}x^2 + 2$

4A. stretched vertically

4B. stretched vertically and translated down

4C. compressed vertically and translated up

A **reflection** flips a figure across a line. When $f(x) = x^2$ or the variable x is multiplied by -1, the graph is reflected across the x- or y-axis.

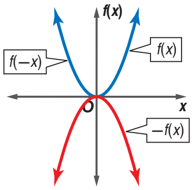
StudyTip

Reflection A reflection of $f(x) = x^2$ across the y-axis results in the same function, because $f(-x) = (-x)^2 = x^2$.

KeyConcept Reflections

The graph of $-f(x)$ is the reflection of the graph of $f(x) = x^2$ across the x-axis.

The graph of $f(-x)$ is the reflection of the graph of $f(x) = x^2$ across the y-axis.



2 Dilations and Reflections

Example 4 shows how to describe the graph of a quadratic function of the form $f(x) = ax^2$ based on its relationship to the parent function $f(x) = x^2$. **Example 5** shows how to describe the graph of a quadratic function of the form $f(x) = -ax^2 + k$ based on its relationship to the parent function $f(x) = x^2$. **Example 6** shows how to choose the equation that correlates to a function shown in a graph. **Example 7** shows how transformations are used in real-world situations.

Additional Example

4 Describe how the graph of each function is related to the graph of $f(x) = x^2$.

a. $d(x) = \frac{1}{3}x^2$ The graph of $y = \frac{1}{3}x^2$ is a vertical compression of the graph of $f(x) = x^2$.

b. $m(x) = 2x^2 + 1$ The graph of $y = 2x^2 + 1$ is stretched vertically and then translated up 1 unit.

DifferentiatedInstruction AL OL

If students need help comparing and contrasting a function's graph and its parent function's graph,

Then have students write and graph three different functions on the same coordinate plane. Using a red pencil, on this same coordinate plane, have students graph the parent function $f(x) = x^2$. Finally, have students create observation notebooks in which to record their thoughts on how each function's graph is similar to or different from the parent function's graph.

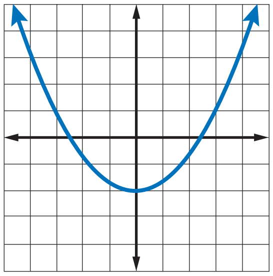
Additional Examples

5 Describe how the graph of each function is related to the graph of $f(x) = x^2$.

a. $g(x) = -3x^2 + 1$ The graph of $g(x) = -3x^2 + 1$ is reflected across the *x*-axis, stretched vertically, and translated up 1 unit.

b. $g(x) = \frac{1}{5}x^2 - 7$ The graph of $g(x) = \frac{1}{5}x^2 - 7$ is compressed vertically and translated down 7 units.

6 **STANDARDIZED TEST PRACTICE** Which is an equation for the function shown in the graph? A



A $y = \frac{1}{3}x^2 - 2$

B $y = 3x^2 + 2$

C $y = -\frac{1}{3}x^2 + 2$

D $y = -3x^2 - 2$

Focus on Mathematical Content

Quadratic Functions Each constant in the quadratic function $f(x) = a(x - h)^2 + k$ has meaning when compared to its parent function $f(x) = x^2$. The value of $|a|$ in the function either compresses or expands (dilates) the graph of the parent function. When the value of *a* is negative, the graph of the parent function is flipped (reflected) over the *x*-axis. The value of *k* in the function moves (translates) the graph of the parent function $|k|$ units up or down. The value of *h* moves the graph $|h|$ units left or right.

WatchOut!

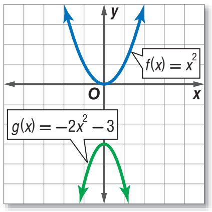
Transformations The graph of $f(x) = -ax^2$ can result in two transformations of the graph of $f(x) = x^2$: a reflection across the *x*-axis if $a > 0$ and either a compression or expansion depending on the absolute value of *a*.

Example 5 Describe and Graph Transformations

Describe how the graph of each function is related to the graph of $f(x) = x^2$.

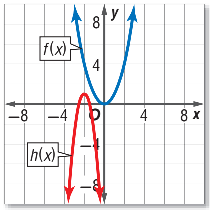
a. $g(x) = -2x^2 - 3$

- $a = -2$, $-2 < 0$, and $|-2| > 1$, so there is a reflection across the *x*-axis and the graph is vertically stretched.
- $k = -3$ and $-3 < 0$, so there is a translation down 3 units.



b. $h(x) = -4(x + 2)^2 + 1$

- $a = -4$, $-4 < 0$, and $|-4| > 1$, so there is a reflection across the *x*-axis and the graph is vertically stretched.
- $h = -2$ and $-2 < 0$, so there is a translation 2 units to the left.
- $k = 1$ and $1 > 0$, so there is a translation up 1 unit.



GuidedPractice

5A. $h(x) = 2(-x)^2 - 9$ **5B.** $g(x) = \frac{1}{5}x^2 + 3$ **5C.** $j(x) = -2(x - 1)^2 - 2$

5A. translated down 9 and stretched vertically

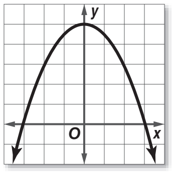
5B. translated up 3 and compressed vertically

5C. reflected over *x*-axis, translated down 2 and 1 right and stretched vertically

You can use what you know about the characteristics of graphs of quadratic equations to match an equation with a graph.

Standardized Test Example 6 Identify an Equation for a Graph

Which is an equation for the function shown in the graph?



A $y = \frac{1}{2}x^2 - 5$ **C** $y = -\frac{1}{2}x^2 + 5$

B $y = -2x^2 - 5$ **D** $y = 2x^2 + 5$

Read the Test Item

You are given a graph. You need to find its equation.

Solve the Test Item

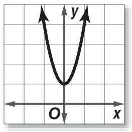
The graph opens downward, so the graph of $y = x^2$ has been reflected across the *x*-axis. The leading coefficient should be negative, so eliminate choices A and D.

The parabola is translated up 5 units, so $k = 5$. Look at the equations. Only choices C and D have $k = 5$. The answer is C.

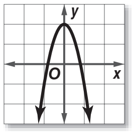
GuidedPractice

6. Which is the graph of $y = -3x^2 + 1$? H

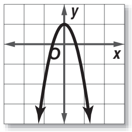
F



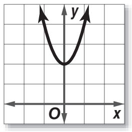
G



H



J



Teach with Tech

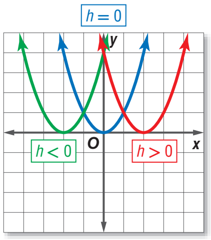
Interactive Whiteboard Drag a coordinate grid onto the board. Sketch the graph of a quadratic function on the board and give students the equation of the graph. Drag the parabola to another location on the board and have students find the new equation. Discuss similarities and differences.

A quadratic function written in the form $f(x) = a(x - h)^2 + k$ is said to be in **vertex form**. Transformations of the parent graph are easily found from an equation in vertex form.

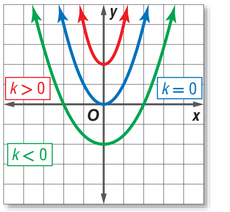
ConceptSummary Transformations of Quadratic Functions

$f(x) = a(x - h)^2 + k$

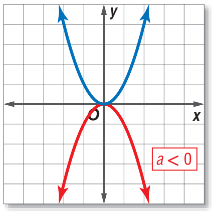
***h*, Horizontal Translation**
h units to the right if *h* is positive
|*h*| units to the left if *h* is negative



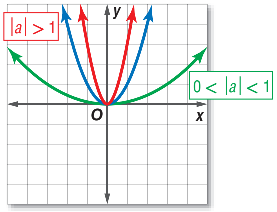
***k*, Vertical Translation**
k units up if *k* is positive
|*k*| units down if *k* is negative



***a*, Reflection**
If $a > 0$, the graph opens up.
If $a < 0$, the graph opens down.



***a*, Dilation**
If $|a| > 1$, the graph is stretched vertically. If $0 < |a| < 1$, the graph is compressed vertically.

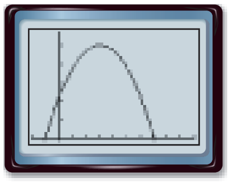


Real-World Example 7 Transformations with a Calculator

FIREWORKS During a firework show, the height *h* in meters of a specific rocket after *t* seconds can be modeled by $h(t) = -4.6(t - 3)^2 + 75$. Graph the function. How is it related to the graph of $f(x) = x^2$?

Four separate transformations are occurring.

The negative sign of the coefficient of x^2 causes a reflection across the x-axis. A dilation occurs, which compresses the graph vertically. There are also translations up 75 units and to the of right 3 units.

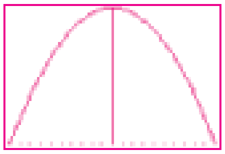


[−2, 10] scl: 1 by [−2, 85] scl: 15

GuidedPractice

7. MONUMENTS The St. Louis Arch resembles a quadratic and can be modeled by $h(x) = -\frac{2}{315}x^2 + 630$. Graph the function. How is it related to the graph of $f(x) = x^2$?

7.



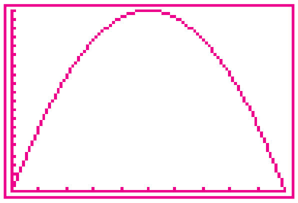
[−10, 10] scl: 1 by [629, 630] scl: 1

Three separate transformations are occurring. The negative sign of the coefficient of x^2 causes a reflection across the x-axis. Then a dilation occurs which stretches the graph vertically. Finally, there is a vertical translation of 630 units up.

Additional Example

7 CANNON During a 4th of July celebration, a cannon is fired. The flight of the cannon ball can be modeled by $d(t) = -4(x - 5)^2 + 100$, where *t* is time in seconds and $d(t)$ is distance in meters.

a. Graph the function.



[0, 10] scl: 1 by [0, 100] scl: 10

b. How is the graph of $d(t)$ related to the graph of $f(x) = x^2$? The graph is reflected over the x-axis, compressed vertically, vertically translated up 100, and horizontally translated right 5.

Tips for New Teachers

Width of the Parabola Students often mistake a greater value for |*a*| to imply a wider parabola. Suggest that students think of the value of *a* in $y = ax^2 + k$ the same way they thought of the value of the slope *m* in $y = mx + b$. A greater value for *m* means a steeper line. A greater value for |*a*| means that the sides of the parabola are steeper.

3 Practice

Formative Assessment

Use Exercises 1–7 to check for understanding.

Use the chart at the bottom of this page to customize assignments for your students.

Exercise Alert

Grid Paper For Exercise 24, students will need grid paper.

Additional Answers

1. translated down 11 units
2. translated right 2 units and compressed vertically
3. reflected across the *x*-axis, translated up 8 units
4. translated up 6 units
5. reflected across the *x*-axis, translated left 3 units and stretched vertically
6. reflected across the *x*-axis, translated down 2 units
8. translated down 10 units
9. reflected across the *x*-axis, translated down 7 units
10. translated right 3 units and up 8 units and stretched vertically
11. compressed vertically, translated up 6 units
12. reflected across the *x*-axis, stretched vertically, translated down 5 units
13. stretched vertically, translated up 3 units
14. compressed vertically, translated down 1.1 unit
15. translated left 1 unit and up 2.6 units and stretched vertically
16. compressed vertically, translated up $\frac{5}{6}$ unit
17. stretched vertically, translated down 6.5 units

Check Your Understanding

Examples 1–5, 7 Describe how the graph of each function is related to the graph of $f(x) = x^2$. **1–6. See margin.**

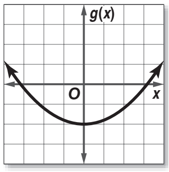
1. $g(x) = x^2 - 11$ **2.** $h(x) = \frac{1}{2}(x - 2)^2$ **3.** $h(x) = -x^2 + 8$

4. $g(x) = x^2 + 6$ **5.** $g(x) = -4(x + 3)^2$ **6.** $h(x) = -x^2 - 2$

Example 6 **7. MULTIPLE CHOICE** Which is an equation for the function shown in the graph? **C**

A $g(x) = \frac{1}{5}x^2 + 2$ C $g(x) = \frac{1}{5}x^2 - 2$

B $g(x) = -5x^2 - 2$ D $g(x) = -\frac{1}{5}x^2 - 2$



Practice and Problem Solving

Examples 1–5, 7 Describe how the graph of each function is related to the graph of $f(x) = x^2$. **8–17. See margin.**

8. $g(x) = -10 + x^2$ **9.** $h(x) = -7 - x^2$ **10.** $g(x) = 2(x - 3)^2 + 8$

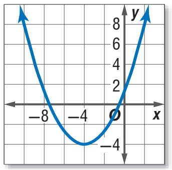
11. $h(x) = 6 + \frac{2}{3}x^2$ **12.** $g(x) = -5 - \frac{4}{3}x^2$ **13.** $h(x) = 3 + \frac{5}{2}x^2$

14. $g(x) = 0.25x^2 - 1.1$ **15.** $h(x) = 1.35(x + 1)^2 + 2.6$

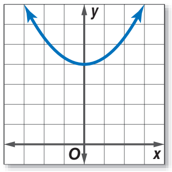
16. $g(x) = \frac{3}{4}x^2 + \frac{5}{6}$ **17.** $h(x) = 1.01x^2 - 6.5$

Example 6 Match each equation to its graph.

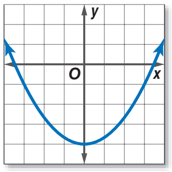
A



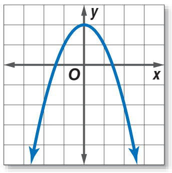
B



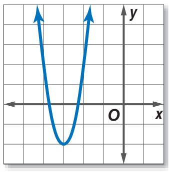
C



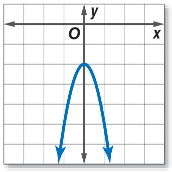
D



E



F



18. $y = \frac{1}{3}x^2 - 4$ **C** **19.** $y = \frac{1}{3}(x + 4)^2 - 4$ **A** **20.** $y = \frac{1}{3}x^2 + 4$ **B**

21. $y = -3x^2 - 2$ **F** **22.** $y = -x^2 + 2$ **D** **23.** $y = (2x + 6)^2 + 2$ **E**

B **24. SQUIRRELS** A squirrel 12 feet above the ground drops an acorn from a tree. The function $h = -16t^2 + 12$ models the height of the acorn above the ground in feet after t seconds. Graph the function, and compare this graph to the graph of its parent function. **See Answer Appendix for graph.**

List the functions in order from the most stretched vertically to the least stretched vertically graph. **27. *h*(*x*), *g*(*x*), *f*(*x*) 28. *f*(*x*), *h*(*x*), *g*(*x*)**

25. $g(x) = 2x^2, h(x) = \frac{1}{2}x^2$ **g(x), h(x)** **26.** $g(x) = -3x^2, h(x) = \frac{2}{3}x^2$ **g(x), h(x)**

27. $g(x) = -4x^2, h(x) = 6x^2, f(x) = 0.3x^2$ **28.** $g(x) = -x^2, h(x) = \frac{5}{3}x^2, f(x) = -4.5x^2$

Differentiated Homework Options

Level	Assignment	Two-Day Option	
AL Basic	8–23, 36, 38–40	9–23 odd	8–22 even, 36, 38–40
OL Core	9–23 odd, 24, 25, 27, 29–31, 33, 35, 36, 38–40	8–23	24–36, 38–40
BL Advanced	24–40		

29. **ROCKS** A rock drops from a cliff 20,000 inches above the ground. At the same time, another rock drops from a cliff 30,000 inches above the ground.

a. Write two functions that model the heights h of the rocks after t seconds. **a. $h = -16t^2 + 20{,}000$ and $h = -16t^2 + 30{,}000$**

b. Which rock will reach the ground first? **b. The rock from the 20,000-in. cliff will reach the ground first.**

30. **SPRINKLERS** The path of water from a sprinkler can be modeled by quadratic functions. The following functions model paths for three different sprinklers.

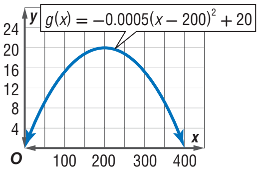
Sprinkler A: $y = -0.35x^2 + 3.5$
Sprinkler B: $y = -0.21x^2 + 1.7$
Sprinkler C: $y = -0.08x^2 + 2.4$

a. Which sprinkler will send water the farthest? Explain. **a. Sprinkler C because the graph is compressed vertically the most.**

b. Which sprinkler will send water the highest? Explain. **Sprinkler A because it is translated up the most.**

c. Which sprinkler will produce the narrowest path? Explain. **Sprinkler A because it is expanded the least.**

31. **GOLF** The path of a drive can be modeled by a quadratic function where $g(x)$ is the vertical distance in yards of the ball from the ground and x is the horizontal distance in yards.



a. How can you obtain $g(x)$ from the graph of $f(x) = x^2$.

b. A second golfer hits a ball from the red tee, which is 30 yards closer to the hole. What function $h(x)$ can be used to describe the second golfer's shot? **$h(x) = 0.0005(x - 230)^2 + 20$**

31a. The graph of $g(x)$ is the graph of $f(x)$ translated 200 yards right, compressed vertically, reflected in the x-axis, and translated up 20 yards.

Describe the transformations to obtain the graph of $g(x)$ from the graph of $f(x)$.

32. $f(x) = x^2 + 3$
$g(x) = x^2 - 2$

32. Translate the graph of $f(x)$ down 5 units.

33. $f(x) = x^2 - 4$
$g(x) = (x - 2)^2 + 7$

33. Translate the graph of $f(x)$ up 11 units and to the right 2 units.

34. $f(x) = -6x^2$
$g(x) = -3x^2$ **Compress vertically the graph of $f(x)$.**

35. **COMBINING FUNCTIONS** An engineer created a self-refueling generator that burns fuel according to the function $g(t) = -t^2 + 10t + 200$, where t represents the time in hours and $g(t)$ represents the number of gallons remaining.

a. How long will it take for the generator to run out of fuel? **20 h**

b. The engine self-refuels at a rate of 40 gallons per hour. Write a linear function $h(t)$ to represent the refueling of the generator. **$h(t) = 40t$**

c. Find $T(t) = g(t) + h(t)$. What does this new function represent? **35c. $T(t) = -t^2 + 50t + 200$; the fuel in the tank after t hours with refueling**

d. Will the generator run out of fuel? If so, when? **Yes; after about 53 hours 43 minutes.**

H.O.T. Problems Use Higher-Order Thinking Skills

36. **REASONING** Are the following statements *sometimes*, *always*, or *never* true? Explain. **See margin.**

a. The graph of $y = x^2 + k$ has its vertex at the origin.

b. The graphs of $y = ax^2$ and its reflection over the x-axis are the same width.

c. The graph of $y = x^2 + k$, where $k \geq 0$, and the graph of a quadratic with vertex at $(0, -3)$ have the same maximum or minimum point.

37. **CHALLENGE** Write a function of the form $y = ax^2 + k$ with a graph that passes through the points $(-2, 3)$ and $(4, 15)$. **$y = x^2 - 1$**

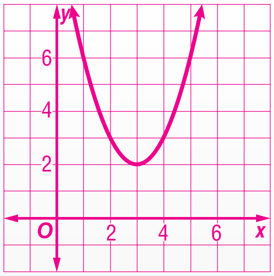
38. **REASONING** Determine whether all quadratic functions that are reflected across the y-axis produce the same graph. Explain your answer. **See margin.**

39. **OPEN ENDED** Write a quadratic function that opens downward and is wider than the parent graph. **Sample answer: $f(x) = -\frac{1}{2}x^2$**

40. **WRITING IN MATH** Describe how the values of a and k affect the graphical and tabular representations for the functions $y = ax^2$, $y = x^2 + k$, and $y = ax^2 + k$. **See margin.**

DifferentiatedInstruction OL BL

Extension All of the functions graphed and analyzed in this lesson had their vertex located on the y-axis. Ask students to graph $y = (x - 3)^2 + 2$ and then tell how the graph is related to the graph of $y = x^2$. It is a parabola that is moved 3 units to the right and 2 units up.



4 Assess

Ticket Out the Door Write five different quadratic functions of the form $f(x) = ax^2 + k$ on pieces of paper. Give each student one. Have students tell you how the graph of each function is related to the graph of $y = x^2$ as they walk out the door.

Additional Answers

36a. Sometimes; this only occurs if $k = 0$. For any other value, the graph will be translated up or down.

36b. Always; the reflection does not affect the width. Both graphs are dilated by a factor of a.

36c. Never; if the graph with a vertex at (0, -3) opens upward, it will have a lower minimum. If it opens downward, it will have a maximum. The first graph has a minimum.

38. Sample answer: Not all reflections over the y-axis produce the same graph. If the vertex of the original graph is not on the y-axis, the graph will not have the y-axis as its axis of symmetry and its reflection across the y-axis will be a different parabola.

40. Sample answer: For $y = ax^2$, the parent graph is stretched vertically if $a > 1$ or compressed vertically if $0 < a < 1$. The y-values in the table will all be multiplied by a factor of a. For $y = x^2 + k$, the parent graph is translated up if k is positive and moved down if k is negative. The y-values in the table will all have the constant k added to them or subtracted from them. For $y = ax^2 + k$, the graph will either be stretched vertically or compressed vertically based upon the value of a and then will be translated up or down depending on the value of k. The y-values in the table will be multiplied by a factor of a and the constant k added to them.

1 Focus

Objective

- Complete the square in a quadratic expression to find the maximum or minimum value of the related function.

2 Teach

Working in Cooperative Groups

Divide the class into pairs. Work through Activity 1 as a class. Then ask students to work with their partners to complete Activities 2 and 3.

Practice Have students complete Exercises 1–8 and 11.

LAB 11

Algebra Lab

Finding the Maximum or Minimum Value

In Lesson 9-3, we learned about the vertex form of the equation of a quadratic function. You will now learn how to write equations in vertex form and use them to identify key characteristics of the graphs of quadratic functions.

Common Core State Standards
A.SSE.3b, A.REI.4b, F.IF.8a

Activity 1 Find a Minimum

Write $y = x^2 + 4x - 10$ in vertex form. Identify the axis of symmetry, extrema, and zeros. Then graph the function.

Step 1 Complete the square to write the function in vertex form.

$y = x^2 + 4x - 10$ Original function

$y + 10 = x^2 + 4x$ Add 10 to each side.

$y + 10 + 4 = x^2 + 4x + 4$ Since $\left(\frac{4}{2}\right)^2 = 4$, add 4 to each side.

$y + 14 = (x + 2)^2$ Factor $x^2 + 4x + 4$.

$y = (x + 2)^2 - 14$ Subtract 14 from each side to write in vertex form.

Step 2 Identify the axis of symmetry and extrema based on the equation in vertex form. The vertex is at (h, k) or $(-2, -14)$. Since there is no negative sign before the x^2-term, the parabola opens up and has a minimum at $(-2, -14)$. The equation of the axis of symmetry is $x = -2$.

Step 3 Solve for x to find the zeros.

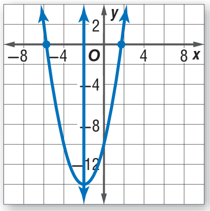
$(x + 2)^2 - 14 = 0$ Vertex form, $y = 0$

$(x + 2)^2 = 14$ Add 14 to each side.

$x + 2 = \pm\sqrt{14}$ Take square root of each side.

$x \approx -5.74 \text{ or } 1.74$ Subtract 2 from each side.

The zeros are approximately -5.74 and 1.74.



Step 4 Use the key features to graph the function.

There may be a negative coefficient before the quadratic term. When this is the case, the parabola will open down and have a maximum.

Activity 2 Find a Maximum

Write $y = -x^2 + 6x - 5$ in vertex form. Identify the axis of symmetry, extrema, and zeros. Then graph the function.

Step 1 Complete the square to write the equation of the function in vertex form.

$y = -x^2 + 6x - 5$ Original function

$y + 5 = -x^2 + 6x$ Add 5 to each side.

$y + 5 = -(x^2 - 6x)$ Factor out −1.

$y + 5 - 9 = -(x^2 - 6x + 9)$ Since $\left(\frac{6}{2}\right)^2 = 9$, add −9 to each side.

$y - 4 = -(x - 3)^2$ Factor $x^2 - 6x + 9$.

$y = -(x - 3)^2 + 4$ Add 4 to each side to write in vertex form.

Step 2 Identify the axis of symmetry and extrema based on the equation in vertex form. The vertex is at (h, k) or $(3, 4)$. Since there is a negative sign before the x^2-term, the parabola opens down and has a maximum at $(3, 4)$. The equation of the axis of symmetry is $x = 3$.

Step 3 Solve for x to find the zeros.

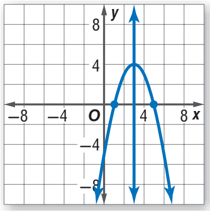
$0 = -(x - 3)^2 + 4$ Vertex form, $y = 0$

$(x - 3)^2 = 4$ Add $(x - 3)^2$ to each side.

$x - 3 = \pm 2$ Take the square root of each side.

$x = 5$ or 1 Add 3 to each.

Step 4 Use the key features to graph the function.



Analyze the Results

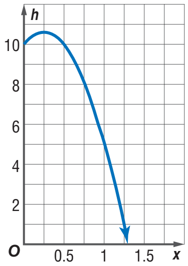
1. Why do you need to complete the square to write the equation of a quadratic function in vertex form? **See margin.**

Write each function in vertex form. Identify the axis of symmetry, extrema, and zeros. Then graph the function. **2–10. See Answer Appendix.**

2. $y = x^2 + 6x$
3. $y = x^2 - 8x + 6$
4. $y = x^2 + 2x - 12$
5. $y = x^2 + 6x + 8$
6. $y = x^2 - 4x + 3$
7. $y = x^2 - 2.4x - 2.2$
8. $y = -4x^2 + 16x - 11$
9. $y = 3x^2 - 12x + 5$
10. $y = -x^2 + 6x - 5$

Activity 3 Use Extrema in the Real World

DIVING **Alexis jumps from a diving platform upward and outward before diving into the pool. The function $h = -9.8t^2 + 4.9t + 10$, where h is the height of the diver in meters above the pool after t seconds approximates Alexis's dive. Graph the function, then find the maximum height that she reaches and the equation of the axis of symmetry.**



Step 1 Graph the function.

Step 2 Complete the square to write the eqution of the function in vertex form.

$h = -9.8t^2 + 4.9t + 10$

$h = -9.8(t - 0.25)^2 + 10.6125$

Step 3 The vertex is at $(0.25, 10.6125)$, so the maximum height is 10.6125 meters. The equation of the axis of symmetry is $x = 0.25$.

Exercise

11. **SOFTBALL** Jenna throws a ball in the air. The function $h = -16t^2 + 40t + 5$, where h is the height in feet and t represents the time in seconds, approximates Jenna's throw. Graph the function, then find the maximum height of the ball and the equation of the axis of symmetry. When does the ball hit the ground? **See margin.**

3 Assess

Formative Assessment

Use Exercises 9 and 10 to assess each student's knowledge of vertex form and finding zeros, the line of symmetry, and extrema.

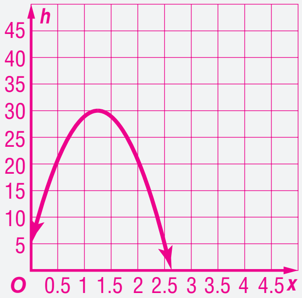
From Concrete to Abstract

Ask students to summarize how to write an equation in vertex form and find the zeros, the line of symmetry, and extrema.

Additional Answers

1. Sample answer: In vertex form, the x only appears once. You must use completing the square to create a perfect square trinomial, so that it can be factored and reduce the x-terms to one.

11.



30 ft; $t \approx 1.25$; 2.62 seconds

1 Focus

Objective Use a graphing calculator to investigate families of exponential functions.

Materials

- TI-83/84 Plus or other graphing calculator

Teaching Tips

- Remind students that to set their viewing window to standard, use ZOOM 6. Otherwise, use the window settings shown.
- For Activity 2, remind students that to enter functions like $y = \left(\frac{1}{2}\right)^x$, they will need to type (1 ÷ 2) ^ X,T,θ,*n* ENTER in the Y= list.

2 Teach

Working in Cooperative Groups

Divide the class into pairs. Have one student work through Activity 1 and the other work through Activity 2. Ask students to describe to each other what they see on the calculator screens. Have them switch activities and see if they notice any other similarities or differences.

Then ask students to work with their partners to complete Activities 3 and 4.

Practice Have students complete Exercises 1 and 2.

LAB 12

Graphing Technology Lab

Family of Exponential Functions

An **exponential function** is a function of the form $y = ab^x$, where $a \neq 0$, $b > 0$, and $b \neq 1$. You have studied the effects of changing parameters in linear functions. You can use a graphing calculator to analyze how changing the parameters *a* and *b* affects the graphs in the family of exponential functions.

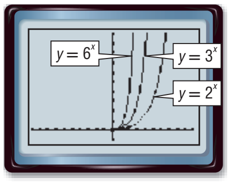
Common Core State Standards
F.IF.7e, F.BF.3

Activity 1 *b* in $y = b^x$, $b > 1$

Graph the set of equations on the same screen.
Describe any similarities and differences among the graphs.

$y = 2^x$, $y = 3^x$, $y = 6^x$

Enter the equations in the Y= list and graph.

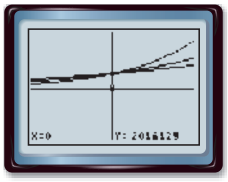


[−10, 10] scl: 1 by [−10, 100] scl: 10

There are many similarities in the graphs. The domain for each function is all real numbers, and the range is all positive real numbers. The functions are increasing over the entire domain. The graphs do not display any line symmetry.

Use the ZOOM feature to investigate the key features of the graphs.

Zooming in twice on a point near the origin allows closer inspection of the graphs. The *y*-intercept is 1 for all three graphs.



[−0.625, 0.625] scl: 1 by
[−3.25..., 3.63...] scl: 10

Tracing along the graphs reveals that there are no *x*-intercepts, no maxima and no minima.

The graphs are different in that the graphs for the equations in which *b* is greater are steeper.

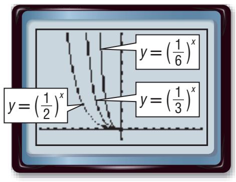
The effect of *b* on the graph is different when $0 < b < 1$.

Activity 2 *b* in $y = b^x$, $0 < b < 1$

Graph the set of equations on the same screen.
Describe any similarities and differences among the graphs.

$y = \left(\frac{1}{2}\right)^x$, $y = \left(\frac{1}{3}\right)^x$, $y = \left(\frac{1}{6}\right)^x$

The domain for each function is all real numbers, and the range is all positive real numbers. The function values are all positive and the functions are decreasing over the entire domain. The graphs display no line symmetry. There are no *x*-intercepts, and the *y*-intercept is 1 for all three graphs. There are no maxima or minima.



[−10, 10] scl: 1 by [−10, 100] scl: 10

However, the graphs in which *b* is lesser are steeper.

WatchOut!

Common Misconceptions Be sure students do not confuse polynomial functions and exponential functions. While $y = x^2$ and $y = 2^x$ each have an exponent, $y = x^2$ is a polynomial function, and $y = 2^x$ is an exponential function.

Activity 3 a in $y = ab^x$, $a > 0$

Graph each set of equations on the same screen. Describe any similarities and differences among the graphs.

$y = 2^x$, $y = 3(2^x)$, $y = \frac{1}{6}(2^x)$

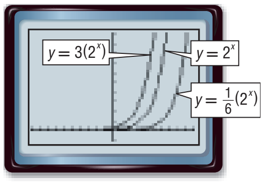
The domain for each function is all real numbers, and the range is all positive real numbers. The functions are increasing over the entire domain. The graphs do not display any line symmetry.

Use the ZOOM feature to investigate the key features of the graphs.

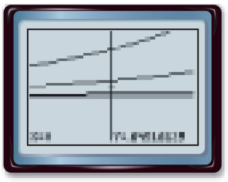
Zooming in twice on a point near the origin allows closer inspection of the graphs.

Tracing along the graphs reveals that there are no x-intercepts, no maxima and no minima.

However, the graphs in which a is greater are steeper. The y-intercept is 1 in the graph of $y = 2^x$, 3 in $y = 3(2^x)$, and $\frac{1}{6}$ in $y = \frac{1}{6}(2^x)$.



[−10, 10] scl: 1 by [−10, 100] scl: 10



[−0.625, 0.625] scl: 1 by
[−2.79..., 4.08...] scl: 10

Activity 4 a in $y = ab^x$, $a < 0$

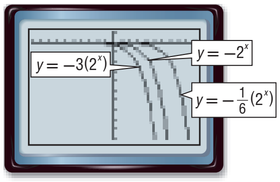
Graph each set of equations on the same screen. Describe any similarities and differences among the graphs.

$y = -2^x$, $y = -3(2^x)$, $y = -\frac{1}{6}(2^x)$

The domain for each function is all real numbers, and the range is all negative real numbers. The functions are decreasing over the entire domain. The graphs do not display any line symmetry.

There are no x-intercepts, no maxima and no minima.

However, the graphs in which the absolute value of a is greater are steeper. The y-intercept is −1 in the graph of $y = -2^x$, −3 in $y = -3(2^x)$, and $-\frac{1}{6}$ in $y = -\frac{1}{6}(2^x)$.



[−10, 10] scl: 1 by [−100, 10] scl: 10

Model and Analyze **1–3. See margin.**

1. How does b affect the graph of $y = ab^x$ when $b > 1$ and when $0 < b < 1$? Give examples.
2. How does a affect the graph of $y = ab^x$ when $a > 0$ and when $a < 0$? Give examples.
3. Make a conjecture about the relationship of the graphs of $y = 3^x$ and $y = \left(\frac{1}{3}\right)^x$. Verify your conjecture by graphing both functions.

3 Assess

Formative Assessment

Use Exercise 3 to assess each student's knowledge of graphing exponential functions.

From Concrete to Abstract

Ask students to summarize how to use technology to find the solutions to exponential equations and inequalities.

Additional Answers

1. The value of b affects the steepness of the graph. When $b > 1$, the greater the value of b the steeper the graph. When $0 < b < 1$, the lesser the value of b the steeper the graph. Sample answer: The graph of $y = 5^x$ is steeper than the graph of $y = 3^x$. The graph of $y = \left(\frac{1}{5}\right)^x$ is steeper than the graph of $y = \left(\frac{1}{3}\right)^x$.
2. The value of a affects the steepness and direction of opening of the graph. The greater the absolute value of a, the steeper the graph. When $a > 0$ the graph opens up, and when $a < 0$ the graph opens down. Sample answer: The graph of $y = 4(2^x)$ is steeper than the graph of $y = 3(2^x)$. The graph of $y = 4(2^x)$ opens up and the graph of $y = -4(2^x)$ opens down.
3. Sample answer: The graph of $y = \left(\frac{1}{3}\right)^x$ is the graph of $y = 3^x$ reflected over the y-axis.

1 Focus

Objective Use a graphing calculator to solve exponential equations and inequalities.

Materials

- TI-Nspire Technology

Teaching Tips

- For Activity 1, remind students that to enter $3^x + 4$, they will need to use the ^ key for the exponent and use the down arrow before entering the $+ 4$.
- When changing the windows settings, use the **tab** key to move from one field to another.
- In Activity 2, student will need to use the **tab** key to move the curser to the entry line to type **f2(*x*)**.

2 Teach

Working in Cooperative Groups

Divide the class into pairs. Work through Activity 1 and Activity 2 as a class. Then ask students to work with their partners to complete Exercises 1–9 and Activities 3 and 4.

Practice Have students complete Exercises 10–12.

LAB 13

Graphing Technology Lab

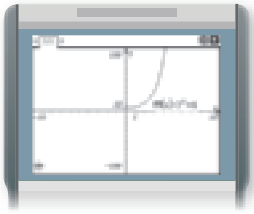
Solving Exponential Equations and Inequalities

You can use TI-Nspire Technology to solve exponential equations and inequalities by graphing and by using tables.

Common Core State Standards
A.REI.11

Activity 1 Graph an Exponential Equation

Graph $y = 3^x + 4$ using a graphing calculator.



Step 1 Add a new **Graphs** page.

Step 2 Enter $3^x + 4$ as **f1(*x*)**.

Step 3 Use the **Window Settings** option from the **Window/Zoom** menu to adjust the window so that x is from -10 to 10 and y is from -100 to 100. Keep the scales as **Auto**.

To solve an equation by graphing, graph both sides of the equation and locate the point(s) of intersection.

Activity 2 Solve an Exponential Equation by Graphing

Solve $2^{x-2} = \frac{3}{4}$.

Step 1 Add a new **Graphs** page.

Step 2 Enter 2^{x-2} as **f1(*x*)** and $\frac{3}{4}$ as **f2(*x*)**.

Step 3 Use the **Intersection Point(s)** tool from the **Points & Lines** menu to find the intersection of the two graphs. Select the graph of **f1(*x*)** **enter** and then the graph of **f2(*x*)** **enter**.

The graphs intersect at about (1.58, 0.75). Therefore, the solution of $2^{x-2} = \frac{3}{4}$ is 1.58.

Exercises

Use a graphing calculator to solve each equation.

1. $\left(\frac{1}{3}\right)^{x-1} = \frac{3}{4}$ **≈1.26**

2. $2^{2x-1} = 2x$ **0.5, 1**

3. $\left(\frac{1}{2}\right)^{2x} = 2^{2x}$ **0**

4. $5^{\frac{1}{3}x+2} = -x$ **≈−3.61**

5. $\left(\frac{1}{8}\right)^{2x} = -2x + 1$ **0, ≈0.409**

6. $2^{\frac{1}{4}x-1} = 3^{x+1}$ **≈−1.94**

7. $2^{3x-1} = 4^x$ **1**

8. $4^{2x-3} = 5^{-x+1}$ **≈1.32**

9. $3^{2x-4} = 2^x + 1$ **3**

Activity 3 Solve an Exponential Equation by Using a Table

Solve $2\left(\frac{1}{2}\right)^{x+2} = \frac{1}{4}$ using a table.

Step 1 Add a new **Lists & Spreadsheet** page.

Step 2 Label column A as x. Enter values from -4 to 4 in cells A1 to A9.

Step 3 In column B in the formula row, enter the left side of the rational equation. In column C of the formula row, enter $= \frac{1}{4}$. Specify **Variable Reference** when prompted.

Scroll until you see where the values in Columns B and C are equal. This occurs at $x = 1$. Therefore, the solution of $2\left(\frac{1}{2}\right)^{x+2} = \frac{1}{4}$ is 1.

You can also use a graphing calculator to solve exponential inequalities.

Activity 4 Solve an Exponential Inequality

Solve $4^{x-3} \le \left(\frac{1}{4}\right)^{2x}$.

Step 1 Add a new **Graphs** page.

Step 2 Enter the left side of the inequality into **f1(*x*)**. Press **del**, select $\ge$, and enter 4^{x-3}. Enter the right side of the inequality into **f2(*x*)**. Press **tab del** $\le$, and enter $\left(\frac{1}{4}\right)^{2x}$.

The x-values of the points in the region where the shading overlap is the solution set of the original inequality. Therefore the solution of $4^{x-3} \le \left(\frac{1}{4}\right)^{2x}$ is $x \le 1$.

Exercises

Use a graphing calculator to solve each equation or inequality.

10. $\left(\frac{1}{3}\right)^{3x} = 3^x$ **0**

11. $\left(\frac{1}{6}\right)^{2x} = 4^x$ **0**

12. $3^{1-x} \le 4^x$ **$x \ge 0.442$**

13. $4^{3x} \le 2x + 1$ **$-0.409 \le x \le 0$**

14. $\left(\frac{1}{4}\right)^x > 2^{x+4}$ **$x < -1.33$**

15. $\left(\frac{1}{3}\right)^{x-1} \ge 2^x$ **$x \le 0.613$**

3 Assess

Formative Assessment

Use Exercises 13–15 to assess each student’s knowledge of solving exponential equations and inequalities.

From Concrete to Abstract

Ask students to summarize the use of technology to find the solutions to exponential equations and inequalities.

1 Focus

Objective Use properties of rational exponents to transform expressions for exponential functions into equivalent forms to solve problems.

2 Teach

Working in Cooperative Groups

Organize students into groups of 2, mixing abilities. Then have groups complete the Activity and Exercises 1–3.

Teaching Tip

Point out to students that the annual interest formula is approximated as a monthly interest rate using $\frac{1}{12} \cdot 12$ because there are 12 months in a year.

Practice Have students complete Exercise 4.

3 Assess

Formative Assessment

Use Exercise 4 to assess whether students understand how to use the properties of exponents to write equivalent expressions in order to compare interest rates.

From Concrete to Abstract

Ask students to justify that $A = P\left(1 + \frac{r}{n}\right)^{nt}$ is approximately equivalent to $A = P[(1 + r)^{\frac{1}{n}}]^{nt}$ graphically by fixing the values for P, r and n. Depending on the values chosen, students should see that the graphs of the two functions nearly coincide for a large interval of their domain.

Additional Answers

3. About 3.0% per year; this rate is greater than the 2.5% per year offered by Plan B.

LAB 14 Algebra Lab
Transforming Exponential Expressions

You can use the properties of rational exponents to transform exponential functions into other forms in order to solve real-world problems.

Common Core State Standards
A.SSE.3c, F.IF.8b, F.BF.3

Activity Write Equivalent Exponential Expressions

Monique is trying to decide between two savings account plans. Plan A offers 0.25% interest compounded monthly, while Plan B offers 2.5% interest compounded annually. Which is the better plan? Explain.

In order to compare the plans, we must compare rates with the same compounding frequency. One way to do this is to compare the approximate monthly interest rates of each plan, also called the *effective* monthly interest rate. While you can use the compound interest formula to find this rate, you can also use the properties of exponents.

Write a function to represent the amount A Monique would earn after t years with Plan B. For convenience, let the initial amount of Monique's investment be $1.

$y = a(1 + r)^t$ Equation for exponential growth

$A(t) = 1(1 + 0.025)^t$ $y = A(t)$, $a = 1$, $r = 2.5\%$ or 0.025

$= 1.025^t$ Simplify.

Now write a function equivalent to $A(t)$ that represents 12 compoundings per year, with a power of $12t$, instead of 1 per year, with a power of $1t$.

$A(t) = 1.025^{1t}$ Original function

$= 1.025^{\left(\frac{1}{12} \cdot 12\right)t}$ $1 = \frac{1}{12} \cdot 12$

$= \left(1.025^{\frac{1}{12}}\right)^{12t}$ Power of a Power

$\approx 1.0021^{12t}$ $(1.025)^{\frac{1}{12}} = \sqrt[12]{1.025}$ or about 1.0021

From this equivalent function, we can determine that the effective monthly interest by Plan B is about 0.0021 or about 0.21% per month. This rate is less than the monthly interest rate of 0.25% per month offered by Plan A, so Plan A is the better plan.

Model and Analyze

1. Use the compound interest formula $A = P\left(1 + \frac{r}{n}\right)^m$ to determine the effective monthly interest rate for Plan B. How does this rate compare to the rate calculated using the method in the Activity above? **About 0.21% per month; they are approximately the same.**

2. Write a function to represent the amount A Monique would earn after t months by Plan A. Then use the properties of exponents to write a function equivalent to $A(t)$ that represents the amount earned after t years. **$A(t) = (1.0025)^t$; $A(t) \approx (1.030)^{\frac{1}{12}t}$**

3. From the expression you wrote in Exercise 2, identify the effective annual interest rate by Plan A. Use this rate to explain why Plan A is the better plan. **See margin.**

4. Suppose Plan A offered 1.5% interest per quarter. Use the properties of exponents to explain which is the better plan. **See margin.**

4. The function $A(t) = (1.015)^t$ gives the amount $A(t)$ earned after t quarters.

$A(t) = (1.015)^{1t}$ Original function

$= (1.015)^{\left(\frac{1}{4} \cdot 4\right)t}$ $1 = \frac{1}{4} \cdot 4$

$= \left[(1.015)^{\frac{1}{4}}\right]^{4t}$ Power of a Power

$\approx (1.0037)^{4t}$ $(1.015)^{\frac{1}{4}} = \sqrt[4]{1.015}$ or about 1.0037

The effective annual interest rate is about 0.0037 or 3.7% per month, which is greater than the annual interest rate of 2.5% per month offered by Plan A, so Plan B is the better plan.

LAB 15

Algebra Lab

Average Rate of Change of Exponential Functions

You know that the rate of change of a linear function is the same for any two points on the graph. The rate of change of an exponential function is not constant.

Common Core State Standards
F.IF.6

Activity Evaluating Investment Plans

John has \$2000 to invest in one of two plans. Plan 1 offers to increase his principal by \$75 each year, while Plan 2 offers to pay 3.6% interest compounded monthly. The dollar value of each investment after t years is given by $A_1 = 2000 + 75t$ and $A_2 = 2000(1.003)^{12t}$, respectively. Use the function values, the average rate of change, and the graphs of the equations to interpret and compare the plans.

Step 1 Copy and complete the table below by finding the missing values for A_1 and A_2.

t	0	1	2	3	4	5
A_1	2000	2075	2150	2225	2300	2375
A_2	2000	2073.2	2149.08	2227.74	2309.27	2393.79

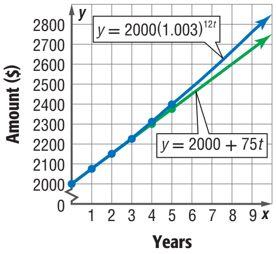
Step 2 Find the average rate of change for each plan from $t = 0$ to 1, $t = 3$ to 4, and $t = 0$ to 5.

Plan 1: $\frac{2075 - 2000}{1 - 0}$ or 75 $\quad \frac{2300 - 2225}{4 - 3}$ or 75 $\quad \frac{2375 - 2000}{5 - 0}$ or 75

Plan 2: $\frac{2073.2 - 2000}{1 - 0}$ or 73.2 $\quad \frac{2309.27 - 2227.74}{4 - 3}$ or about 82 $\quad \frac{2393.79 - 2000}{5 - 0}$ or about 79

Step 3 Graph the ordered pairs for each function. Connect each set of points with a smooth curve.

Step 4 Use the graph and the rates of change to compare the plans. Both graphs have a rate of change for the first year of about \$75 per year. From year 3 to 4, Plan 1 continues to increase at \$75 per year, but Plan 2 grows at a rate of more than \$81 per year. The average rate of change over the first five years for Plan 1 is \$75 per year and for Plan 2 is over \$78 per year. This indicates that as the number of years increases, the investment in Plan 2 grows at an increasingly faster pace. This is supported by the widening gap between their graphs.



3. Sample answer: The value of the equipment decreases at a slower rate as the number of years increases.

Exercises

The value of a company's piece of equipment decreases over time due to depreciation. The function $y = 16{,}000(0.985)^{2t}$ represents the value after t years.

1. What is the average rate of change over the first five years? **−\$448.86 per year**
2. What is the average rate of change of the value from year 5 to year 10? **−\$386 per year**
3. What conclusion about the value can we make based on these average rates of change?
4. Copy and complete the table for $y = x^4$.

x	−3	−2	−1	0	1	2	3
y	81	16	1	0	1	16	81

Compare and interpret the average rate of change for $x = -3$ to 0 and for $x = 0$ to 3.

4. Sample answer: The average rate of change for $x = -3$ to 0 is −27 while the average rate of change for $x = 0$ to 3 is 27. This would indicate that the graph of the function was going down and then changed to going up.

From Concrete to Abstract

After students have completed Exercise 4, have them discuss what characteristics of a graph they can determine by examining the rate of change for a function.

1 Focus

Objective Calculate and interpret the average rate of change of an exponential function.

Materials for Each Student

- grid paper

Easy to Make Manipulatives

Teaching Algebra with Manipulatives
Template for grid paper, p. 1

Teaching Tip

Have students use increments of 25 on the vertical axis. This will give a clearer illustration of the difference in the two graphs.

2 Teach

Working in Cooperative Groups

Put students in groups of two or three, mixing abilities. Have groups complete the activity.

- Discuss how the length of time of the investment affects the comparison of the plans.
- Have students describe the shape of the graph for each plan and discuss how the shape is related to the average rates of change.

Practice Have students complete Exercises 1–4.

3 Assess

Formative Assessment

Use Exercises 1–3 to assess whether students can calculate and interpret an average rate of change.

1 Focus

VerticalAlignment

Before Lesson 16 Wrote explicit formulas to represent arithmetic and geometric sequences.

Lesson 16 Use a recursive formula to list terms in a sequence. Write recursive formulas for arithmetic and geometric sequences.

After Lesson 16 Identify linear, quadratic, and exponential functions from given data.

2 Teach

Scaffolding Questions

Have students read the **Why?** section of the lesson.

Ask:

- How does the total cost of the shuttle service change as a customer is added? Sample answer: The total cost increases by $10.
- Is this sequence *arithmetic*, *geometric*, or neither? arithmetic
- How much would it cost for 9 customers? $105

LESSON 16 Recursive Formulas

Then

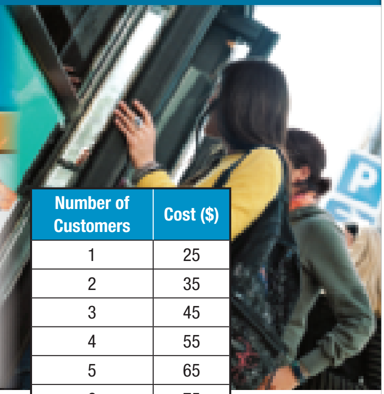
- You wrote explicit formulas to represent arithmetic and geometric sequences.

Now

1. Use a recursive formula to list terms in a sequence.
2. Write recursive formulas for arithmetic and geometric sequences.

Why?

- Clients of a shuttle service get picked up from their homes and driven to premium outlet stores for shopping. The total cost of the service depends on the total number of customers. The costs for first six customers are shown.



Number of Customers	Cost ($)
1	25
2	35
3	45
4	55
5	65
6	75

Patrick SheÁndell o' carroll/PhotoAlto

NewVocabulary
recursive formula

Common Core State Standards
F.IF.3, F.IF.9, F.BF.1a, F.BF.2

1 Using Recursive Formulas

An explicit formula allows you to find any term a_n of a sequence by using a formula written in terms of n. For example, $a_n = 2n$ can be used to find the fifth term of the sequence 2, 4, 6, 8, …: $a_5 = 2(5)$ or 10.

A **recursive formula** allows you to find the nth term of a sequence by performing operations to one or more of the preceding terms. Since each term in the sequence above is 2 greater than the term that preceded it, we can add 2 to the fourth term to find that the fifth term is 8 + 2 or 10. We can then write a recursive formula for a_n.

$$\begin{array}{lll} a_1 = & & = 2 \\ a_2 = & a_1 + 2 \text{ or } 2 + 2 & = 4 \\ a_3 = & a_2 + 2 \text{ or } 4 + 2 & = 6 \\ a_4 = & a_3 + 2 \text{ or } 6 + 2 & = 8 \\ \vdots & \vdots & \\ a_n = & a_{n-1} + 2 & \end{array}$$

A recursive formula for the sequence above is $a_1 = 2$, $a_n = a_{n-1} + 2$, for $n \geq 2$ where n is an integer. The term denoted a_{n-1} represents the term immediately before a_n. Notice that the first term a_1 is given, along with the domain for n.

Example 1 Use a Recursive Formula

Find the first five terms of the sequence in which $a_1 = 7$ and $a_n = 3a_{n-1} - 12$, if $n \geq 2$.

Use $a_1 = 7$ and the recursive formula to find the next four terms.

$$\begin{aligned} a_2 &= 3a_{2-1} - 12 && n = 2 \\ &= 3a_1 - 12 && \text{Simplify.} \\ &= 3(7) - 12 \text{ or } 9 && a_1 = 7 \end{aligned}$$

$$\begin{aligned} a_3 &= 3a_{3-1} - 12 && n = 3 \\ &= 3a_2 - 12 && \text{Simplify.} \\ &= 3(9) - 12 \text{ or } 15 && a_2 = 9 \end{aligned}$$

$$\begin{aligned} a_4 &= 3a_{4-1} - 12 && n = 4 \\ &= 3a_3 - 12 && \text{Simplify.} \\ &= 3(15) - 12 \text{ or } 33 && a_3 = 15 \end{aligned}$$

$$\begin{aligned} a_5 &= 3a_{5-1} - 12 && n = 5 \\ &= 3a_4 - 12 && \text{Simplify.} \\ &= 3(33) - 12 \text{ or } 87 && a_4 = 33 \end{aligned}$$

The first five terms are 7, 9, 15, 33, and 87.

GuidedPractice

1. Find the first five terms of the sequence in which $a_1 = -2$ and $a_n = (-3)\, a_{n-1} + 4$, if $n \geq 2$. **−2, 10, −26, 82, −242**

2 Writing Recursive Formulas

To write a recursive formula for an arithmetic or geometric sequence, complete the following steps.

KeyConcept Writing Recursive Formulas

Step 1 Determine if the sequence is arithmetic or geometric by finding a common difference or a common ratio.

Step 2 Write a recursive formula.

Arithmetic Sequences	$a_n = a_{n-1} + d$, where d is the common difference
Geometric Sequences	$a_n = r \cdot a_{n-1}$, where r is the common ratio

Step 3 State the first term and domain for n.

StudyTip

Defining *n* For the *n*th term of a sequence, the value of *n* must be a positive integer. Although we must still state the domain of *n*, from this point forward, we will assume that *n* is an integer.

Example 2 Write Recursive Formulas

Write a recursive formula for each sequence.

a. 17, 13, 9, 5, …

Step 1 First subtract each term from the term that follows it.

$13 - 17 = -4 \qquad 9 - 13 = -4 \qquad 5 - 9 = -4$

There is a common difference of −4. The sequence is arithmetic.

Step 2 Use the formula for an arithmetic sequence.

$a_n = a_{n-1} + d$ Recursive formula for arithmetic sequence

$a_n = a_{n-1} + (-4)$ $d = -4$

Step 3 The first term a_1 is 17, and $n \geq 2$.

A recursive formula for the sequence is $a_1 = 17$, $a_n = a_{n-1} - 4$, $n \geq 2$.

StudyTip

Domain The domain for *n* is decided by the given terms. Since the first term is already given, it makes sense that the first term to which the formula would apply is the 2nd term of the sequence, or when $n = 2$.

b. 6, 24, 96, 384, …

Step 1 First subtract each term from the term that follows it.

$24 - 6 = 18 \qquad 96 - 24 = 72 \qquad 384 - 96 = 288$

There is no common difference. Check for a common ratio by dividing each term by the term that precedes it.

$\frac{24}{6} = 4 \qquad \frac{96}{24} = 4 \qquad \frac{384}{96} = 4$

There is a common ratio of 4. The sequence is geometric.

Step 2 Use the formula for a geometric sequence.

$a_n = r \cdot a_{n-1}$ Recursive formula for geometric sequence

$a_n = 4a_{n-1}$ $r = 4$

Step 3 The first term a_1 is 6, and $n \geq 2$.

A recursive formula for the sequence is $a_1 = 6$, $a_n = 4a_{n-1}$, $n \geq 2$.

GuidedPractice

2A. 4, 10, 25, 62.5, … $a_1 = 4$, $a_n = 2.5a_{n-1}$, $n \geq 2$.

2B. 9, 36, 63, 90, … $a_1 = 9$, $a_n = a_{n-1} + 27$, $n \geq 2$.

1 Using Recursive Formulas

Example 1 shows how to find the first five terms of a sequence using a recursive formula.

Formative Assessment

Use the Guided Practice exercises after each example to determine students' understanding of concepts.

Additional Example

1 Find the first five terms of the sequence in which $a_1 = -8$ and $a_n = -2a_{n-1} + 5$, if $n \geq 2$. $-8, 21, -37, 79, -153$

Additional Examples also in Interactive Classroom PowerPoint® Presentations

Interactive White Board READY

2 Writing Recursive Formulas

Example 2 shows how to write a recursive formula for a sequence. **Example 3** shows how to write recursive and explicit formulas for a sequence. **Example 4** shows how to translate between recursive and explicit formulas.

Additional Example

2 Write a recursive formula for each sequence.

a. 23, 29, 35, 41, … $a_1 = 23$, $a_n = a_{n-1} + 6$, $n \geq 2$

b. 7, −21, 63, −189, … $a_1 = 7$, $a_n = -3a_{n-1}$, $n \geq 2$

Tips for New Teachers

Notation Recursive formulas are occasionally defined for a_{n+1} and written in terms of a_n. For part **a** of Example 2, the recursive formula can be written as $a_1 = 17$, $a_{n+1} = a_n - 4$, $n \geq 1$.

Additional Examples

3 **CARS** The price of a car depreciates at the end of each year.

Year	Price ($)
1	12,000
2	7200
3	4320
4	2592

a. Write a recursive formula for the sequence. $a_1 = 12{,}000$, $a_n = 0.6a_{n-1}$

b. Write an explicit formula for the sequence. $a_n = 12{,}000(0.6)^{n-1}$

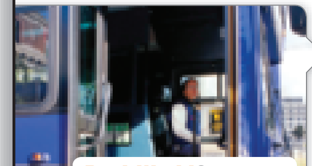
4 **a.** Write a recursive formula for $a_n = 2n - 4$. $a_1 = -2$, $a_n = a_{n-1} + 2, n \geq 2$

b. Write an explicit formula for $a_1 = 84$, $a_n = 1.5a_{n-1}$, $n \geq 2$. $a_n = 84(1.5)^{n-1}$

Tips for New Teachers

Terms The first term of a sequence is occasionally denoted as a_0.

A sequence can be represented by both an explicit formula and a recursive formula.



Real-World Career

Transportation The number of jobs in the transportation industry is expected to grow by an estimated 1.1 million between 2004 and 2014. The specific fields dictate the educational requirements, which include a high school diploma and some form of specialized training.

Source: United States Department of Labor

Example 3 Write Recursive and Explicit Formulas

COST Refer to the beginning of the lesson. Let n be the number of customers.

a. Write a recursive formula for the sequence.

Steps 1 and 2 First subtract each term from the term that follows it.

$35 - 25 = 10$ $\quad 45 - 35 = 10$ $\quad 55 - 45 = 10$

There is a common difference of 10. The sequence is arithmetic.

Step 3 Use the formula for an arithmetic sequence.

$a_n = a_{n-1} + d$ Recursive formula for arithmetic sequence

$a_n = a_{n-1} + 10$ $d = 10$

Step 4 The first term a_1 is 25, and $n \geq 2$.

A recursive formula for the sequence is $a_1 = 25$, $a_n = a_{n-1} + 10$, $n \geq 2$.

b. Write an explicit formula for the sequence.

Step 1 The common difference is 10.

Step 2 Use the formula for the nth term of an arithmetic sequence.

$a_n = a_1 + (n - 1)d$ Formula for the nth term

$= 25 + (n - 1)10$ $a_1 = 25$ and $d = 10$

$= 25 + 10n - 10$ Distributive Property

$= 10n + 15$ Simplify.

An explicit formula for the sequence is $a_n = 10n + 15$.

Guided Practice

3. **SAVINGS** The money that Ronald has in his savings account earns interest each year. He does not make any withdrawals or additional deposits. The account balance at the beginning of each year is $10,000, $10,300, $10,609, $10,927.27, and so on. Write a recursive formula and an explicit formula for the sequence. $a_1 = 10{,}000$, $a_n = 1.03a_{n-1}$, $n \geq 2$; $a_n = 10{,}000(1.03)^{n-1}$

If several successive terms of a sequence are needed, a recursive formula may be useful, whereas if just the nth term of a sequence is needed, an explicit formula may be useful. Thus, it is sometimes beneficial to translate between the two forms.

Study Tip

Geometric Sequence Recall that the formula for the nth term of a geometric sequence is $a_n = a_1r^{n-1}$.

Example 4 Translate between Recursive and Explicit Formulas

a. Write a recursive formula for $a_n = 6n + 3$.

$a_n = 6n + 3$ is an explicit formula for an arithmetic sequence with $d = 6$ and $a_1 = 6(1) + 3$ or 9. Therefore, a recursive formula for a_n is $a_1 = 9$, $a_n = a_{n-1} + 6$, $n \geq 2$.

b. Write an explicit formula for $a_1 = 120$, $a_n = 0.8a_{n-1}$, $n \geq 2$.

$a_n = 0.8a_{n-1}$ is a recursive formula for a geometric sequence with $a_1 = 120$ and $r = 0.8$. Therefore, an explicit formula for a_n is $a_n = 120(0.8)^{n-1}$.

Guided Practice

4A. Write a recursive formula for $a_n = 4(3)^{n-1}$. $a_1 = 4$, $a_n = 3a_{n-1}$, $n \geq 2$.

4B. Write an explicit formula for $a_1 = -16$, $a_n = a_{n-1} - 7$, $n \geq 2$. $a_n = -7n - 9$.

John A. Rizzo/Photodisc/Getty Images

Differentiated Instruction AL OL

Interpersonal Learners Divide the class into groups of two or three students. Have each student write a sequence on one note card and the recursive formula for the sequence on another note card. Repeat the process for 10 sequences. Then, have the students lay the cards face down. Each student should take turns flipping over two cards, attempting to find a match between a sequence and its recursive formula.

Check Your Understanding

Example 1 **Find the first five terms of each sequence.**

1. $a_1 = 16, a_n = a_{n-1} - 3, n \geq 2$
16, 13, 10, 7, 4

2. $a_1 = -5, a_n = 4a_{n-1} + 10, n \geq 2$
–5, –10, –30, –110, –430

Example 2 **Write a recursive formula for each sequence.**

3. 1, 6, 11, 16, …
$a_1 = 1, a_n = a_{n-1} + 5, n \geq 2$

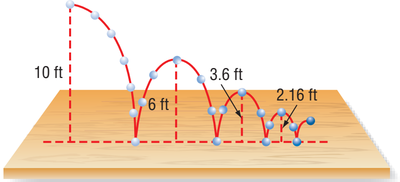
4. 4, 12, 36, 108, …
$a_1 = 4, a_n = 3a_{n-1}, n \geq 2$

Example 3 **5.** **BALL** A ball is dropped from an initial height of 10 feet. The maximum heights the ball reaches on the first three bounces are shown.

a. Write a recursive formula for the sequence.

b. Write an explicit formula for the sequence.

5a. $a_1 = 10, a_n = 0.6a_{n-1}, n \geq 2$
5b. $a_n = 10(0.6)^{n-1}$



Example 4 **For each recursive formula, write an explicit formula. For each explicit formula, write a recursive formula.**

6. $a_1 = 4, a_n = a_{n-1} + 16, n \geq 2$
$a_n = 16n - 12$

7. $a_n = 5n + 8$
$a_1 = 13, a_n = a_{n-1} + 5, n \geq 2$

8. $a_n = 15(2)^{n-1}$
$a_1 = 15, a_n = 2a_{n-1}, n \geq 2$

9. $a_1 = 22, a_n = 4a_{n-1}, n \geq 2$
$a_n = 22(4)^{n-1}$

Practice and Problem Solving

Example 1 **Find the first five terms of each sequence.**

10. $a_1 = 23, a_n = a_{n-1} + 7, n \geq 2$
23, 30, 37, 44, 51

11. $a_1 = 48, a_n = -0.5a_{n-1} + 8, n \geq 2$
48, –16, 16, 0, 8

12. $a_1 = 8, a_n = 2.5a_{n-1}, n \geq 2$
8, 20, 50, 125, 312.5

13. $a_1 = 12, a_n = 3a_{n-1} - 21, n \geq 2$
12, 15, 24, 51, 132

14. $a_1 = 13, a_n = -2a_{n-1} - 3, n \geq 2$
13, –29, 55, –113, 223

15. $a_1 = \frac{1}{2}, a_n = a_{n-1} + \frac{3}{2}, n \geq 2$
$\frac{1}{2}, 2, \frac{7}{2}, 5, \frac{13}{2}$

Example 2 **Write a recursive formula for each sequence.**

16. 12, –1, –14, –27, …
$a_1 = 12, a_n = a_{n-1} - 13, n \geq 2$

17. 27, 41, 55, 69, …
$a_1 = 27, a_n = a_{n-1} + 14, n \geq 2$

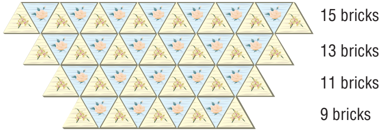
18. 2, 11, 20, 29, …
$a_1 = 2, a_n = a_{n-1} + 9, n \geq 2$

19. 100, 80, 64, 51.2, …
$a_1 = 100, a_n = 0.8a_{n-1}, n \geq 2$

20. 40, –60, 90, –135, …
$a_1 = 40, a_n = -1.5a_{n-1}, n \geq 2$

21. 81, 27, 9, 3, …
$a_1 = 81, a_n = \frac{1}{3}a_{n-1}, n \geq 2$

Example 3 **22.** **BRICK** A landscaper is building a brick patio. Part of the patio includes a pattern constructed from triangles. The first four rows of the pattern are shown. **22a.** $a_1 = 15, a_n = a_{n-1} - 2, n \geq 2$



a. Write a recursive formula for the sequence.

b. Write an explicit formula for the sequence. $a_n = 17 - 2n$

Example 4 **For each recursive formula, write an explicit formula. For each explicit formula, write a recursive formula.**

23. $a_n = 3(4)^{n-1}$
$a_1 = 3, a_n = 4a_{n-1}, n \geq 2$

24. $a_1 = -2, a_n = a_{n-1} - 12, n \geq 2$ $a_n = -12n + 10$

25. $a_1 = 38, a_n = \frac{1}{2}a_{n-1}, n \geq 2$
$a_n = 38\left(\frac{1}{2}\right)^{n-1}$

26. $a_n = -7n + 52$ $a_1 = 45, a_n = a_{n-1} - 7, n \geq 2$

Differentiated Homework Options

Level	Assignment	Two-Day Option	
AL Basic	10–26, 31, 33, 35	11–25 odd	10–26 even, 31, 33, 35
OL Core	11–25 odd, 27–31, 33, 35	10–26	27–31, 33, 35
BL Advanced	27–35		

3 Practice

Formative Assessment

Use Exercises 1–9 to check for understanding.

Use the chart at the bottom of this page to customize assignments for your students.

WatchOut!

Error Analysis In Exercise 31, students should recognize that the sequence is geometric with a common ratio of –1. Therefore, the sequence can be represented as both an explicit formula and a recursive formula.

4 Assess

Ticket Out the Door Have each student create a sequence by writing the first five terms. Then have them write an explicit formula and a recursive formula for the sequence.

Multiple Representations

In Exercise 30, students use logic, analysis, and algebra to explore the Fibonacci sequence and find terms as needed.

Additional Answer

31. Both; sample answer: The sequence can be written as the recursive formula $a_1 = 2$, $a_n = (-1)a_{n-1}$, $n \geq 2$. The sequence can also be written as the explicit formula $a_n = 2(-1)^{n-1}$.

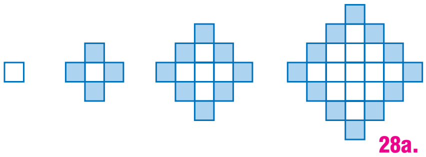
B **27.** **TEXT MESSAGES** Barbara received a chain text message that she forwarded to five of her friends. Each of her friends forwarded the message to five more friends, and so on.

a. Find the first five terms of the sequence representing the number of people who receive the text in the nth round. **1, 5, 25, 125, 625**

b. Write a recursive formula for the sequence. **$a_1 = 1$, $a_n = 5a_{n-1}$, $n \geq 2$**

c. If Barbara represents a_1, find a_8. **78,125**

28. **GEOMETRY** Consider the pattern below. The number of blue boxes increases according to a specific pattern.

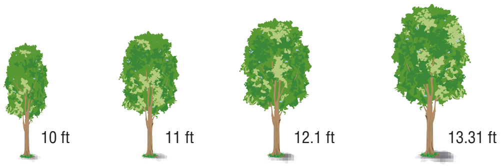


28a. $a_1 = 0$, $a_n = a_{n-1} + 4$, $n \geq 2$

a. Write a recursive formula for the sequence of the number of blue boxes in each figure.

b. If the first box represents a_1, find the number of blue boxes in a_8. **28**

29. **TREE** The growth of a certain type of tree slows as the tree continues to age. The heights of the tree over the past four years are shown.



a. Write a recursive formula for the height of the tree. **$a_1 = 10$, $a_n = 1.1a_{n-1}$, $n \geq 2$**

b. If the pattern continues, how tall will the tree be in two more years? Round your answer to the nearest tenth of a foot. **16.1 ft**

C **30.** **MULTIPLE REPRESENTATIONS** The Fibonacci sequence is neither arithmetic nor geometric and can be defined by a recursive formula. The first terms are 1, 1, 2, 3, 5, 8, …

a. Logical Determine the relationship between the terms of the sequence. What are the next five terms in the sequence? **30b. $a_1 = 1$, $a_2 = 1$, $a_n = a_{n-2} + a_{n-1}$, $n \geq 3$**

b. Algebraic Write a formula for the nth term if $a_1 = 1$, $a_2 = 1$, and $n \geq 3$.

c. Algebraic Find the 15th term. **610**

d. Analytical Explain why the Fibonacci sequence is not an arithmetic sequence. **Sample answer: There is no common difference.**

30a. Sample answer: The first two terms are 1. Starting with the third term, the two previous terms are added together to get the next term; 13, 21, 34, 55, 89.

H.O.T. Problems Use Higher-Order Thinking Skills

31. **ERROR ANALYSIS** Patrick and Lynda are working on a math problem that involves the sequence 2, −2, 2, −2, 2, … . Patrick thinks that the sequence can be written as a recursive formula. Lynda believes that the sequence can be written as an explicit formula. Is either of them correct? Explain. **See margin.**

32. **CHALLENGE** Find a_1 for the sequence in which $a_4 = 1104$ and $a_n = 4a_{n-1} + 16$. **12**

33. **REASONING** Determine whether the following statement is *true* or *false*. Justify your reasoning.

There is only one recursive formula for every sequence.

33. False; sample answer: A recursive formula for the sequence 1, 2, 3, … can be written as $a_1 = 1$, $a_n = a_{n-1} + 1$, $n \geq 2$ or as $a_1 = 1$, $a_2 = 2$, $a_n = a_{n-2} + 2$, $n \geq 3$.

34. **CHALLENGE** Find a recursive formula for 4, 9, 19, 39, 79, … **$a_1 = 4$, $a_n = 2a_{n-1} + 1$, $n \geq 2$**

35. **WRITING IN MATH** Explain the difference between an explicit formula and a recursive formula.

35. Sample answer: In an explicit formula, the nth term a_n is given as a function of n. In a recursive formula, the nth term a_n is found by performing operations to one or more of the terms that precede it.

Differentiated Instruction BL

Extension For Exercise 30, students wrote a recursive formula for the Fibonacci sequence, which is neither arithmetic nor geometric. Have students write a recursive formula for another sequence that is neither arithmetic nor geometric.

LAB 17 Algebra Lab
Inverse Functions

You have discovered that every nonhorizontal linear function has an inverse function. You have learned how to find the inverse of any function by exchanging the coordinates for a set of ordered pairs. In the following activity, we will exchange coordinates to find the inverse of a quadratic function and determine whether the inverse is a function.

Common Core State Standards
F.BF.4a

Activity 1 Exchange Coordinates

Find the inverse of $y = x^2$ by exchanging the coordinates. Is the inverse a function?

Step 1 Make a table of values for $y = x^2$ using x from −3 to 3.

x	−3	−2	−1	0	1	2	3
y	9	4	1	0	1	4	9

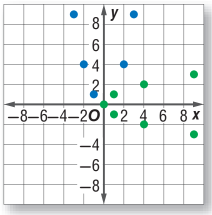
Step 2 Write the coordinates as a set of ordered pairs.

{(−3, 9), (−2, 4), (−1, 1), (0, 0), (1, 1), (2, 4), (3, 9)}

Step 3 Exchange the x- and y-coordinates in each ordered pair to form the inverse.

{(9, −3), (4, −2), **(1, −1)**, (0, 0), **(1, 1)**, (4, 2), (9, 3)}

Step 4 Examine the set of ordered pairs and determine if it would be a function. This set of ordered pairs would not be a function because each x-value is not paired with a unique y-value. For example, there are two y-values when $x = 1$.



You have also learned how to find the inverse of a linear function algebraically. In the next activity, you will find the inverse of the quadratic function from Activity 1.

Activity 2 Use Algebra

Find the inverse of $y = x^2$ algebraically. Check by graphing the function, its inverse, and the line $y = x$.

Step 1 Find the inverse algebraically.

$y = x^2$	Original function
$x = y^2$	Interchange x and y.
$\pm\sqrt{x} = \sqrt{y^2}$	Take the square root of each side.
$\pm\sqrt{x} = y$	Simplify.

The inverse of $y = x^2$ is $y = \pm\sqrt{x}$.

(continued on the next page)

1 Focus

Objective Find the inverse of a quadratic function. Create an inverse function by limiting the domain of the original function.

Materials for Each Group

- grid paper

Easy to Make Manipulatives

Teaching Algebra with Manipulatives
Template for grid paper, p. 1

Teaching Tip

Before students do the activities, review finding the inverse of a linear function. Recall that each inverse is linear. Remind students that the graph of each quadratic function is a parabola. Ask if they think each inverse of a quadratic function will also be a quadratic function.

2 Teach

Working in Cooperative Groups

Put students in groups of two or three, mixing abilities, to complete Activities 1–3 and Exercises 1, 4, and 8.

Ask:

- Will the reflection of the graph of every quadratic function in the line $y = x$ represent a relation that is not a function? yes
- What feature on the graph of a function determines that its inverse will not be a function? If two or more points lie on the same horizontal line.
- Is the graph of every quadratic function symmetric about the y-axis? If not, where is the axis of symmetry found? No; the axis of symmetry is a vertical line through the vertex of the graph.

Practice Have students complete Exercises 2, 5, and 10.

3 Assess

Formative Assessment

Use Exercises 1 and 4 to assess if students can find the inverse of a quadratic function. Use Exercise 10 to assess if students understand how to restrict the domain to create an inverse function.

From Concrete to Abstract

Have students discuss why it might be necessary to have the inverse be a function. Where is this often observed? It might be necessary for each input to have a unique output. This can be observed on graphing calculators.

In later courses, students will study trigonometric functions. Similar to quadratic functions, the domain of a function will often need to be restricted for its inverse to be a function.

Additional Answers

1. {(6, −3), (1, −2), (−2, −1), (−3, 0), (−2, 1), (1, 2), (6, 3)}; not a function

2. {(16, −3), (9, −2), (4, −1), (1, 0), (0, 1), (1, 2), (4, 3)}; not a function

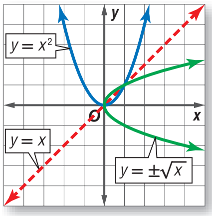
3. {(18, −3), (8, −2), (2, −1), (0, 0), (2, 1), (8, 2), (18, 3)}; not a function

4. {(25, −3), (10, −2), (1, −1), (−2, 0), (1, 1), (10, 2), (25, 3)}; not a function

Algebra Lab
Inverse Functions *Continued*

Step 2 On a coordinate plane, plot and connect the sets of points from Steps 2 and 3 of Activity 1 with a smooth curve to graph $y = x^2$ and its inverse. Graph the line $y = x$.

Step 3 The graph of $y = \pm\sqrt{x}$ does not pass the vertical line test for a function. The inverse is not a function.



Many functions like $y = x^2$ have inverse relations that are not functions. It is often possible to limit the domains of these functions so that their inverses will be functions.

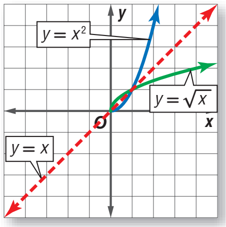
Activity 3 Restricted Domains

Restrict the domain of $y = x^2$ so that its inverse is a function.

Notice from Activity 2 that the graph of $y = x^2$ is symmetric about the y-axis. If we restrict the domain of $y = x^2$ to either $x \geq 0$ or $x \leq 0$, we are left with half of the graph.

For $x \geq 0$, the graph of $y = x^2$ is now the portion of the parabola to the right of the y-axis. Its inverse is its reflection across the line $y = x$, which is the top portion of the graph of $y = \pm\sqrt{x}$.

Since each x-value of this reflection is paired with a unique y-value, the inverse is now a function.



Exercises

Write a set of ordered pairs for the inverse of each function by making a table of values for x from −3 to 3 and exchanging the coordinates. Is the inverse a function? **1–4. See margin.**

1. $y = x^2 - 3$ **2.** $y = (x - 1)^2$

3. $y = 2x^2$ **4.** $y = 3x^2 - 2$

Find the inverse of each function algebraically. Is the inverse a function?

5. $y = x^2 + 2$ **$y = \pm\sqrt{x - 2}$; not a function** **6.** $y = (x - 1)^2$ **$y = 1 \pm \sqrt{x}$; not a function**

7. $y = (x + 3)^2 - 4$ **$y = \pm\sqrt{x + 4} - 3$; not a function** **8.** $y = 4x^2 + 2$ **$y = \pm\frac{\sqrt{x - 2}}{4}$; not a function**

Name a restricted domain for each function for which its inverse would be a function.

9. $y = x^2 - 1$ **Sample answer: $x \geq 0$** **10.** $y = (x + 2)^2$ **Sample answer: $x \geq -2$**

11. $y = (x - 2)^2 + 1$ **Sample answer: $x \geq 2$** **12.** $y = 3x^2 - 1$ **Sample answer: $x \geq 0$**

LAB 18

Algebra Lab
Rational and Irrational Numbers

A set is closed under an operation if for any numbers in the set, the result of the operation is also in the set. A set may be closed under one operation and not closed under another.

Common Core State Standards
N.RN.3

Activity 1 Closure of Rational Numbers and Irrational Numbers

Are the sets of rational and irrational numbers closed under multiplication? under addition?

Step 1 To determine if each set is closed under multiplication, examine several products of two rational factors and then two irrational factors.

Rational: $5 \times 2 = 10$; $-3 \times 4 = -12$; $3.7 \times 0.5 = 1.85$; $\frac{3}{4} \times \frac{2}{3} = \frac{1}{2}$

Irrational: $\pi \times \sqrt{2} = \sqrt{2}\pi$; $\sqrt{3} \times \sqrt{7} = \sqrt{21}$; $\sqrt{5} \times \sqrt{5} = 5$

The product of each pair of rational numbers is rational. However, the products of pairs of irrational numbers are both irrational and rational. Thus, it appears that the set of rational numbers is closed under multiplication, but the set of irrational numbers is not.

Step 2 Repeat this process for addition.

Rational: $3 + 8 = 11$; $-4 + 7 = 3$; $3.7 + 5.82 = 9.52$; $\frac{2}{5} + \frac{1}{4} = \frac{13}{20}$

Irrational: $\sqrt{3} + \pi = \sqrt{3} + \pi$; $3\sqrt{5} + 6\sqrt{5} = 9\sqrt{5}$; $\sqrt{12} + \sqrt{50} = 2\sqrt{3} + 5\sqrt{2}$

The sum of each pair of rational numbers is rational, and the sum of each pair of irrational numbers is irrational. Both sets are closed under addition.

Activity 2 Rational and Irrational Numbers

What kind of numbers are the product and sum of a rational and irrational number?

Step 1 Examine the products of several pairs of rational and irrational numbers.

$3 \times \sqrt{8} = 6\sqrt{2}$; $\frac{3}{4} \times \sqrt{2} = \frac{3\sqrt{2}}{4}$; $1 \times \sqrt{7} = \sqrt{7}$; $0 \times \sqrt{5} = 0$

The product is rational only when the rational factor is 0. The product of each nonzero rational number and irrational number is irrational.

Step 2 Find the sums of several pairs of a rational and irrational number.

$5 + \sqrt{3} = 5 + \sqrt{3}$; $\frac{2}{3} + \sqrt{5} = \frac{2 + 3\sqrt{5}}{3}$; $-4 + \sqrt{6} = -1(4 - \sqrt{6})$

The sum of each rational and irrational number is irrational.

Analyze the Results

1. The difference of any two unique rational numbers is always a rational number. The difference of any two unique irrational numbers is always irrational. The difference of any rational and irrational numbers is always irrational.

1. What kinds of numbers are the difference of two unique rational numbers, two unique irrational numbers, and a rational and an irrational number?
2. Is the quotient of every rational and irrational number always another rational or irrational number? If not, provide a counterexample. **No; sample answer: $\frac{\sqrt{5}}{0}$ does not equal a real number.**
3. **CHALLENGE** Recall that rational numbers are numbers that can be written in the form $\frac{a}{b}$, where a and b are integers and $b \neq 0$. Using $\frac{a}{b}$ and $\frac{c}{d}$ show that the sum and product of two rational numbers must always be a rational number. **See margin.**

Additional Answer

3. $\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$ and $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$ The sum and product of two integers is always an integer, so $(ad + bc)$, ac, and bd are all integers. By definition of a rational number, the sum and product of the two rational numbers are rational.

From Concrete to Abstract

After students complete Exercise 3, ask the class for possible ways that irrational numbers could be represented and used to test the product prediction from Activity 1.

1 Focus

Objective Investigate the products and sums of two rational numbers, two irrational numbers, and a rational and irrational number.

Teaching Tip

Emphasize that using examples with actual numbers cannot actually prove the conclusions made in the activities. This would need to be done algebraically. To increase the reliability of each conclusion, examples should be done using all types of numbers in the set(s).

2 Teach

Working in Cooperative Groups

Have students work in pairs, mixing abilities. Have them first make a list of subsets for rational and irrational numbers. After reading Activities 1 and 2, have them test more pairs of numbers from the subsets they listed.

Ask:

- Did you find any examples that did not agree with the conclusion in each activity? no
- What would finding one example that did not support the conclusion indicate? One example would be a counter example that would prove that the statement is false.

Practice Have students complete Exercises 1–3.

3 Assess

Formative Assessment

Use Exercises 1 and 2 to assess if students can investigate operations completed with specific types of numbers.

1 Focus

Objective Simplify radical expressions with indices greater than 2 and with variables and/or rational numbers in the radicand.

Teaching Tip

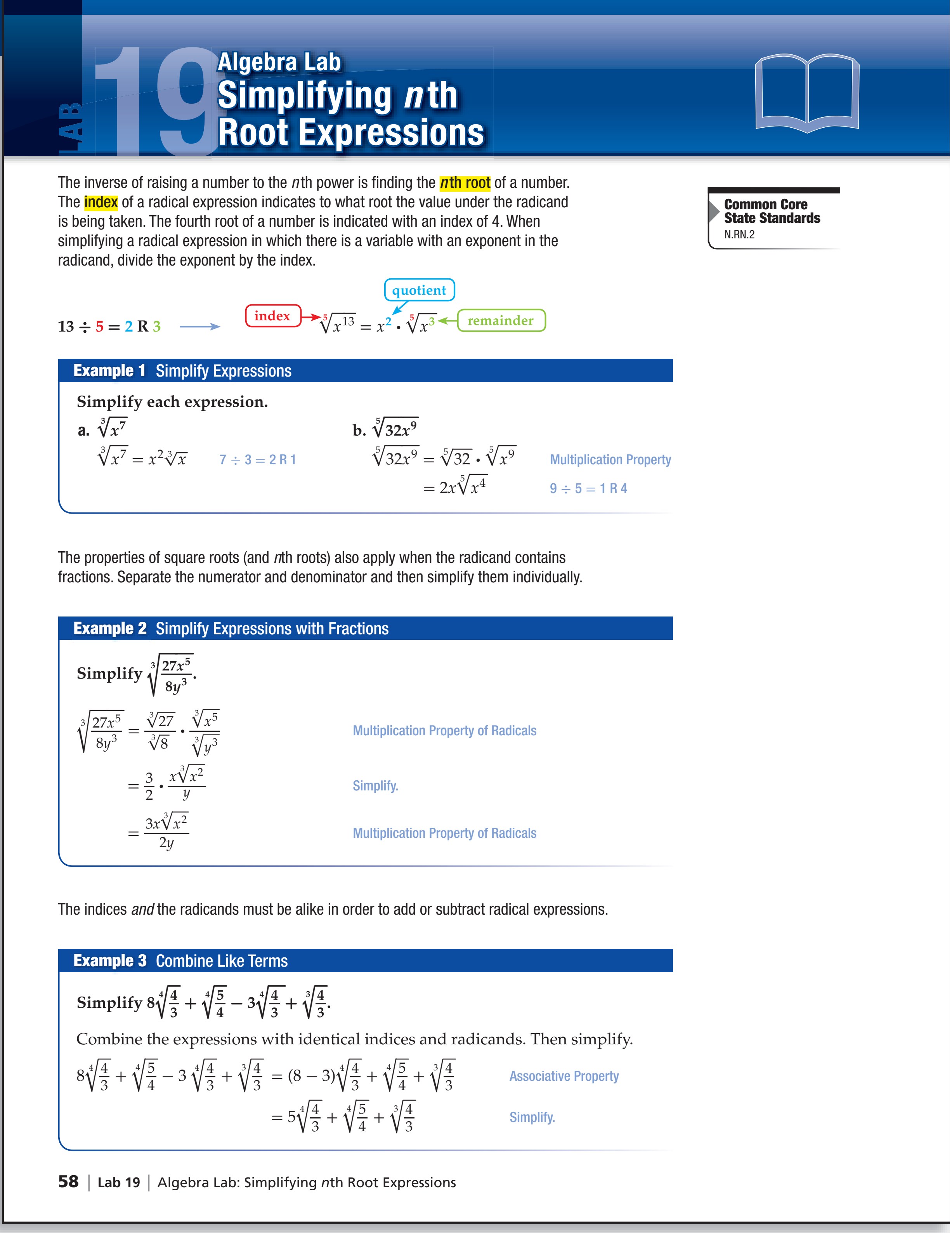
For Example 3, discuss the similarity between rules for adding radical expressions with the rules of adding and subtracting like terms in polynomials.

2 Teach

Working in Cooperative Groups

Put students in groups of two or three, mixing abilities.

- Have them write all of the new rules discussed in this lab.
- Have them include one example for when each rule is used and one example for when each rule is violated.



LAB 19

Algebra Lab
Simplifying *n*th Root Expressions

Common Core State Standards
N.RN.2

The inverse of raising a number to the *n*th power is finding the ***n*th root** of a number. The **index** of a radical expression indicates to what root the value under the radicand is being taken. The fourth root of a number is indicated with an index of 4. When simplifying a radical expression in which there is a variable with an exponent in the radicand, divide the exponent by the index.

$$13 \div 5 = 2 \text{ R } 3 \longrightarrow \text{index} \rightarrow \sqrt[5]{x^{13}} = \overset{\text{quotient}}{x^2} \cdot \sqrt[5]{x^3} \leftarrow \text{remainder}$$

Example 1 Simplify Expressions

Simplify each expression.

a. $\sqrt[3]{x^7}$

$\sqrt[3]{x^7} = x^2\sqrt[3]{x}$ $\quad 7 \div 3 = 2 \text{ R } 1$

b. $\sqrt[5]{32x^9}$

$\sqrt[5]{32x^9} = \sqrt[5]{32} \cdot \sqrt[5]{x^9}$ Multiplication Property

$= 2x\sqrt[5]{x^4}$ $\quad 9 \div 5 = 1 \text{ R } 4$

The properties of square roots (and *n*th roots) also apply when the radicand contains fractions. Separate the numerator and denominator and then simplify them individually.

Example 2 Simplify Expressions with Fractions

Simplify $\sqrt[3]{\frac{27x^5}{8y^3}}$.

$\sqrt[3]{\frac{27x^5}{8y^3}} = \frac{\sqrt[3]{27}}{\sqrt[3]{8}} \cdot \frac{\sqrt[3]{x^5}}{\sqrt[3]{y^3}}$ Multiplication Property of Radicals

$= \frac{3}{2} \cdot \frac{x\sqrt[3]{x^2}}{y}$ Simplify.

$= \frac{3x\sqrt[3]{x^2}}{2y}$ Multiplication Property of Radicals

The indices *and* the radicands must be alike in order to add or subtract radical expressions.

Example 3 Combine Like Terms

Simplify $8\sqrt[4]{\frac{4}{3}} + \sqrt[4]{\frac{5}{4}} - 3\sqrt[4]{\frac{4}{3}} + \sqrt[3]{\frac{4}{3}}$.

Combine the expressions with identical indices and radicands. Then simplify.

$8\sqrt[4]{\frac{4}{3}} + \sqrt[4]{\frac{5}{4}} - 3\sqrt[4]{\frac{4}{3}} + \sqrt[3]{\frac{4}{3}} = (8 - 3)\sqrt[4]{\frac{4}{3}} + \sqrt[4]{\frac{5}{4}} + \sqrt[3]{\frac{4}{3}}$ Associative Property

$= 5\sqrt[4]{\frac{4}{3}} + \sqrt[4]{\frac{5}{4}} + \sqrt[3]{\frac{4}{3}}$ Simplify.

When multiplying radical expressions, ensure that the indices are the same. Then multiply the radicands and simplify if possible. Once none of the remaining terms can be combined or simplified, the expression is considered simplified.

Example 4 Simplify Expressions with Products

Simplify $5\sqrt[4]{6} \cdot 2\sqrt[4]{12} \cdot \sqrt[3]{10}$.

Multiply the radicands with identical indexes.

$5\sqrt[4]{6} \cdot 2\sqrt[4]{12} \cdot \sqrt[3]{10} = (5 \cdot 2)(\sqrt[4]{6} \cdot \sqrt[4]{12}) \cdot \sqrt[3]{10}$ Associative Property

$= 10 \cdot (\sqrt[4]{6} \cdot \sqrt[4]{12}) \cdot \sqrt[3]{10}$ Multiply.

$= 10\sqrt[4]{72}\sqrt[3]{10}$ Multiply.

The properties of radical expressions still hold when variables are in the radicand.

Example 5 Simplify Expressions with Several Operations

Simplify $6\sqrt[4]{x} \cdot \sqrt[4]{x^3} + 3(\sqrt[3]{x} + 2\sqrt[3]{x})$.

Follow the order of operations and the properties of radical expressions.

$6\sqrt[4]{x} \cdot \sqrt[4]{x^3} + 3(\sqrt[3]{x} + 2\sqrt[3]{x}) = 6\sqrt[4]{x} \cdot \sqrt[4]{x^3} + 3(3\sqrt[3]{x})$ Add like terms.

$= 6\sqrt[4]{x \cdot x^3} + 3(3\sqrt[3]{x})$ Associative Property

$= 6\sqrt[4]{x^4} + 9\sqrt[3]{x}$ Multiply.

$= 6x + 9\sqrt[3]{x}$ Simplify.

Exercises

Simplify each expression. **9. $2\sqrt[5]{a^4} - 10\sqrt[5]{a}$**

1. $\sqrt[3]{c^6}$ **c^2**

2. $\sqrt[4]{16d^9}$ **$2d^2\sqrt[4]{d}$**

3. $\sqrt[3]{9} \cdot \sqrt[3]{6} \cdot \sqrt[3]{3}$ **$3\sqrt[3]{6}$**

4. $\sqrt[3]{\frac{8a^4}{125b^7}}$ **$\frac{2a\sqrt[3]{a}}{5b^2\sqrt[3]{b}}$**

5. $\sqrt[5]{\frac{32x^4}{5y^6z^5}}$ **$\frac{2\sqrt[5]{x^4}}{yz\sqrt[5]{5y}}$**

6. $\sqrt[4]{\frac{3}{2}} + 5\sqrt[4]{\frac{3}{2}} - 2\sqrt[4]{\frac{2}{3}}$ **$6\sqrt[4]{\frac{3}{2}} - 2\sqrt[4]{\frac{2}{3}}$**

7. $3\sqrt[4]{6} \cdot 4\sqrt[3]{6} \cdot 5\sqrt[4]{8}$ **$4\sqrt[3]{6} \cdot 30\sqrt[4]{3}$**

8. $3\sqrt[4]{x^2} + 2\sqrt[4]{x} \cdot 4\sqrt[4]{x}$ **$11\sqrt[4]{x^2}$**

9. $\sqrt[5]{a} \cdot 2\sqrt[5]{a^3} - 2(\sqrt[5]{a} + 4\sqrt[5]{a})$

10. $\sqrt[4]{\frac{x}{4}} + 5\sqrt[4]{\frac{x}{4}} - 2\sqrt[4]{\frac{2x}{3}}$ **$6\sqrt[4]{\frac{x}{4}} - 2\sqrt[4]{\frac{2x}{3}}$**

11. $\sqrt[4]{\frac{8a^2}{15b^3}} \cdot 3\sqrt[4]{\frac{2a^3}{27b}}$ **$\frac{2a}{b}\sqrt[4]{\frac{a}{5}}$**

12. $\sqrt[4]{\frac{16x^3}{81y^5}} + 3\sqrt[4]{\frac{x^3}{y}} + \sqrt[3]{\frac{16x}{y^8}}$ **$\frac{2+9y}{3y}\sqrt[4]{\frac{x^3}{y}} + \frac{2}{y^2}\sqrt[3]{\frac{2x}{y^2}}$**

Think About It

13. Provide an example in which two radical expressions with *unlike* radicands can be combined by addition. **Sample answer: $\sqrt[3]{81} + \sqrt[3]{24} = 3\sqrt[3]{3} + 2\sqrt[3]{3} = 5\sqrt[3]{3}$**

14. Provide an example in which two radical expressions with identical indices and with like variables in the radicand *cannot* be combined by addition. **Sample answer: $\sqrt[3]{x} + \sqrt[3]{x^2}$**

Practice Have students complete Exercises 1–14.

Teaching Tip

Remind students that the product of two or more radical expressions may be able to be simplified, as is done in Exercise 3.

3 Assess

Formative Assessment

Use Exercises 13 and 14 to assess whether students understand the rules of simplifying radical expressions.

Ticket Out the Door

Make several copies each of five different operations involving two radical expressions. As students leave the room, ask them to determine whether the expressions can be combined.

1 Focus

Objective Use graphing technology to solve rational equations.

Materials

- TI-Nspire Technology

Teaching Tips

- For Activity 1 step 3, remind students that to enter $\frac{5}{x+2}$, they will need to put parentheses around the denominator. Then they will need to use the **tab** key to move the curser to the entry line to type $\frac{3}{x}$ into **f2*(x)*.**
- When changing the windows settings, use the **tab** key to move from one field to another.
- A spreadsheet program can also be used to complete Activity 2.
- For Activity 3, remind students that they cannot edit a line once they press **enter**. However, they can use **ctrl c** and **ctrl x** to copy and paste a line, and then make edits.

2 Teach

Working in Cooperative Groups

Divide the class into pairs. Work through Activity 1 as a class. Then ask students to work with their partners to complete Exercises 1 to 10 and Activities 2 and 3.

Practice Have students complete Exercises 11 to 15.

LAB 20

Graphing Technology Lab
Solving Rational Equations

You can use TI-Nspire Technology to solve rational equations by graphing, by using tables, and by using a computer algebra system (CAS).

Common Core State Standards
A.REI.11

To solve by graphing, graph both sides of the equation and locate the point(s) of intersection.

Activity 1 Solve a Rational Equation by Graphing

Solve $\frac{5}{x+2} = \frac{3}{x}$ by graphing.

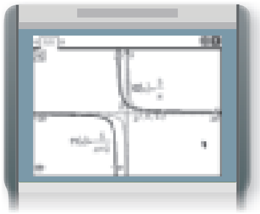
Step 1 Add a new **Graphs** page.

Step 2 Use the **Window Settings** option from the **Window/Zoom** menu to adjust the window to −20 to 20 for both x and y. Set both scales to 2.

Step 3 Enter $\frac{5}{x+2}$ into **f1(*x*)** and $\frac{3}{x}$ into **f2(*x*)**.

Step 4 Change the thickness of the graph of **f1(*x*)** by selecting the graph of **f1(*x*)** and the **ctrl menu Attributes** option.

Step 5 Use the **Intersection Point(s)** tool from the **Points & Lines** menu to find the intersection of the two graphs. Select the graph of **f1(*x*) enter** and then the graph of **f2(*x*) enter**.



The graphs intersect at (3, 1). This means that $\frac{5}{x+2}$ and $\frac{3}{x}$ both equal 1 when $x = 3$. Thus, the solution of $\frac{5}{x+2} = \frac{3}{x}$ is $x = 3$.

Exercises

Use a graphing calculator to solve each equation.

1. $\frac{5}{x} + \frac{4}{x} = 10$ **$\frac{9}{10}$**
2. $\frac{12}{x} + \frac{3}{4} = \frac{3}{2}$ **16**
3. $\frac{6}{x} + \frac{3}{2x} = 12$ **$\frac{5}{8}$**
4. $\frac{4}{x} + \frac{3}{4x} = \frac{1}{8}$ **38**
5. $\frac{4}{x} + \frac{x-2}{2x} = x$ **$-\frac{3}{2}$ or 2**
6. $\frac{3}{3x-2} + \frac{5}{x} = 0$ **$\frac{5}{9}$**
7. $\frac{2x+1}{2} + \frac{3}{2x} = \frac{2}{x}$ **−1 or $\frac{1}{2}$**
8. $\frac{x}{x+2} + x = \frac{5x+8}{x+2}$ **4**
9. $\frac{1}{2x} + \frac{5}{x} = \frac{3}{x-1}$ **$\frac{11}{5}$**
10. $\frac{4x-3}{x-2} + \frac{2x+5}{x-2} = 6$ **no solution**

Activity 2 Solve a Rational Equation by Using a Table

Solve $\frac{2x+1}{3} = \frac{x+2}{2}$ using a table.

Step 1 Add a new **Lists & Spreadsheet** page.

Step 2 Label column A as x. Enter values from −4 to 4 in cells A1 to A9.

Step 3 In column B in the formula row, enter the left side of the rational equation, with parenthesis around the binomials. In column C in the formula row, enter the right side of the rational equation, with parenthesis around the binomials. Specify **Variable Reference** when prompted.

Scroll until you see where the values in Columns B and C are equal. This occurs at $x = 4$. Therefore the solution of $\frac{2x+1}{3} = \frac{x+2}{2}$ is 4.

You can also use a computer algebra system (CAS) to solve rational equations.

Activity 3 Solve a Rational Equation by Using a CAS

Solve $\frac{x-3}{x} - \frac{x-4}{x-2} = \frac{1}{x}$ using a CAS.

Step1 Add a new **Calculator** page.

Step 2 To solve, select the **Solve** tool from the **Algebra** menu. Enter the left side of the equation with parenthesis around the binomials. Enter = and the right side of the equation. Then type a comma, followed by x, and then **enter**.

The solution of 4 is displayed.

Exercises

Use a table or CAS to solve each equation.

11. $\frac{2}{x} + \frac{2+x}{2} = \frac{x+3}{2}$ **4**

12. $\frac{4}{x-2} = -\frac{1}{x+3}$ **−2**

13. $\frac{3}{x+2} + \frac{4}{x-1} = 0$ $\mathbf{-\frac{5}{7}}$

14. $\frac{1}{x+1} + \frac{2}{x-1} = 0$ $\mathbf{-\frac{1}{3}}$

15. $\frac{2}{x+4} + \frac{4}{x-1} = 0$ $\mathbf{-\frac{7}{3}}$

16. $\frac{1}{x-2} + \frac{x+2}{4} = 2x$ **0 or** $\mathbf{\frac{16}{7}}$

17. $\frac{2x}{x+3} + \frac{x+1}{2} = x$ **−1 or 3**

18. $\frac{2}{x-3} + \frac{3}{x-2} = \frac{4}{x}$ $\mathbf{\frac{-7 \pm \sqrt{145}}{2}}$

19. $\frac{x^2}{x+1} + \frac{x}{x-1} = x$ **0**

3 Assess

Tips for New Teachers

Using Tables Point out to students that the table method only works when their table includes the *x*-value(s) of the solution(s). If a solution is not found with a table with integer values of *x*, students should adjust their *x*-values or use another method to find any solutions.

Formative Assessment

Use Exercises 16–19 to assess each student's knowledge of solving rational equations.

From Concrete to Abstract

Ask students to summarize the use of technology to find the solutions to rational equations.

1 Focus

VerticalAlignment

Before Lesson 21 You calculated measures of central tendency and variation.

Lesson 21 Describe the shape of a distribution. Use the shapes of distributions to select appropriate statistics.

After Lesson 21 Compare sets of data using appropriate statistics.

2 Teach

Scaffolding Questions

Have students read the **Why?** section of the lesson.

Ask:

- Why is Sarah's average time so much different than her median time? Sample answer: The times that Sarah recorded after pulling a muscle in her back probably raised her average time. Her median time was less affected by the higher times.
- Which time is a better representation of Sarah's 100-meter dash time? Explain. Sample answer: Her median time is probably a better representation since it is less affected by any outliers or times that are not a true reflection of Sarah's normal times.

LESSON 21 Distributions of Data

Then

- You calculated measures of central tendency and variation.

Now

1 Describe the shape of a distribution.

2 Use the shapes of distributions to select appropriate statistics.

Why?

- While training for the 100-meter dash, Sarah pulled a muscle in her lower back. After being cleared for practice, she continued to train. Sarah's median time was about 12.34 seconds, but her average time dropped to about 12.53 seconds.

NewVocabulary

distribution
negatively skewed distribution
symmetric distribution
positively skewed distribution

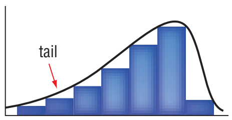
Common Core State Standards
S.ID.1, S.ID.2, S.ID.3

1 Describing Distributions

A **distribution** of data shows the observed or theoretical frequency of each possible data value. Recall that a histogram is a type of bar graph used to display data that have been organized into equal intervals. A histogram is useful when viewing the overall distribution of the data within a set over its range. You can see the shape of the distribution by drawing a curve over the histogram.

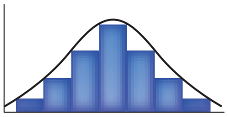
KeyConcept Symmetric and Skewed Distributions

Negatively Skewed Distribution



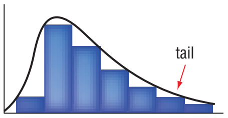
The majority of the data are on the right.

Symmetric Distribution



The data are evenly distributed.

Positively Skewed Distribution



The majority of the data are on the left.

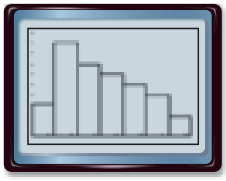
Example 1 Distribution Using a Histogram

Use a graphing calculator to construct a histogram for the data, and use it to describe the shape of the distribution.

25, 22, 31, 25, 26, 35, 18, 39, 22, 32, 34, 26, 42, 23, 40, 36, 18, 30
26, 30, 37, 23, 19, 33, 24, 29, 39, 21, 43, 25, 34, 24, 26, 30, 21, 22

First, press STAT ENTER and enter each data value. Then, press 2nd [STAT PLOT] ENTER ENTER and choose ılıl. Press ZOOM [ZoomStat] to adjust the window.

The graph is high on the left and has a tail on the right. Therefore, the distribution is positively skewed.



[17, 45] scl: 4 by [0, 10] scl: 1

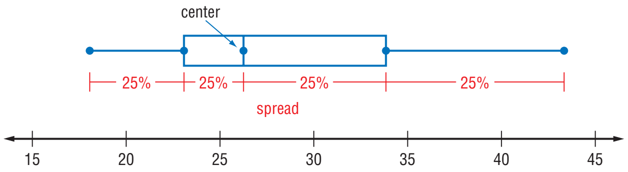
GuidedPractice

1. Use a graphing calculator to construct a histogram for the data, and use it to describe the shape of the distribution. **See Answer Appendix.**

8, 11, 15, 25, 21, 26, 20, 12, 32, 20, 31, 14, 19, 27, 22, 21, 14, 8
6, 23, 18, 16, 28, 25, 16, 20, 29, 24, 17, 35, 20, 27, 10, 16, 22, 12

Matthew Leete/The Image Bank/Getty Images

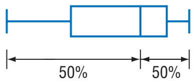
A box-and-whisker plot can also be used to identify the shape of a distribution. Recall from Lesson 0-13 that a box-and-whisker plot displays the spread of a data set by dividing it into four quartiles. The data from Example 1 are displayed below.



Notice that the left whisker is shorter than the right whisker, and that the line representing the median is closer to the left whisker. This represents a peak on the left and a tail to the right.

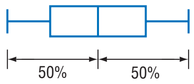
KeyConcept Symmetric and Skewed Box-and-Whisker Plots

Negatively Skewed



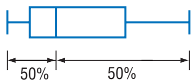
The left whisker is longer than the right. The median is closer to the shorter whisker.

Symmetric



The whiskers are the same length. The median is in the center of the data.

Positively Skewed



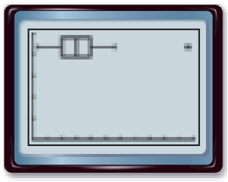
The right whisker is longer than the left. The median is closer to the shorter whisker.

Example 2 Distribution Using a Box-and-Whisker Plot

Use a graphing calculator to construct a box-and-whisker plot for the data, and use it to determine the shape of the distribution.

9, 17, 15, 10, 16, 2, 17, 19, 10, 18, 14, 8, 20, 20, 3, 21, 12, 11
5, 26, 15, 28, 12, 5, 27, 26, 15, 53, 12, 7, 22, 11, 8, 16, 22, 15

Enter the data as L1. Press 2nd [STAT PLOT] ENTER ENTER and choose ⊢□⊣. Adjust the window to the dimensions shown.



[0, 55] scl: 5 by [0, 5] scl: 1

The lengths of the whiskers are approximately equal, and the median is in the middle of the data. This indicates that the data are equally distributed to the left and right of the median. Thus, the distribution is symmetric.

StudyTip

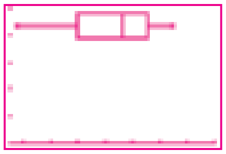
Outliers In Example 2, notice that the outlier does not affect the shape of the distribution.

GuidedPractice

2. Use a graphing calculator to construct a box-and-whisker plot for the data, and use it to describe the shape of the distribution.

40, 50, 35, 48, 43, 31, 52, 42, 54, 38, 50, 46, 49, 43, 40, 50, 32, 53
51, 43, 47, 41, 49, 50, 34, 54, 51, 44, 54, 39, 47, 35, 51, 44, 48, 37

2.



[30, 60] scl: 4 by [0, 5] scl: 1

negatively skewed

1 Describing Distributions

Example 1 shows how to describe the shape of a distribution using a histogram. **Example 2** shows how to describe the shape of a distribution using a box-and-whisker plot.

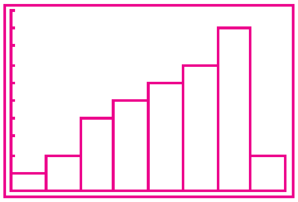
Formative Assessment

Use the Guided Practice exercises after each example to determine students' understanding of concepts.

Additional Examples

1 Use a graphing calculator to construct a histogram for the data, and use it to describe the shape of the distribution.

9, 18, 22, 12, 24, 25, 19, 25, 2
5, 28, 12, 22, 19, 28, 15, 23, 6
8, 27, 17, 14, 22, 21, 13, 24, 21
9, 25, 16, 24, 16, 25, 27, 21, 10

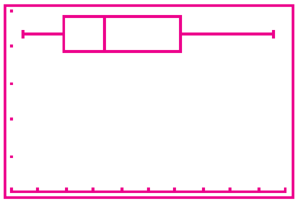


[0, 32] scl: 4 by [0, 10] scl: 1

negatively skewed

2 Use a graphing calculator to construct a box-and-whisker plot for the data, and use it to determine the shape of the distribution.

9, 18, 22, 12, 24, 25, 19, 25, 2
5, 28, 12, 22, 19, 28, 15, 23, 6
8, 27, 17, 14, 22, 21, 13, 24, 21
9, 25, 16, 24, 16, 25, 27, 21, 10



[50, 100] scl: 5 by [0, 5] scl: 1

positively skewed

DifferentiatedInstruction AL OL BL

Interpersonal Learners Have students work in pairs to think of ways to help them remember how to recognize the different shapes of the distributions from histograms and box-and-whisker plots.

2 Analyzing Distributions

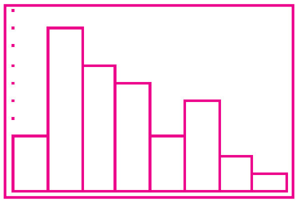
Example 3 shows how to choose appropriate statistics to describe a set of data using a histogram. **Example 4** shows how to choose appropriate statistics to describe a set of data using a box-and-whisker plot.

Additional Example

3 Describe the center and spread of the data using either the mean and standard deviation or the five-number summary. Justify your choice by constructing a histogram for the data.

78, 68, 72, 71, 79, 67, 71, 78, 70
80, 76, 82, 82, 70, 84, 72, 71, 85
67, 86, 74, 86, 73, 72, 77, 87, 70
66, 88, 75, 72, 76, 71, 90, 69, 94

Sample answer: The distribution is skewed, so use the five-number summary. The range is 94 – 66 or 28. The median is 74.5, and half of the data are between 71 and 82.



[64, 96] scl: 4 by [0, 10] scl: 1

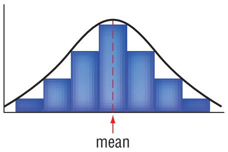
Additional Examples also in Interactive Classroom PowerPoint® Presentations

IWB Interactive White Board READY

2 Analyzing Distributions

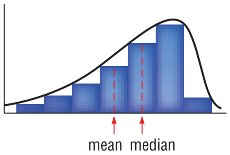
You have learned that data can be described using statistics. The mean and median describe the center. The standard deviation and quartiles describe the spread. You can use the shape of the distribution to choose the most appropriate statistics that describe the center and spread of a set of data.

When a distribution is symmetric, the mean accurately reflects the center of the data. However, when a distribution is skewed, this statistic is not as reliable.

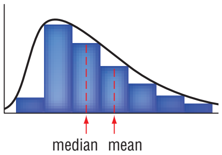


In Lesson 0-12, you discovered that outliers can have a strong effect on the mean of a data set, while the median is less affected. So, when a distribution is skewed, the mean lies away from the majority of the data toward the tail. The median is less affected and stays near the majority of the data.

Negatively Skewed Distribution



Positively Skewed Distribution



When choosing appropriate statistics to represent a set of data, first determine the shape of the distribution.

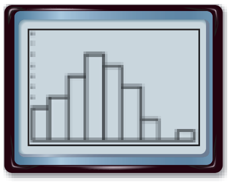
- If the distribution is relatively symmetric, the mean and standard deviation can be used.
- If the distribution is skewed or has outliers, use the five-number summary.

Example 3 Choose Appropriate Statistics

Describe the center and spread of the data using either the mean and standard deviation or the five-number summary. Justify your choice by constructing a histogram for the data.

21, 28, 16, 30, 25, 34, 21, 47, 18, 36, 24, 28, 30, 15, 33, 24, 32, 22
27, 38, 23, 29, 15, 27, 33, 19, 34, 29, 23, 26, 19, 30, 25, 13, 20, 25

Use a graphing calculator to create a histogram. The graph is high in the middle and low on the left and right. Therefore, the distribution is symmetric.



[12, 48] scl: 4 by [0, 10] scl: 1

The distribution is symmetric, so use the mean and standard deviation to describe the center and spread. Press STAT ▶ ENTER ENTER.

1-Var Stats

The mean $\overline{x}$ is about 26.1 with standard deviation σ of about 7.1.

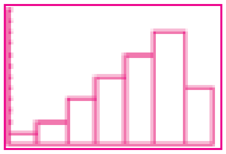
TechnologyTip

Bin Width On a graphing calculator, each bar is called a *bin*. The width of each bin can be adjusted by pressing WINDOW and changing Xscl. View the histogram using different bin widths and compare the results to determine the appropriate bin width.

DifferentiatedInstruction OL BL

Interpersonal Learners Have students work in pairs to generate a set of data, create a histogram using the data, and describe the shape of the distribution. Then have the students describe the center and spread of the data using either the mean and standard deviation or the five-number summary.

3. Sample answer: The distribution is skewed, so use the five-number summary. The range is 32 – 2 or 30. The median is 24, and half of the data are between 18 and 26.



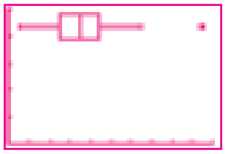
[0, 35] scl: 5 by [0, 12] scl: 1

Real-WorldLink

Volunteers in the Peace Corps must be at least 18 years old, and more than 90% of volunteers have college degrees. Volunteers work in another country for 27 months and are placed in host countries that have the greatest needs for skilled volunteers.

Source: Peace Corps

4. Sample answer: The distribution is symmetric, so use the mean and standard deviation to describe the center and spread. The mean amount raised was $36.70 with a standard deviation of about $18.58.



[0, 100] scl: 10 by [0, 5] scl: 1

Paul Burns/Digital Vision/Getty Images

GuidedPractice

3. Describe the center and spread of the data using either the mean and standard deviation or the five-number summary. Justify your choice by creating a histogram for the data.

19, 2, 25, 14, 24, 20, 27, 30, 14, 25, 19, 32, 21, 31, 25, 16, 24, 22
29, 6, 26, 32, 17, 26, 24, 26, 32, 10, 28, 19, 26, 24, 11, 23, 19, 8

A box-and-whisker plot is helpful when viewing a skewed distribution since it is constructed using the five-number summary.

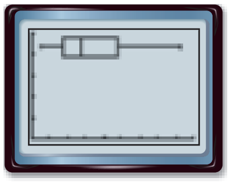
Real-World Example 4 Choose Appropriate Statistics

COMMUNITY SERVICE **The number of community service hours each of Ms. Tucci's students completed is shown. Describe the center and spread of the data using either the mean and standard deviation or the five-number summary. Justify your choice by constructing a box-and-whisker plot for the data.**

Community Service Hours

6	13	8	7	19	12	2	19	11	22	7	33	13
3	8	10	5	25	16	6	14	7	20	10	30	

Use a graphing calculator to create a box-and-whisker plot. The right whisker is longer than the left and the median is closer to the left whisker. Therefore, the distribution is positively skewed.



[0, 36] scl: 4 by [0, 5] scl: 1

The distribution is positively skewed, so use the five-number summary. The range is 33 – 2 or 31. The median number of hours completed is 11, and half of the students completed between 7 and 19 hours.

```
1-Var Stats
↑n=25
 minX=2
 Q1=7
 Med=11
 Q3=19
 maxX=33
```

GuidedPractice

4. **FUNDRAISER** The money raised per student in Mr. Bulanda's 5th period class is shown. Describe the center and spread of the data using either the mean and standard deviation or the five-number summary. Justify your choice by creating a box-and-whisker plot for the data.

Money Raised per Student (dollars)

41	27	52	18	42	32	16	95	27	65
36	45	5	34	50	15	62	38	57	20
38	21	33	58	25	42	31	8	40	28

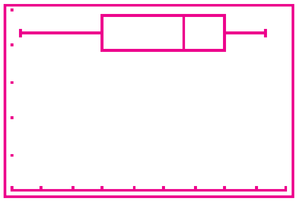
Additional Example

4 **BOWLING** The averages for the bowlers on five teams are shown below. Describe the center and spread of the data using either the mean and standard deviation or the five-number summary. Justify your choice by constructing a box-and-whisker plot for the data.

Bowling Average

142	180	161	131	201
179	152	177	196	148
198	123	203	170	187
159	193	176	137	183

Sample answer: The distribution is skewed, so use the five-number summary. The range is 203 – 123 or 80. The median bowling average is 176.5, and half of the bowlers have an average between 150 and 190.



[120, 210] scl: 10 by [0, 5] scl: 1

3 Practice

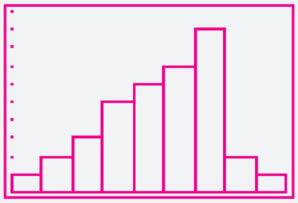
Formative Assessment

Use Exercises 1–4 to check for understanding.

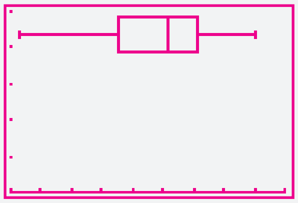
Use the chart at the bottom of this page to customize assignments for your students.

Additional Answers

1.



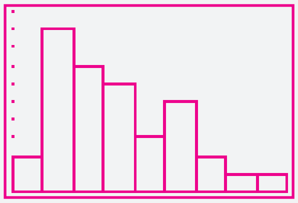
[56, 92] scl: 4 by [0, 10] scl: 1



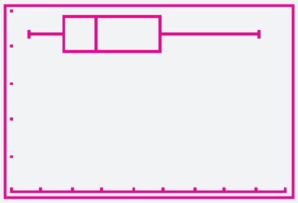
[56, 92] scl: 4 by [0, 5] scl: 1

negatively skewed

2.



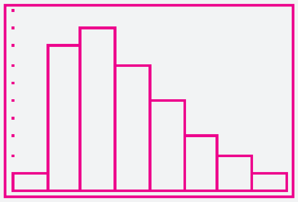
[10, 100] scl: 10 by [0, 10] scl: 1



[10, 100] scl: 10 by [0, 5] scl: 1

positively skewed

3. Sample answer: The distribution is skewed, so use the five-number summary. The range is 92 − 52 or 40. The median is 65, and half of the data are between 59.5 and 74.



[48, 96] scl: 6 by [0, 10] scl: 1

Check Your Understanding

Examples 1–2 **Use a graphing calculator to construct a histogram and a box-and-whisker plot for the data. Then describe the shape of the distribution.** **1–2. See margin.**

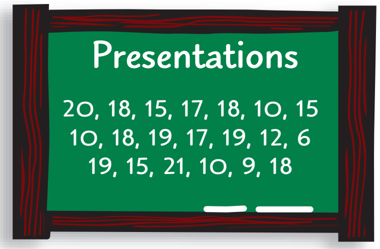
1. 80, 84, 68, 64, 57, 88, 61, 72, 76, 80, 83, 77, 78, 82, 65, 70, 83, 78
73, 79, 70, 62, 69, 66, 79, 80, 86, 82, 73, 75, 71, 81, 74, 83, 77, 73

2. 30, 24, 35, 84, 60, 42, 29, 16, 68, 47, 22, 74, 34, 21, 48, 91, 66, 51
33, 29, 18, 31, 54, 75, 23, 45, 25, 32, 57, 40, 23, 32, 47, 67, 62, 23

Example 3 **Describe the center and spread of the data using either the mean and standard deviation or the five-number summary. Justify your choice by constructing a histogram for the data.** **See margin.**

3. 58, 66, 52, 75, 60, 56, 78, 63, 59, 54, 60, 67, 72, 80, 68, 88, 55, 60
59, 61, 82, 70, 67, 60, 58, 86, 74, 61, 92, 76, 58, 62, 66, 74, 69, 64

Example 4 **4.** **PRESENTATIONS** The length of the students' presentations in Ms. Monroe's 2nd period class are shown. Describe the center and spread of the data using either the mean and standard deviation or the five-number summary. Justify your choice by constructing a box-and-whisker plot for the data. **See Answer Appendix.**



Presentations

20, 18, 15, 17, 18, 10, 15
10, 18, 19, 17, 19, 12, 6
19, 15, 21, 10, 9, 18

Practice and Problem Solving

Examples 1–2 **Use a graphing calculator to construct a histogram and a box-and-whisker plot for the data. Then describe the shape of the distribution.** **5–6. See Answer Appendix.**

5. 55, 65, 70, 73, 25, 36, 33, 47, 52, 54, 55, 60, 45, 39, 48, 55, 46, 38
50, 54, 63, 31, 49, 54, 68, 35, 27, 45, 53, 62, 47, 41, 50, 76, 67, 49

6. 42, 48, 51, 39, 47, 50, 48, 51, 54, 46, 49, 36, 50, 55, 51, 43, 46, 37
50, 52, 43, 40, 33, 51, 45, 53, 44, 40, 52, 54, 48, 51, 47, 43, 50, 46

Example 3 **Describe the center and spread of the data using either the mean and standard deviation or the five-number summary. Justify your choice by constructing a histogram for the data.** **7–8. See Answer Appendix.**

7. 32, 44, 50, 49, 21, 12, 27, 41, 48, 30, 50, 23, 37, 16, 49, 53, 33, 25
35, 40, 48, 39, 50, 24, 15, 29, 37, 50, 36, 43, 49, 44, 46, 27, 42, 47

8. 82, 86, 74, 90, 70, 81, 89, 88, 75, 72, 69, 91, 96, 82, 80, 78, 74, 94
85, 77, 80, 67, 76, 84, 80, 83, 88, 92, 87, 79, 84, 96, 85, 73, 82, 83

Example 4 **9.** **WEATHER** The daily low temperatures for New Carlisle over a 30-day period are shown. Describe the center and spread of the data using either the mean and standard deviation or the five-number summary. Justify your choice by constructing a box-and-whisker plot for the data. **See Answer Appendix.**

Temperature (°F)														
48	50	55	53	57	53	44	61	57	49	51	58	46	54	57
50	55	47	57	48	58	53	49	56	59	52	48	55	53	51

Differentiated Homework Options

Level	Assignment	Two-Day Option	
AL Basic	5–9, 15–17	5–9 odd	6, 8, 15–17
OL Core	5–11, 15–17	5–9	10, 11, 15–17
BL Advanced	10–17		

B **10.** **TRACK** Refer to the beginning of the lesson. Sarah's 100-meter dash times are shown. **a–c. See Answer Appendix.**

100-meter dash (seconds)				
12.20	12.35	13.60	12.24	12.72
12.18	12.06	12.41	12.28	13.06
12.87	12.04	12.38	12.20	13.12
12.30	13.27	12.93	12.16	12.02
12.50	12.14	11.97	12.24	13.09
12.46	12.33	13.57	11.96	13.34

a. Use a graphing calculator to create a box-and-whisker plot. Describe the center and spread of the data.

b. Sarah's slowest time prior to pulling a muscle was 12.50 seconds. Use a graphing calculator to create a box-and-whisker plot that *does not* include the times that she ran after pulling the muscle. Then describe the center and spread of the new data set.

c. What effect does removing the times recorded after Sarah pulled a muscle have on the shape of the distribution and on how you should describe the center and spread?

11. **MENU** The prices for entrees at a restaurant are shown.

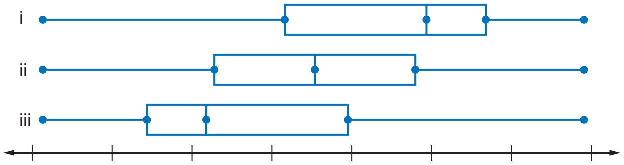
Entree Prices ($)				
9.00	11.25	16.50	9.50	13.00
18.50	7.75	11.50	13.75	9.75
8.00	16.50	12.50	10.25	17.75
13.00	10.75	16.75	8.50	11.50

a. Use a graphing calculator to create a box-and-whisker plot. Describe the center and spread of the data. **a, b. See Answer Appendix.**

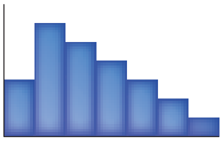
b. The owner of the restaurant decides to eliminate all entrees that cost more than $15. Use a graphing calculator to create a box-and-whisker plot that reflects this change. Then describe the center and spread of the new data set.

H.O.T. Problems Use Higher-Order Thinking Skills

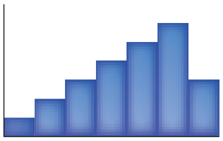
C **CHALLENGE** **Identify the histogram that corresponds to each of the following box-and-whisker plots.**



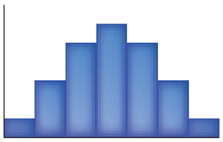
12. **iii**



13. **i**



14. **ii**



15. **WRITING IN MATH** Research and write a definition for a *bimodal distribution*. How can the measures of center and spread of a bimodal distribution be described? **15–17. See Answer Appendix.**

16. **OPEN ENDED** Give an example of a set of real-world data with a distribution that is symmetric and one with a distribution that is not symmetric.

17. **WRITING IN MATH** Explain why the mean and standard deviation are used to describe the center and spread of a symmetrical distribution and the five-number summary is used to describe the center and spread of a skewed distribution.

4 Assess

Ticket Out the Door Have students explain the difference between *positively skewed, negatively skewed,* and *symmetric* sets of data and give an example of each.

DifferentiatedInstruction OL BL

Extension Have students generate a set of data that is neither skewed nor symmetrical. Discuss situations that may result in data that resembles such distributions.

1 Focus

VerticalAlignment

Before Lesson 22 You calculated measures of central tendency and variation.

Lesson 22 Determine the effect that transformations of data have on measures of central tendency and variation.

After Lesson 22 Summarize data from simulations.

2 Teach

Scaffolding Questions

Have students read the **Why?** section of the lesson.

Ask:

- How can Tom calculate the effect of each bonus? Sample answer: For the extra $5 per day, Tom can add $5 to each daily total. For the 10% increase, Tom can multiply each daily total by 1.10.
- When is it to Tom's benefit that he accepts the extra $5 over the 10% increase and vice versa? Explain. Sample answer: On days when Tom makes less than $50, he should take the extra $5 because 10% of any value less than $50 will be less than $5. On days when Tom makes more than $50, he should take the 10% increase because 10% of any value greater than $50 will be greater than $5. When Tom makes $50, the bonuses are equal.

LESSON 22 Comparing Sets of Data

Then

- You calculated measures of central tendency and variation.

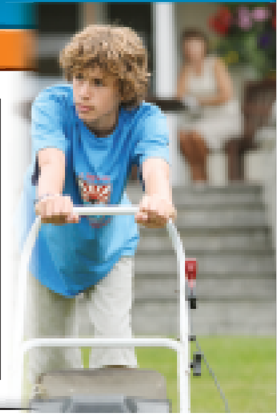
Now

1. Determine the effect that transformations of data have on measures of central tendency and variation.
2. Compare data using measures of central tendency and variation.

Why?

- Tom gets paid hourly to do landscaping work. Because he is such a good employee, Tom is planning to ask his boss for a bonus. Tom's initial pay for a month is shown. He is trying to decide whether he should ask for an extra $5 per day or a 10% increase in his daily wages.

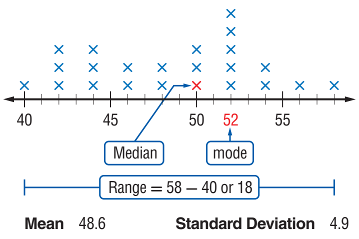
Tom's Pay ($)		
44	52	50
40	48	46
44	52	54
58	42	52
54	50	52
42	52	46
56	48	44
50	42	



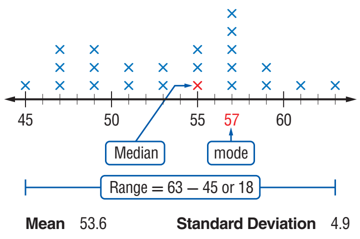
NewVocabulary
linear transformation

1 Transformations of Data To see the effect that an extra $5 per day would have on Tom's daily pay, we can find the new daily pay values and compare the measures of center and variation for the two sets of data. The new data can be found by performing a *linear transformation*. A **linear transformation** is an operation performed on a data set that can be written as a linear function. Tom's daily pay after the $5 bonus can be found using $y = 5 + x$, where x represents his original daily pay and y represents his daily pay after the bonus.

Tom's Earnings Before Extra $5



Tom's Earnings With Extra $5



Notice that each value was translated 5 units to the right. Thus, the mean, median, and mode increased by 5. Since the new minimum and maximum values also increased by 5, the range remained the same. The standard deviation is unchanged because the amount by which each value deviates from the mean stayed the same.

These results occur when any positive or negative number is added to every value in a set of data.

KeyConcept Transformations Using Addition

If a real number k is added to every value in a set of data, then:

- the mean, median, and mode of the new data set can be found by adding k to the mean, median, and mode of the original data set, and
- the range and standard deviation will not change.

Peter Cade/Iconica/Getty Images

Common Core State Standards
S.ID.1, S.ID.2, S.ID.3

Technology Tip

1-Var Stats To quickly calculate the mean $\bar{x}$, median **Med**, standard deviation σ, and range of a data set, enter the data as **L1** in a graphing calculator, and then press STAT ▶ ENTER ENTER. Subtract **minX** from **maxX** to find the range.

Example 1 Transformation Using Addition

Find the mean, median, mode, range, and standard deviation of the data set obtained after adding 7 to each value.

13, 5, 8, 12, 7, 4, 5, 8, 14, 11, 13, 8

Method 1 Find the mean, median, mode, range, and standard deviation of the original data set.

Mean	9	Mode	8	Standard Deviation	3.3
Median	8	Range	10		

Add 7 to the mean, median, and mode. The range and standard deviation are unchanged.

Mean	16	Mode	15	Standard Deviation	3.3
Median	15	Range	10		

Method 2 Add 7 to each data value.

20, 12, 15, 19, 14, 11, 12, 15, 21, 18, 20, 15

Find the mean, median, mode, range, and standard deviation of the new data set.

Mean	16	Mode	15	Standard Deviation	3.3
Median	15	Range	10		

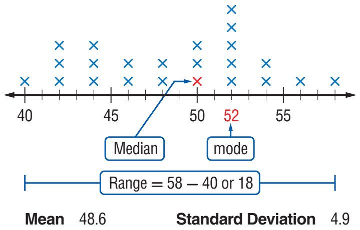
Guided Practice

1. Find the mean, median, mode, range, and standard deviation of the data set obtained after adding −4 to each value. **30, 32.5, 22, 38, 10.1**

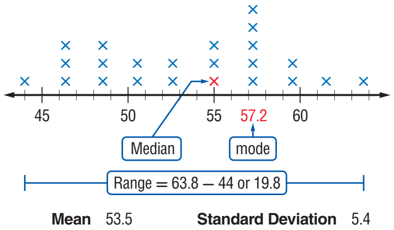
27, 41, 15, 36, 26, 40, 53, 38, 37, 24, 45, 26

To see the effect that a daily increase of 10% has on the data set, we can multiply each value by 1.10 and recalculate the measures of center and variation.

Tom's Earnings Before Extra 10%



Tom's Earnings With Extra 10%



Notice that each value did not increase by the same amount, but did increase by a factor of 1.10. Thus, the mean, median, and mode increased by a factor of 1.10. Since each value was increased by a constant percent and not by a constant amount, the range and standard deviation both changed, also increasing by a factor of 1.10.

Key Concept Transformations Using Multiplication

If every value in a set of data is multiplied by a constant k, $k > 0$, then the mean, median, mode, range, and standard deviation of the new data set can be found by multiplying each original statistic by k.

Differentiated Instruction AL OL BL ELL

Interpersonal Learners Have students work in pairs to perform transformations on data sets. Have each student create a data set and then determine an operation that should be done to each value. The students should exchange data sets, and then find the mean, median, mode, range, and standard deviation of the original data set and of the new data set after the operation is performed.

1 Transformations of Data

Example 1 shows how to calculate descriptive statistics for a set of data that has been transformed using addition. **Example 2** shows how to calculate descriptive statistics for a set of data that has been transformed using multiplication.

Formative Assessment

Use the Guided Practice exercises after each example to determine students' understanding of concepts.

Additional Examples

1 Find the mean, median, mode, range, and standard deviation of the data set obtained after adding 12 to each value.

73, 78, 61, 54, 88, 90, 63, 78, 80, 61, 86, 78 86.2, 90, 90, 36, 11.3

2 Find the mean, median, mode, range, and standard deviation of the data set obtained after multiplying each value by 2.5.

4, 2, 3, 1, 4, 6, 2, 3, 7, 5, 1, 4 mean: 8.75; median: 8.75; mode: 10; range: 15; standard deviation: 4.5

Additional Examples also in Interactive Classroom PowerPoint® Presentations

IWB **Interactive White Board READY**

Tips for New Teachers

Transformations Using Multiplication If every value in a set of data is multiplied by a negative constant k, then the mean, median, and mode of the new data set can be found by multiplying the mean, median, and mode of the original data set by k. The range and standard deviation of the new data set can be found by multiplying the range and standard deviation of the original data set by $|k|$. Students will explore this in Exercise 22.

2 Comparing Distributions

Example 3 shows how to use histograms to compare the distributions of two data sets. **Example 4** shows how to use box-and-whisker plots to compare the distributions of two data sets.

Additional Example

3 **GAMES** Brittany and Justin are playing a computer game. Their high scores for each game are shown below.

Brittany's Scores
29, 43, 54, 58, 39, 44, 39, 53, 32, 48, 39, 49, 38, 31, 41, 44, 44, 45, 48, 31

Justin's Scores
48, 26, 28, 53, 39, 28, 30, 58, 45, 37, 30, 31, 40, 32, 30, 44, 33, 35, 43, 35

a. Use a graphing calculator to create a histogram for each set of data. Then describe the shape of each distribution. See bottom margin for graphs; Brittany, symmetric; Justin, positively skewed

b. Compare the distributions using either the means and standard deviations or the five-number summaries. Justify your choice. Sample answer: One distribution is symmetric and the other is skewed, so use the five-number summaries. Both distributions have a maximum of 58, but Brittany's minimum score is 29 compared to Justin's minimum scores of 26. The median for Brittany's scores is 43.5 and the upper quartile for Justin's scores is 43.5. This means that 50% of Brittany's scores are between 43.5 and 58, while only 25% of Justin's scores fall within this range. Therefore, we can conclude that overall, Brittany's scores are higher than Justin's scores.

Since the medians for both bonuses are equal and the means are approximately equal, Tom should ask for the bonus that he thinks he has the best chance of receiving.

Example 2 Transformation Using Multiplication

Find the mean, median, mode, range, and standard deviation of the data set obtained after multiplying each value by 3.

21, 12, 15, 18, 16, 10, 12, 19, 17, 18, 12, 22

Find the mean, median, mode, range, and standard deviation of the original data set.

Mean	16	Mode	12	Standard Deviation	3.7
Median	16.5	Range	12		

Multiply the mean, median, mode, range, and standard deviation by 3.

Mean	48	Mode	36	Standard Deviation	11.1
Median	49.5	Range	36		

GuidedPractice

2. Find the mean, median, mode, range, and standard deviation of the data set obtained after multiplying each value by 0.8. **44, 44.4, 48, 13.6, 4.2**

63, 47, 54, 60, 55, 46, 51, 60, 58, 50, 56, 60

2 Comparing Distributions

Recall that when choosing appropriate statistics to represent data, you should first analyze the shape of the distribution. The same is true when comparing distributions.

- Use the mean and standard deviation to compare two symmetric distributions.
- Use the five-number summaries to compare two skewed distributions or a symmetric distribution and a skewed distribution.

Example 3 Compare Data Using Histograms

QUIZ SCORES **Robert and Elaine's quiz scores for the first semester of Algebra 1 are shown below.**

Robert's Quiz Scores	Elaine's Quiz Scores
85, 95, 70, 87, 78, 82, 84, 84, 85, 99, 88, 74, 75, 89, 79, 80, 92, 91, 96, 81	89, 76, 87, 86, 92, 77, 78, 83, 83, 82, 81, 82, 84, 85, 85, 86, 89, 93, 77, 85

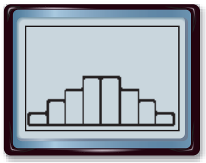
TechnologyTip

Histograms To create a histogram for a set of data in L2, press 2nd [STAT PLOT] ENTER ENTER, choose 📊, and enter L2 for Xlist.

a. **Use a graphing calculator to construct a histogram for each set of data. Then describe the shape of each distribution.**

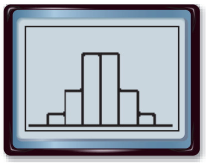
Enter Robert's quiz scores as L1 and Elaine's quiz scores as L2.

Robert's Quiz Scores



[69, 101] scl: 4 by [0, 8] scl: 1

Elaine's Quiz Scores



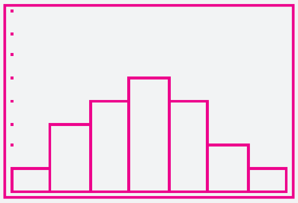
[69, 101] scl: 4 by [0, 8] scl: 1

Both distributions are high in the middle and low on the left and right. Therefore, both distributions are symmetric.

Additional Answer (Additional Example)

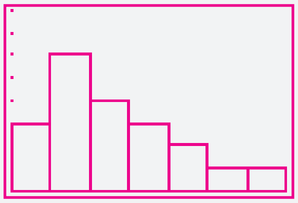
3a.

Brittany's Scores



[25, 60] scl: 5 by [0, 8] scl: 1

Justin's Scores

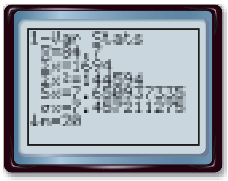


[25, 60] scl: 5 by [0, 8] scl: 1

b. Compare the data sets using either the means and standard deviations or the five-number summaries. Justify your choice.

Both distributions are symmetric, so use the means and standard deviations to describe the centers and spreads.

Robert's Quiz Scores



Elaine's Quiz Scores

The means for the students' quiz scores are approximately equal, but Robert's quiz scores have a much higher standard deviation than Elaine's quiz scores. This means that Elaine's quiz scores are generally closer to her mean than Robert's quiz scores are to his mean.

GuidedPractice

COMMUTE The students in two of Mr. Martin's classes found the average number of minutes that they each spent traveling to school each day.

3A. Use a graphing calculator to construct a histogram for each set of data. Then describe the shape of each distribution.

3B. Compare the data sets using either the means and standard deviations or the five-number summaries. Justify your choice. **See margin.**

2nd Period (minutes)
8, 4, 18, 7, 13, 26, 12, 6, 20, 5, 9, 24, 8, 16, 31, 13, 17, 10, 8, 22, 12, 25, 13, 11, 18, 12, 16, 22, 25, 33

7th Period (minutes)
21, 4, 20, 13, 22, 6, 10, 23, 13, 25, 14, 16, 19, 21, 19, 8, 20, 18, 9, 14, 21, 17, 19, 22, 4, 19, 21, 26

Box-and-whisker plots are useful for comparisons of data because they can be displayed on the same screen.

Real-World Example 4 Compare Data Using Box-and-Whisker Plots

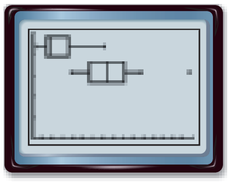
FOOTBALL **Kurt's total rushing yards per game for his junior and senior seasons are shown.**

Junior Season (yards)					
16	20	72	4	25	18
34	10	42	17	56	12

Senior Season (yards)					
77	54	109	60	156	72
39	83	73	101	46	80

a. Use a graphing calculator to construct a box-and-whisker plot for each set of data. Then describe the shape of each distribution.

Enter Kurt's rushing yards from his junior season as **L1** and his rushing yards from his senior season as **L2**. Graph both box-and-whisker plots on the same screen by graphing **L1** as **Plot1** and **L2** as **Plot2**.



[0, 160] scl: 10 by [0, 5] scl: 1

For Kurt's junior season, the right whisker is longer than the left, and the median is closer to the left whisker. The distribution is positively skewed.

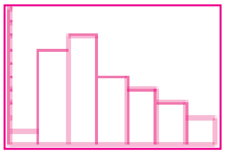
For Kurt's senior season, the lengths of the whiskers are approximately equal, and the median is in the middle of the data. The distribution is symmetric.

TechnologyTip

Multiple Data Sets In order to calculate statistics for a set of data in **L2**, press STAT ▶ ENTER 2nd [L2] ENTER.

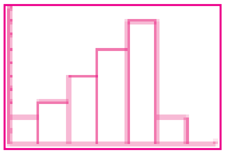
3A.

2nd Period



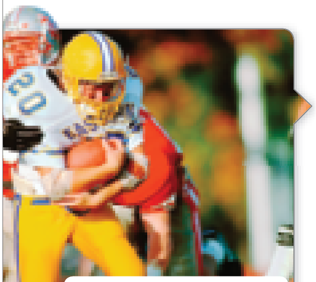
[0, 35] scl: 5 by [0, 10] scl: 1

7th Period



[0, 35] scl: 5 by [0, 10] scl: 1

2nd period, positively skewed; 7th period, negatively skewed



Real-World Link

Only about 2% of all high school student athletes attend a college with the aid of athletic scholarships.

Source: NCAA

CORBIS

Additional Answer

3B. Sample answer: The distributions are skewed, so use the five-number summaries. Both classes have a minimum of 4, but 2nd period has a median of 13 and 7th period has a lower quartile of 13. This means that the lower 50% of the data from 2nd period spans the same range as the lower 25% of the data from 7th period. The upper 50% of the data from 2nd period spans from 13 to 33, while the upper 75% of data from 7^{th} period spans from 13 to 26. Therefore, we can conclude that while the median for 7th period is significantly higher than the median for 2nd period, the data from 2nd period is dispersed over a wider range than the data from 7th period.

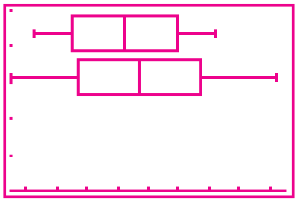
Additional Example

4 **FISHING** Steve and Kurt went fishing for the weekend. The weights of the fish they each caught are shown below.

Steve's Fish (pounds)
1.6, 2.1, 2.6, 1.3, 2.7, 3.2, 1.4, 2.3, 3.5, 1.9, 2.2, 2.7, 3.5, 1.4, 3.7, 3.4, 1.8, 2.5, 3.0

Kurt's Fish (pounds)
1.1, 3.2, 2.3, 3.7, 1.7, 2.7, 2.1, 4.0, 1.0, 2.9, 2.9, 1.2, 3.3, 2.3, 4.5, 2.4, 3.9

a. Use a graphing calculator to create a box-and-whisker plot for each data set. Then describe the shape of the distribution for each data set.



[1, 4.6] scl: 0.4 by [0, 5] scl: 1

both symmetric

b. Compare the distributions using either the means and standard deviations or the five-number summaries. Justify your choice. The distributions are symmetric, so use the means and standard deviations. The mean weight for Steve's fish is about 2.5 pounds with standard deviation of about 0.8 pound. The mean weight for Kurt's fish is about 2.7 pounds with standard deviation of about 1 pound. While the mean weight for Kurt's fish is greater, the weights of Kurt's fish also have more variability. This means the weights for Steve's fish are generally closer to his mean than the weights for Kurt's fish.

3 Practice

Formative Assessment

Use Exercises 1–6 to check for understanding.

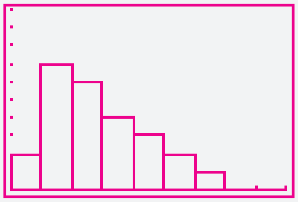
Use the chart at the bottom of this page to customize assignments for your students.

Additional Answer (Guided Practice)

4B. Sample answer: One distribution is symmetric and the other is skewed, so use the five-number summaries. The maximum for Vanessa's junior season is 21, while the median for her senior season is 22. This means that in half of her games during her senior season, Vanessa scored more points than in any of the games during her junior season. Therefore, we can conclude that overall, Vanessa scored more points during her senior season.

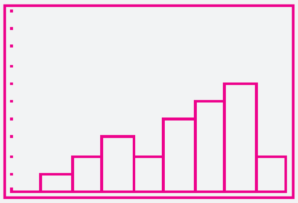
Additional Answers

5a. Kyle's Distances



[17, 19.25] scl: 0.25 by [0, 10] scl: 1

Mark's Distances



[17, 19.25] scl: 0.25 by [0, 10] scl: 1

Kyle, positively skewed; Mark, negatively skewed

5b. Sample answer: The distributions are skewed, so use the five-number summaries. Kyle's upper quartile is 17.98, while Mark's lower quartile is 18.065. This means that 75% of Mark's distances are greater than 75% of Kyle's distances. Therefore, we can conclude that overall, Mark's distances are higher than Kyle's.

b. **Compare the data sets using either the means and standard deviations or the five-number summaries. Justify your choice.**

One distribution is symmetric and the other is skewed, so use the five-number summaries to compare the data.

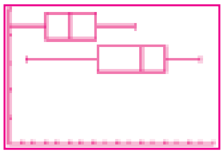
The upper quartile for Kurt's junior season was 38, while the minimum for his senior season was 39. This means that Kurt rushed for more yards in every game during his senior season than 75% of the games during his junior season.

The maximum for Kurt's junior season was 72, while his median for his senior season was 75. This means that in half of his games during his senior year, he rushed for more yards than in any game during his junior season. Overall, we can conclude that Kurt rushed for many more yards during his senior season than during his junior season.

StudyTip

Box-and-Whisker Plots Recall that a box-and-whisker plot displays the spread of a data set by dividing it into four quartiles. Each quartile accounts for 25% of the data.

4A.



[0, 34] scl: 2 by [0, 5] scl: 1

junior season, symmetric; senior season, negatively skewed

GuidedPractice

BASKETBALL The points Vanessa scored per game during her junior and senior seasons are shown.

4A. Use a graphing calculator to construct a histogram for each set of data. Then describe the shape of each distribution.

4B. Compare the data sets using either the means and standard deviations or the five-number summaries. Justify your choice. **See margin.**

Junior Season (points)
10, 12, 6, 10, 13, 8, 12, 3, 21, 14, 7, 0, 15, 6, 16, 8, 17, 3, 17, 2

Senior Season (points)
10, 32, 3, 22, 20, 30, 26, 24, 5, 22, 28, 32, 26, 21, 6, 20, 24, 18, 12, 25

Check Your Understanding

Example 1 **Find the mean, median, mode, range, and standard deviation of each data set that is obtained after adding the given constant to each value.**

1. 10, 13, 9, 8, 15, 8, 13, 12, 7, 8, 11, 12; + (−7)
3.5, 3.5, 1, 8, 2.4

2. 38, 36, 37, 42, 31, 44, 37, 45, 29, 42, 30, 42; + 23
60.8, 60.5, 65, 16, 5.5

Example 2 **Find the mean, median, mode, range, and standard deviation of each data set that is obtained after multiplying each value by the given constant.**

3. 6, 10, 3, 7, 4, 9, 3, 8, 5, 11, 2, 1; × 3
17.3, 16.5, 9, 30, 9.4

4. 42, 39, 45, 44, 37, 42, 38, 37, 41, 49, 42, 36; × 0.5
20.5, 20.8, 21, 6.5, 1.8

Example 3 **5.** **TRACK** Mark and Kyle's long jump distances are shown.

Kyle's Distances (ft)
17.2, 18.28, 18.56, 17.28, 17.36, 18.08, 17.43, 17.71, 17.46, 18.26, 17.51, 17.58, 17.41, 18.21, 17.34, 17.63, 17.55, 17.26, 17.18, 17.78, 17.51, 17.83, 17.92, 18.04, 17.91

Mark's Distances (ft)
18.88, 19.24, 17.63, 18.69, 17.74, 19.18, 17.92, 18.96, 18.19, 18.21, 18.46, 17.47, 18.49, 17.86, 18.93, 18.73, 18.34, 18.67, 18.56, 18.79, 18.47, 18.84, 18.87, 17.94, 18.7

a. Use a graphing calculator to construct a histogram for each set of data. Then describe the shape of each distribution. **a–b. See margin.**

b. Compare the data sets using either the means and standard deviations or the five-number summaries. Justify your choice.

Differentiated Homework Options

Level	Assignment	Two-Day Option	
AL Basic	7–18, 22–25	7–17 odd	8–18 even, 22–25
OL Core	7–17 odd, 19, 20, 22–25	7–18	19, 20, 22–25
BL Advanced	19–25		

Example 4

6. **TIPS** Miguel and Stephanie are servers at a restaurant. The tips that they earned to the nearest dollar over the past 15 workdays are shown.

Miguel's Tips ($)	Stephanie's Tips ($)
14, 68, 52, 21, 63, 32, 43, 35, 70, 37, 42, 16, 47, 38, 48	34, 52, 43, 39, 41, 50, 46, 36, 37, 47, 39, 49, 44, 36, 50

a. Use a graphing calculator to construct a box-and-whisker plot for each set of data. Then describe the shape of each distribution. **a, b. See margin.**

b. Compare the data sets using either the means and standard deviations or the five-number summaries. Justify your choice.

Practice and Problem Solving

Extra Practice is on page R12.

Example 1

Find the mean, median, mode, range, and standard deviation of each data set that is obtained after adding the given constant to each value.

7. 52, 53, 49, 61, 57, 52, 48, 60, 50, 47; + 8
60.9, 60, 60, 14, 4.7

8. 101, 99, 97, 88, 92, 100, 97, 89, 94, 90; + (−13)
81.7, 82.5, 84, 13, 4.5

9. 27, 21, 34, 42, 20, 19, 18, 26, 25, 33; + (−4)
22.5, 21.5, no mode, 24, 7.4

10. 72, 56, 71, 63, 68, 59, 77, 74, 76, 66; + 16
84.2, 85.5, no mode, 21, 6.8

Example 2

Find the mean, median, mode, range, and standard deviation of each data set that is obtained after multiplying each value by the given constant.

11. 11, 7, 3, 13, 16, 8, 3, 11, 17, 3; × 4
36.8, 38, 12, 56, 20.0

12. 64, 42, 58, 40, 61, 67, 58, 52, 51, 49; × 0.2
10.8, 11, 11.6, 5.4, 1.7

13. 33, 37, 38, 29, 35, 37, 27, 40, 28, 31; × 0.8
26.8, 27.2, 29.6, 10.4, 3.5

14. 1, 5, 4, 2, 1, 3, 6, 2, 5, 1; × 6.5
19.5, 16.3, 6.5, 32.5, 11.6

Example 3

15. **BOOKS** The page counts for the books that the students chose are shown.

1st Period	6th Period
388, 439, 206, 438, 413, 253, 311, 427, 258, 511, 283, 578, 291, 358, 297, 303, 325, 506, 331, 482, 343, 372, 456, 267, 484, 227	357, 294, 506, 392, 296, 467, 308, 319, 485, 333, 352, 405, 359, 451, 378, 490, 379, 401, 409, 421, 341, 438, 297, 440, 500, 312, 502

a. Use a graphing calculator to construct a histogram for each set of data. Then describe the shape of each distribution. **a, b. See margin.**

b. Compare the data sets using either the means and standard deviations or the five-number summaries. Justify your choice.

16. **TELEVISIONS** The prices for a sample of televisions are shown.

The Electronics Superstore	Game Central
46, 25, 62, 45, 30, 43, 40, 46, 33, 53, 35, 38, 39, 40, 52, 42, 44, 48, 50, 35, 32, 55, 28, 58	53, 49, 26, 61, 40, 50, 42, 35, 45, 48, 31, 48, 33, 50, 35, 55, 38, 50, 42, 53, 44, 54, 48, 58

a. Use a graphing calculator to construct a histogram for each set of data. Then describe the shape of each distribution. **a, b. See Answer Appendix.**

b. Compare the data sets using either the means and standard deviations or the five-number summaries. Justify your choice.

Example 4

17. **BRAINTEASERS** The time that it took Leon and Cassie to complete puzzles is shown.

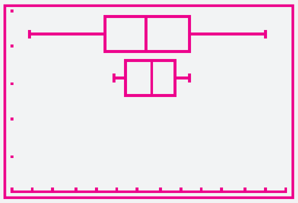
Leon's Times (minutes)	Cassie's Times (minutes)
4.5, 1.8, 3.2, 5.1, 2.0, 2.6, 4.8, 2.4, 2.2, 2.8, 1.8, 2.2, 3.9, 2.3, 3.3, 2.4	2.3, 5.8, 4.8, 3.3, 5.2, 4.6, 3.6, 5.7, 3.8, 4.2, 5.0, 4.3, 5.5, 4.9, 2.4, 5.2

a. Use a graphing calculator to construct a box-and-whisker plot for each set of data. Then describe the shape of each distribution. **a, b. See Answer Appendix.**

b. Compare the data sets using either the means and standard deviations or the five-number summaries. Justify your choice.

Additional Answers

6a.



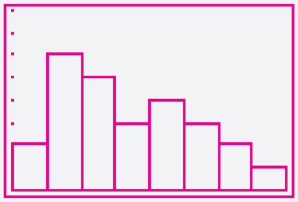
[10, 75] scl: 5 by [0, 5] scl: 1

both symmetric

6b. Sample answer: The distributions are symmetric, so use the means and standard deviations. The mean for Miguel's tips is about $41.73 with standard deviation of about $16.64. The mean for Stephanie's tips is about $42.87 with standard deviation of about $5.73. While the means only differ by $1.14, the standard deviations show that Stephanie's tips are generally closer to her mean than Miguel's tips.

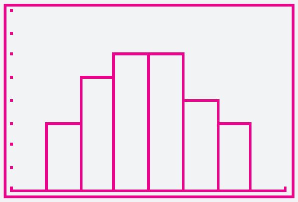
15a.

1st Period



[200, 600] scl: 50 by [0, 8] scl: 1

6th Period



[200, 600] scl: 50 by [0, 8] scl: 1

1st period, positively skewed; 6th period, symmetric

15b. Sample answer: One distribution is symmetric and the other is skewed, so use the five-number summaries. The lower quartile for 1st period is 291 pages, while the minimum for 6th period is 294 pages. This means that the lower 25% of data for 1st period is lower than any data from 6th period. The range for 1st period is 578 − 206 or 372 pages. The range for 6th period is 506 − 294 or 212 pages. The median for 1st period is about 351 pages, while the median for 6th period is 392 pages. This means that, while the median for 6th period is greater, 1st period's pages have a greater range and include greater values than 1st period.

18. **DANCE** The total amount of money that a sample of students spent to attend the homecoming dance is shown.

Boys (dollars)
114, 98, 131, 83, 91, 64, 94, 77, 96, 105, 72, 108, 87, 112, 58, 126

Girls (dollars)
124, 74, 105, 133, 85, 162, 90, 109, 94, 102, 98, 171, 138, 89, 154, 76

a. Use a graphing calculator to construct a box-and-whisker plot for each set of data. Then describe the shape of each distribution. **a, b. See Answer Appendix.**

b. Compare the data sets using either the means and standard deviations or the five-number summaries. Justify your choice.

Rhonda's Pay ($)		
45	55	53
47	53	54
44	56	59
63	47	53
60	57	62
44	50	45
60	53	49
62	47	

B 19. **LANDSCAPING** Refer to the beginning of the lesson. Rhonda, another employee that works with Tom, earned the following over the past month.

a. Find the mean, median, mode, range, and standard deviation of Rhonda's earnings. **52.96, 53, 53, 19, 6.06**

b. A $5 bonus had been added to each of Rhonda's daily earnings. Find the mean, median, mode, range, and standard deviation of Rhonda's earnings before the $5 bonus. **47.96, 48, 48, 19, 6.06**

20. **SHOPPING** The items Lorenzo purchased are shown.

a. Find the mean, median, mode, range, and standard deviation of the prices. **18.95, 16.05, no mode, 40.66, 11.62**

b. A 7% sales tax was added to the price of each item. Find the mean, median, mode, range, and standard deviation of the items without the sales tax. **17.71, 15, no mode, 38, 10.86**

Baseball hat	$14.98
Jeans	$24.61
T-shirt	$12.84
T-shirt	$16.05
Backpack	$42.80
Folders	$2.14
Sweatshirt	$19.26

H.O.T. Problems Use Higher-Order Thinking Skills

C 21. **CHALLENGE** A salesperson has 15 SUVs priced between $33,000 and $37,000 and 5 luxury cars priced between $44,000 and $48,000. The average price for all of the vehicles is $39,250. The salesperson decides to reduce the prices of the SUVs by $2000 per vehicle. What is the new average price for all of the vehicles? **$37,750**

22. **REASONING** If every value in a set of data is multiplied by a constant k, $k < 0$, then how can the mean, median, mode, range, and standard deviation of the new data set be found? **See Answer Appendix.**

23. **WRITING IN MATH** Compare and contrast the benefits of displaying data using histograms and box-and-whisker plots. **23–25. See Answer Appendix.**

24. **REASONING** If k is added to every value in a set of data, and then each resulting value is multiplied by a constant m, $m > 0$, how can the mean, median, mode, range, and standard deviation of the new data set be found? Explain your reasoning.

25. **WRITING IN MATH** Explain why the mean and standard deviation are used to compare the center and spread of two symmetrical distributions and the five-number summary is used to compare the center and spread of two skewed distributions or a symmetric distribution and a skewed distribution.

WatchOut!

Transformations In Exercises 19 and 20, the data given has already been transformed. Students will need to subtract $5 from each of Rhonda's daily earnings, and will need to subtract 7% sales tax from each item Lorenzo purchased.

4 Assess

Ticket Out the Door Have students summarize how the mean, median, mode, range, and standard deviation of a new data set can be found after a transformation is performed on the original data set.

DifferentiatedInstruction OL BL

Extension Have students research the Internet to find data about two cities in the United States. This could include population, median household incomes, or other data. Ask students to make box-and-whisker plots of each data set and compare them. Their analyses should include a comparison using either the means and standard deviations or the five-number summaries.

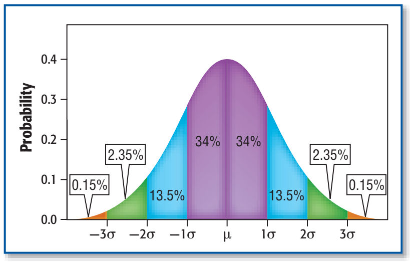
LAB 23

Graphing Technology Lab
The Normal Curve

Common Core State Standards
S.ID.2

When there are a large number of values in a data set, the frequency distribution tends to cluster around the mean of the set in a distribution (or shape) called a **normal distribution**. The graph of a normal distribution is called a **normal curve**. Since the shape of the graph resembles a bell, the graph is also called a *bell curve*.

Data sets that have a normal distribution include reaction times of drivers that are the same age, achievement test scores, and the heights of people that are the same age.



You can use a graphing calculator to graph and analyze a normal distribution if the mean and standard deviation of the data are known.

Activity 1 Graph a Normal Distribution

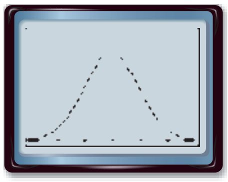
HEIGHT **The mean height of 15-year-old boys in the city where Isaac lives is 67 inches, with a standard deviation of 2.8 inches. Use a normal distribution to represent these data.**

Step 1 Set the viewing window. WINDOW

- Xmin = 67 – 3 × 2.8 ENTER **58.6**
- Xmax = 67 + 3 × 2.8 ENTER **75.4**
- Xscl = 2.8 ENTER
- Ymin = 0 ENTER
- Ymax = 1 ÷ (2 × 2.8) ENTER **.17857142...**
- Yscale = 1 ENTER

Step 2 By entering the mean and standard deviation into the calculator, we can graph the corresponding normal curve. Enter the values using the following keystrokes.

KEYSTROKES: Y= 2nd **[DISTR]** ENTER X,T,θ,*n* , 67 , 2.8) GRAPH



[58.6, 75.4] scl: 2.8 by [0, 0.17857142] scl: 1

(continued on the next page)

1 Focus

Objective Use a graphing calculator to explore normal distribution curves.

Materials for Each Student

- TI-83/84 Plus or other graphing calculator

Teaching Tip

Before starting students on the activity, discuss normal distributions using the graphic on the page. Some other examples of normally distributed random variables are lengths of newborn babies, blood pressure, lengths of objects made by machines, and useful lives of some manufactured items.

2 Teach

Working in Cooperative Groups

Have students work in pairs, mixing abilities to work through Activities 1 and 2.

- For a continuous normally distributed variable, the height of the curve represents only a probability density.
- The total area under the curve is equal to 1.
- The probability that a variable takes on a value within a given interval is equal to the area under the curve between the endpoints of the curve interval.
- The probability that the height (to the nearest inch) of a particular 15-year old is 67 inches is equal to the area under the normal curve from 66.5 inches to 67.5 inches.

Practice Have students complete Exercises 1–6.

3 Assess

Formative Assessment

Use Exercise 2 to assess whether students can use the **normalpdf** command to find normal probabilities.

From Concrete to Abstract

Exercise 4 asks students to find an open ended probability using a different normal distribution.

Graphing Technology Lab
The Normal Curve *Continued*

The probability of a range of values is the area under the curve.

Activity 2 Analyze a Normal Distribution

Use the graph to answer questions about the data. What is the probability that Isaac will be at most 67 inches tall when he is 15?

The sum of all the y-values up to $x = 67$ would give us the probability that Isaac's height will be less than or equal to 67 inches. This is also the area under the curve. We will shade the area under the curve from negative infinity to 67 inches and find the area of the shaded portion of the graph.

Step 1 Use the **ShadeNorm** function.

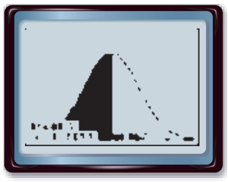
KEYSTROKES: 2nd [DISTR] ▶ ENTER

Step 2 Shade the graph.

Next enter the lowest value, highest value, mean, and standard deviation.

On the TI-84 Plus, -1×10^{99} represents negative infinity.

KEYSTROKES: (–) 1 2nd [EE] 99 , 67 , 67 , 2.8) ENTER



[58.6, 75.4] scl: 2.8 by [0, 0.17857142] scl: 1

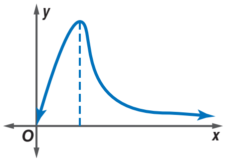
The area is given as 0.5. The probability that Isaac will be 67 inches tall is 0.5 or 50%. Since the mean value is 67, we expect the probability to be 50%.

Exercises

1. What is the probability that Isaac will be at least 6 feet tall when he is 15? **about 4%**
2. What is the probability that Isaac will be between 65 and 68 inches? **about 40%**
3. The **z-score** represents the number of standard deviations that a given data value is from the mean. The z-score for a data value X is given by $z = \frac{X - \mu}{\sigma}$, where μ is the mean and σ is the standard deviation. Find and interpret the z-score of a height of 73 inches. **About 2.14; 73 in. is 2.14 standard deviations above the mean.**
4. Find and interpret the z-score of a height of 61 inches. **About 2.14; 61 in. is 2.14 standard deviations below the mean.**

Extension

Refer to the curve at the right.



5. Compare this curve to the normal curve in Activity 1. **5. Both curves have a bell shape. This curve has a longer tail at the right.**
6. Describe where an outlier of the data set would be graphed on this curve. **6. On this curve, an outlier is plotted at the right, where the y-values are smaller.**

LAB 24

Algebra Lab
Two-Way Frequency Tables

Joana sent out a survey to the freshmen and sophomores, asking if they were planning on attending the dance. One way of organizing her responses is to use a two-way frequency table. A **two-way frequency table** or *contingency table* is used to show the frequencies of data from a survey or experiment classified according to two categories, with the rows indicating one category and the columns indicating the other.

For Joana's survey, the two categories are *class* and *attendance*. These categories can be split into subcategories: *freshman* and *sophomore* for *class*, and *attending* and *not attending* for *attendance*.

Class	Attending	Not Attending	Totals
Freshman			
Sophomore			
Totals			

subcategories

Activity 1 Two-Way Frequency Table

DANCE **Sixty-six freshmen responded to the survey, with 32 saying that they would be attending. Of the 84 sophomores that responded, 46 said they would attend. Organize the data in a two-way table.**

Step 1 Find the values for every combination of subcategories. One combination is freshmen/not attending. Since 32 of 66 freshmen are attending, 66 − 32 or 34 freshmen are *not* attending. These combinations are called **joint frequencies.**

Step 2 Place every combination in the corresponding cell.

Step 3 Find the totals of each subcategory and place them in their corresponding cell. These values are called **marginal frequencies.**

Step 4 Find the sum of each set of marginal frequencies. These two sums should be equal. Place the value in the bottom right corner.

Class	Attending	Not Attending	Totals
Freshman	32	34	66
Sophomore	46	38	84
Totals	78	72	150

joint frequencies

marginal frequencies

marginal frequencies

Analyze the Results **4–7. See margin.**

1. How many students responded to the survey? **150**
2. How many of the students that were surveyed are attending the dance? **78**
3. How many of the surveyed sophomores are not attending the dance? **38**
4. What does each of the joint frequencies represent?
5. What does each of the marginal frequencies represent?
6. **WORK** Heather sent out a survey asking who was working during the holiday. Of the 50 boys who responded, 34 said *yes*. Of the 45 girls who responded, 21 said *no*. Create a two-way frequency table of the results.
7. **SOCCER** Pamela asked if anyone would be interested in a co-ed soccer team. Of the 28 boys who responded, 18 said that they would play and 4 were undecided. Of the 22 girls who responded, 6 said they did not want to play and 3 were undecided. Create a two-way frequency table of the results.

Common Core State Standards
S.ID.5

(continued on the next page)

1 Focus

Objective Use two-way frequency tables to find marginal, joint, and conditional relative frequencies.

Teaching Tips

Using Media Two-way frequency tables are often used to present data in newspapers and other media. Have students find and discuss examples.

Marginal Frequencies Point out to students that the marginal frequencies are on the outside of the table, like the margins of a sheet of paper.

2 Teach

Working in Cooperative Groups

Have student work in mixed ability pairs. Have students alternate the steps of the activities.

Additional Answers

4. The joint frequencies represent every viable combination of subcategories. These include the frequency of freshmen attending, freshmen not attending, sophomores attending, and sophomores not attending.
5. The marginal frequencies represent the total frequency of every subcategory. In other words, they indicate how many times that particular subcategory occurred.
6.

Gender	Working	Not Working	Totals
Boys	34	16	50
Girls	24	21	45
Totals	58	37	95

7.

Gender	Playing	Not Playing	Undecided	Totals
Boys	18	6	4	28
Girls	13	6	3	22
Totals	31	12	7	50

Practice Have students complete Activities 1 and 2 and Exercises 1–13.

3 Assess

Formative Assessment

Use Exercises 8 and 12 to assess each student's ability to use two-way frequency tables to determine probabilities.

From Concrete to Abstract

Ask students to summarize what they learned about two-way frequency tables.

Additional Answers

9. Each conditional relative frequency represents the probability that one subcategory occurred given the knowledge that another subcategory already is known. For example, the probability that a person is not attending if you already know she is a freshman.

10. The sums of the columns (for attending) do not equal 100% because the conditional relative frequency table is organized by class. The probability of freshmen attending and the probability of sophomores attending are not mutually exclusive. The sums of the rows equal 100% because those events *are* mutually exclusive.

Algebra Lab
Two-Way Frequency Tables *Continued*

A **relative frequency** is the ratio of the number of observations in a category to the total number of observations. Relative frequencies are also probabilities. To create a relative frequency two-way table, divide each of the values by the total number of observations and replace them with their corresponding decimals or percents.

Class	Attending	Not Attending	Totals
Freshman	$\frac{32}{150} \approx 21.3\%$	22.7%	44%
Sophomore	30.7%	25.3%	56%
Totals	52%	48%	100%

A **conditional relative frequency** is the ratio of the joint frequency to the marginal frequency. For example, given that a student is a freshman, what is the conditional relative frequency that he or she is going to the dance? In other words, what is the probability that a freshman is going to the dance?

Activity 2 Two-Way Conditional Relative Frequency Table

DANCE **Joana wants to determine the conditional relative frequencies (or probabilities) given the fact that she knows the class of the respondents.**

Step 1 Refer to the table in Activity 1. A total of 66 freshmen responded, and 32 said *yes*. Therefore, the conditional relative frequency that a respondent said *yes* given that the respondent is a freshman is $\frac{32}{66}$.

Step 2 Place every conditional relative frequency in the corresponding cell.

Step 3 The conditional relative frequencies for each row should sum to 100%.

Conditional Relative Frequencies by Class

Class	Attending	Not Attending	Totals
Freshman	$\frac{32}{66} \approx 48\%$	$\frac{34}{66} \approx 52\%$	100%
Sophomore	$\frac{46}{84} \approx 55\%$	$\frac{38}{84} \approx 45\%$	100%

Analyze the Results **9–11. See margin.**

8. Given that a respondent was a sophomore, what is the probability that he or she said *no*? **45%**

9. What does each of the conditional relative frequencies represent?

10. Why do you think that the columns do not sum to 100%?

11. Create a two-way conditional relative frequency table for the category *attendance*.

12. Given that a respondent was not attending, what is the probability that he or she is a freshman? **47%**

13. **ACTIVITIES** The managers, staff, and assistants were given three options for the holiday activity: a potluck, a dinner at a restaurant, and a gift exchange. Five of the 11 managers want a dinner, while 3 want a potluck. Eleven of the 45 staff members want a gift exchange, while 18 want a dinner. Ten of the 32 assistants want a dinner, while 8 of them want a gift exchange.

- **a.** Create a two-way frequency table. **a–b. See margin.**
- **b.** Convert the two-way frequency table into a relative frequency table.
- **c.** Create two conditional relative frequency tables: one for the activities and one for the employees. **See Answer Appendix.**

11.

Conditional Relative Frequencies by Attending

Class	Attending	Not Attending
Freshman	41%	47%
Sophomore	59%	53%
Totals	100%	100%

13a.

Employees	Potluck	Dinner	Gifts	Totals
Managers	3	5	3	11
Staff	16	18	11	45
Assistants	14	10	8	32
Totals	33	33	22	88

13b.

Employees	Potluck	Dinner	Gifts	Totals
Managers	3.4%	5.7%	3.4%	12.5%
Staff	18.2%	20.4%	12.5%	51.1%
Assistants	15.9%	11.4%	9.1%	36.4%
Totals	37.5%	37.5%	25%	100%

Additional Exercises

Use with Lesson 1-1.

1. **SMARTPHONES** A certain smartphone family plan costs $55 per month plus additional usage costs. If x is the number of cell phone minutes used above the plan amount and y is the number of megabytes of data used above the plan amount, interpret the following expressions.
   - **a.** $0.25x$
   - **b.** $2y$
   - **c.** $0.25x + 2y + 55$ **total monthly cost**

   a-b. See Answer Appendix.

Use with Lesson 1-2.

2. **SPORTS** Kamilah sells tickets at Duke University's athletic ticket office. If p represents a preferred season ticket, b represents a blue zone ticket, and g represents a general admission ticket, interpret and then evaluate the following expressions. **a-c. See Answer Appendix.**
   - **a.** $45b$
   - **b.** $15p + 35g$
   - **c.** $6p + 11b + 22g$

Use with Lesson 1-3.

3. **RETAIL** The table shows prices on children's clothing. **a-b. See Answer Appendix.**

Shorts	Shirts	Tank Tops
$7.99	$8.99	$6.99
$5.99	$4.99	$2.99

   - **a.** Interpret the expression 5(8.99) + 2(2.99) + 7(5.99).
   - **b.** Write and evaluate three different expressions that represent 8 pairs of shorts and 8 tops.
   - **c.** If you buy 8 shorts and 8 tops, you receive a discount of 15%. Find the greatest and least amount of money you can spend on the 16 items at the sale. **$101.86; $61.06**

Use with Lesson 1-4.

4. **FOOD** Kenji is picking up take-out food for his study group.

a. the cost of four sandwiches, three soups, three salads, and five drinks

Menu	
Item	Cost ($)
sandwich	2.49
cup of soup	1.29
side salad	0.99
drink	1.49

   - **a.** Interpret the expression 4(2.49) + 3(1.29) + 3(0.99) + 5(1.49).
   - **b.** How much would it cost if Kenji bought four of each item on the menu? **$25.04**

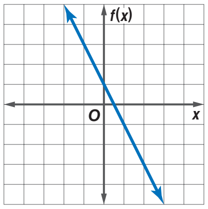
Use with Lesson 1-7.

5. **CELL PHONE PICTURES** The cost of sending cell phone pictures is given by $y = 0.25x$, where x is the number of pictures that you send.
   - **a.** Write the equation in function notation. Interpret the function in terms of the context.
   - **b.** Find $f(5)$ and $f(12)$. What do these values represent?
   - **c.** Determine the domain and range of this function.

   a-c. See Answer Appendix.

6. **EDUCATION** The average national math test scores $f(t)$ for 17-year-olds can be represented as a function of the national science scores t by $f(t) = 0.8t + 72$.
   - **a.** Graph this function. Interpret the function in terms of the context. **See Answer Appendix.**
   - **b.** What is the science score that corresponds to a math score of 308? **295**
   - **c.** What is the domain and range of this function? **The domain is the set of science scores. The range is the set of math scores.**

7. **ERROR ANALYSIS** Corazon thinks $f(x)$ and $g(x)$ are representations of the same function. Maggie disagrees. Who is correct? Explain your reasoning.



x	$g(x)$
−1	1
0	−1
1	−3
2	−5
3	−7

Maggie; the graph represents $f(x) = -2x + 1$ and the table represents $g(x) = -2x - 1$.

Use with Lesson 2-2.

8. **CHALLENGE** Solve each equation for x. Assume that $a \neq 0$.
   - **a.** $ax = 12$ $x = \frac{12}{a}$
   - **b.** $x + a = 15$ $x = 15 - a$
   - **c.** $-5 = x - a$ $x = a - 5$
   - **d.** $\frac{1}{a}x = 10$ $x = 10a$

Use with Explore 2-3.

9. **CHALLENGE** Solve each equation for x. Assume that a $\neq 0$.
   - **a.** $ax + 7 = 5$ $x = \frac{-2}{a}$
   - **b.** $\frac{1}{a}x - 4 = 9$ $x = 13a$
   - **c.** $2 - ax = -8$ $x = \frac{10}{a}$

Use with Lesson 2-4.

10. **REASONING** Solve $5x + 2 = ax - 1$ for x. Assume that $a \neq 0$. Describe each step. **See Answer Appendix.**

Additional Exercises

Use with Lesson 3-5.

11. **SPORTS** To train for an upcoming marathon, Olivia plans to run 3 miles per day for the first week, and then increase the daily distance by a half a mile each of the following weeks.

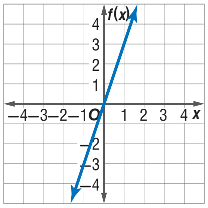
a. Write an equation to represent the nth term of the sequence. $A_n = 2.5 + 0.5n$

b. If the pattern continues, during which week will she run 10 miles per day? week 15

c. Is it reasonable to think that this pattern will continue indefinitely? Explain. Sample answer: No; eventually the number of miles ran per day will become unrealistic.

Use with Lesson 3-6.

12. **ERROR ANALYSIS** Quentin thinks that $f(x)$ and $g(x)$ are both proportional. Claudia thinks they are not proportional. Is either of them correct? Explain your reasoning.



x	$g(x)$
−2	−7
−1	−4
0	−1
1	2
2	5

See Answer Appendix.

Use with Explore 4-2.

13. **COMBINING FUNCTIONS** The parents of a college student open an account for her with a deposit of $5000, and they set up automatic deposits of $100 to the account every week.

a. Write a function $d(t)$ to express the amount of money in the account t weeks after the initial deposit. $d(t) = 5000 + 100t$

b. The student plans on spending $600 the first week and $250 in each of the following weeks for room and board and other expenses. Write a function $w(t)$ to express the amount of money taken out of the account each week. $w(t) = 600 + 250t$

c. Find $B(t) = d(t) - w(t)$. What does this new function represent? See Answer Appendix.

d. Will the student run out of money? If so, when? Yes; in about 30 wk

Use with Lesson 4-3.

Write an equation for the line described in standard form.

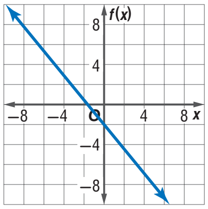
14. through $(-1, 7)$ and $(8, -2)$ $x + y = 6$

15. through $(-4, 3)$ with y-intercept 0 $3x + 4y = 0$

16. with x-intercept 4 and y-intercept 5 $5x + 4y = 20$

17. **ERROR ANALYSIS** Juana thinks that $f(x)$ and $g(x)$ have the same slope but different intercepts. Sabrina thinks that $f(x)$ and $g(x)$ describe the same line. Is either of them correct? Explain your reasoning.

The graph of $g(x)$ is the line that passes through $(3, -7)$ and $(-6, 4)$.



Juana; $f(x)$ and $g(x)$ both have the same slope. However, the x- and y- intercepts are different.

Use with Lesson 4-4.

18. **REASONING** Which key features of the graphs of two parallel lines are the same, and which are different? Which key features of the graphs of two perpendicular lines are the same, and which are different? See Answer Appendix.

Use with Explore 5-2.

19. **CHALLENGE** Solve each inequality for x. Assume that $a > 0$.

a. $-ax < 5$ $x > -\frac{5}{a}$

b. $\frac{1}{a}x \geq 8$ $x \geq 8a$

c. $-6 \geq ax$ $x \leq -\frac{6}{a}$

Use with Lesson 5-3.

20. **CHALLENGE** Solve each inequality for x. Assume that $a > 0$.

a. $ax + 4 \geq -ax - 5$ $\left\{x \middle| x \geq -\frac{9}{2a}\right\}$

b. $2 - ax < x$ $\left\{x \middle| x > \frac{2}{1 + a}\right\}$

c. $-\frac{2}{a}x + 3 > -9$ $\left\{x \middle| x < \frac{a}{6}\right\}$

Use with Explore 5-4.

21. **CHALLENGE** Solve each inequality for x. Assume a is constant and $a > 0$.

a. $-3 < ax + 1 \leq 5$ $x > -\frac{4}{a}$ and $x \leq \frac{4}{a}$

b. $-\frac{1}{a}x + 6 < 1$ or $2 - ax > 8$ $x < -\frac{6}{a}$ or $x > 5a$

Additional Exercises

Use with Lesson 5-6.

22. **CHALLENGE** Write a linear inequality for which (−1, 2), (1, 1), and (3, −4) are solutions but (0, 1) is not.
Sample answer: $y \geq -2x + 2$

Use with Lesson 7-2.

23. **PROBABILITY** The probability of rolling a die and getting an even number is $\frac{1}{2}$. If you roll the die twice, the probability of getting an even number both times is $\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)$ or $\left(\frac{1}{2}\right)^2$. **probability of all evens on 4 rolls**

- **a.** What does $\left(\frac{1}{2}\right)^4$ represent?
- **b.** Write an expression to represent the probability of rolling a die d times and getting an even number every time. Write the expression as a power of 2.
  $\left(\frac{1}{2}\right)^d$; 2^{-d}

Use with Lesson 7-3.

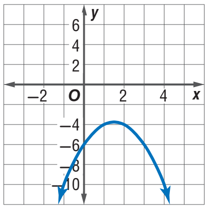
24. **SMARTPHONES** A recent cell phone study showed that company A's phone processes up to 7.95×10^5 bits of data every second. Company B's phone processes up to 1.41×10^6 bits of data every second. Evaluate and interpret $\frac{1.41 \times 10^6}{7.95 \times 10^5}$. **See Answer Appendix.**

25. **HEALTH** A ponderal index p is a measure of a person's body based on height h in meters and mass m in kilograms. One such formula is $p = 100m^{\frac{1}{3}}h^{-1}$. If a person who is 182 centimeters tall has a ponderal index of about 2.2, how much does the person weigh in kilograms?
64.2 kg

Use with Lesson 9-1.

26. **ERROR ANALYSIS** Jade thinks that the parabolas represented by the graph and the description have the same axis of symmetry. Chase disagrees. Who is correct? Explain your reasoning. **Jade; the line of symmetry for both parabolas is $x = 2$.**

a parabola that opens downward, passing through (0, 6) and having a vertex at (2, 2)



27. **WRITING IN MATH** Use tables and graphs to compare and contrast an exponential function $f(x) = ab^x + c$, where $a \neq 0$, $b > 0$, and $b \neq 1$, a quadratic function $g(x) = ax^2 + c$, and a linear function $h(x) = ax + c$. Include intercepts, portions of the graph where the functions are increasing, decreasing, positive, or negative, relative maxima and minima, symmetries, and end behavior. Which function eventually exceeds the others? **See Answer Appendix.**

Use with Lesson 9-5.

28. **MULTIPLE REPRESENTATIONS** In this problem, you will investigate writing a quadratic equation with given roots. If p is a root of $0 = ax^2 + bx + c$, then $(x - p)$ is a factor of $ax^2 + bx + c$.

- **a.** **TABULAR** Copy and complete the first two columns of the table.

Roots	Factors	Equation
2, 5	$(x - 2), (x - 5)$	$(x - 2)(x - 5) = 0$ $x^2 - 7x + 10 = 0$
1, 9	**$(x - 1), (x - 9)$**	**$x^2 - 10x + 9 = 0$**
−1, 3	**$(x + 1), (x - 3)$**	**$x^2 - 2x - 3 = 0$**
0, 6	**$x, (x - 6)$**	**$x^2 - 6x = 0$**
$\frac{1}{2}$, 7	**$\left(x - \frac{1}{2}\right), (x - 7)$**	**$2x^2 - 15x + 7 = 0$**
$-\frac{2}{3}$, 4	**$\left(x + \frac{2}{3}\right), (x - 4)$**	**$3x^2 - 10x - 8 = 0$**

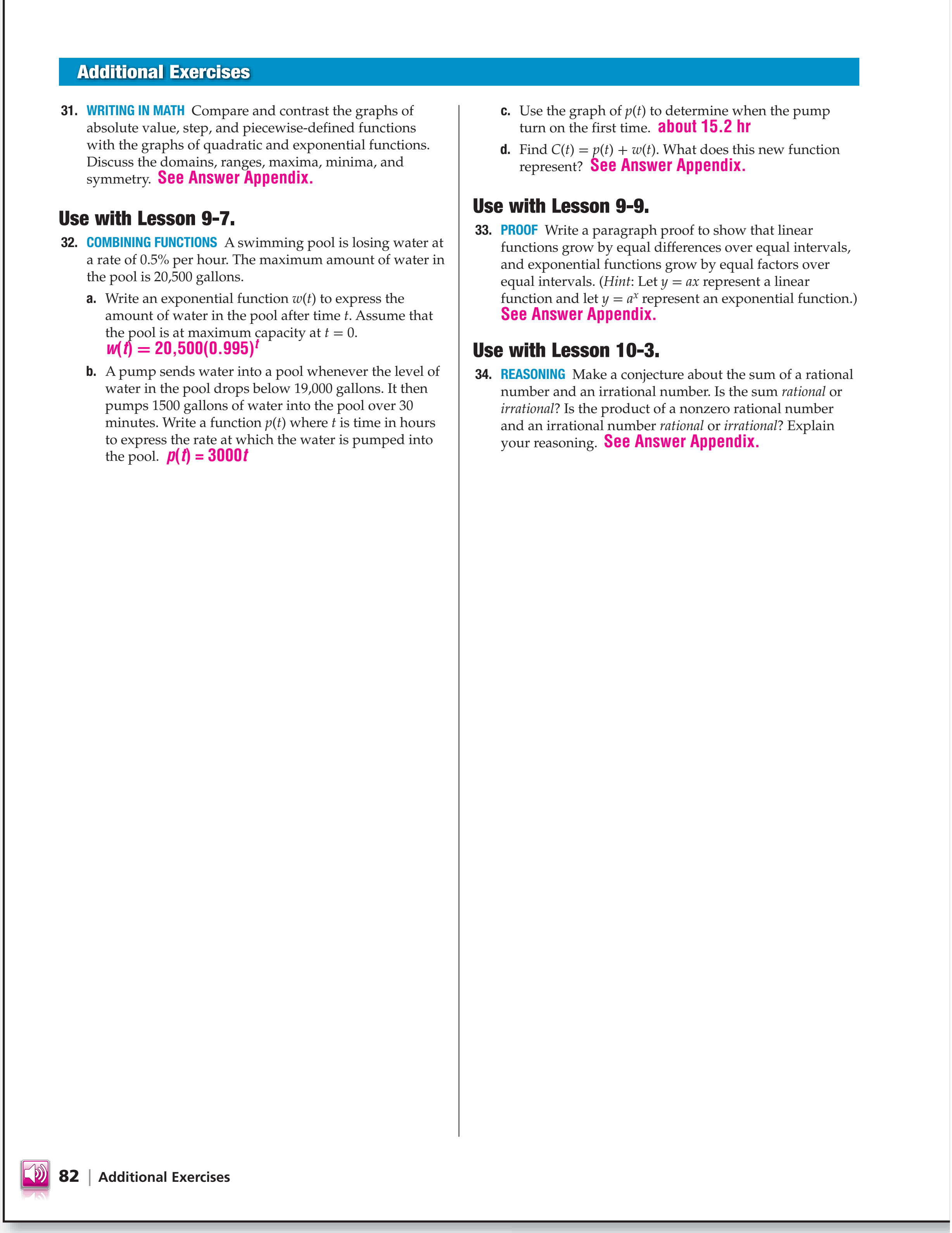
- **b.** **ALGEBRAIC** Multiply the factors to write each equation with integral coefficients. Use the equations to complete the last column of the table. Write each equation.
- **c.** **ANALYTICAL** How could you write an equation with three roots? Test your conjecture by writing an equation with roots 1, 2, and 3. Is the equation quadratic? Explain. **See Answer Appendix.**

Use with Lesson 9-6.

29. **FINANCIAL LITERACY** Daniel deposited \$500 into a savings account and after 8 years, his investment is worth \$807.07. The equation $A = d(1.005)^{12t}$ models the value of Daniel's investment A after t years with an initial deposit d.

- **a.** What would the value of Daniel's investment be if he had deposited \$1000? **about \$1614.14**
- **b.** What would the value of Daniel's investment be if he had deposited \$250? **about \$403.54**
- **c.** Interpret $d(1.005)^{12t}$ to explain how the amount of the original deposit affects the value of Daniel's investment. **See Answer Appendix.**

30. **REASONING** Use tables and graphs to compare and contrast an exponential function $f(x) = ab^x + c$, where $a \neq 0$, $b > 0$, and $b \neq 1$, and a linear function $g(x) = ax + c$. Include intercepts, intervals where the functions are increasing, decreasing, positive, or negative, relative maxima and minima, symmetry, and end behavior.
See Answer Appendix.



Additional Exercises

31. **WRITING IN MATH** Compare and contrast the graphs of absolute value, step, and piecewise-defined functions with the graphs of quadratic and exponential functions. Discuss the domains, ranges, maxima, minima, and symmetry. **See Answer Appendix.**

Use with Lesson 9-7.

32. **COMBINING FUNCTIONS** A swimming pool is losing water at a rate of 0.5% per hour. The maximum amount of water in the pool is 20,500 gallons.

- **a.** Write an exponential function $w(t)$ to express the amount of water in the pool after time t. Assume that the pool is at maximum capacity at $t = 0$. $w(t) = 20{,}500(0.995)^t$
- **b.** A pump sends water into a pool whenever the level of water in the pool drops below 19,000 gallons. It then pumps 1500 gallons of water into the pool over 30 minutes. Write a function $p(t)$ where t is time in hours to express the rate at which the water is pumped into the pool. $p(t) = 3000t$
- **c.** Use the graph of $p(t)$ to determine when the pump turn on the first time. **about 15.2 hr**
- **d.** Find $C(t) = p(t) + w(t)$. What does this new function represent? **See Answer Appendix.**

Use with Lesson 9-9.

33. **PROOF** Write a paragraph proof to show that linear functions grow by equal differences over equal intervals, and exponential functions grow by equal factors over equal intervals. (*Hint*: Let $y = ax$ represent a linear function and let $y = a^x$ represent an exponential function.) **See Answer Appendix.**

Use with Lesson 10-3.

34. **REASONING** Make a conjecture about the sum of a rational number and an irrational number. Is the sum *rational* or *irrational*? Is the product of a nonzero rational number and an irrational number *rational* or *irrational*? Explain your reasoning. **See Answer Appendix.**

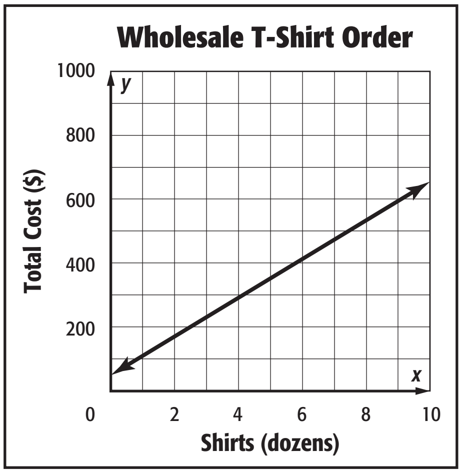
NAME ______________________________ DATE ____________ PERIOD ____________

Lesson 2 Practice

Interpreting Graphs of Functions

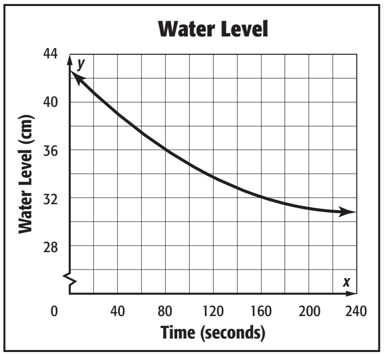
Identify the function graphed as *linear* or *nonlinear*. Then estimate and interpret the intercepts of the graph, any symmetry, where the function is positive, negative, increasing, and decreasing, the *x*-coordinate of any relative extrema, and the end behavior of the graph.

1.



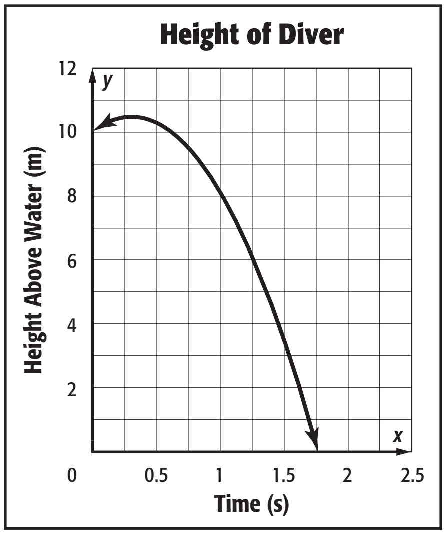
Linear; *y*-intercept is 50, so the set up cost is $50; no *x*-intercept, so at no time is the cost $0; no line symmetry; positive and increasing for *x* > 0, so the cost is always positive will increase as more shirts are ordered.

2.



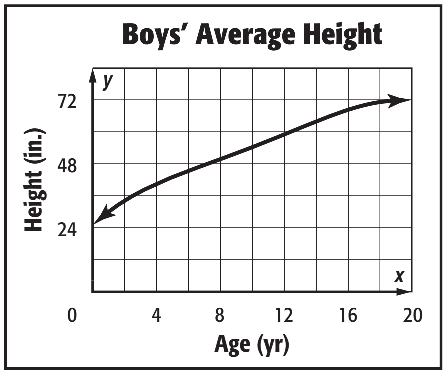
Nonlinear; *y*-intercept is about 43, so water level was about 43 cm when time started; no *x*-intercept, so the water level did not reach 0; no line symmetry; water level was always positive and decreased the entire time; graph appears to level off or begin to increase as *x* increases.

3.



Nonlinear; *y*-intercept is 10, so diver started at 10 m; *x*-intercept of about 1.8, so diver entered the water after about 1.8 sec.; no line symmetry; height was positive for *x* < 1.8 and negative for *x* > 1.8, so diver was above the water until 1.8 sec.; the height increased until max. of 10.5 at 0.3 sec., then it decreased; diver would continue to go down for some time, then would come up.

4.



Nonlinear; *y*-intercept is 24, so the average boy is 24 inches at birth; no *x*-intercept; no line symmetry; always positive, so heights are always positive; appears to be a maximum of about 72 at about 19, this means that an average boy reaches his maximum height of 72 inches at age 19.

Copyright © Glencoe/McGraw-Hill, a division of The McGraw-Hill Companies, Inc.

NAME ______________________ DATE __________ PERIOD __________

Lesson 5 Practice

Regression and Median-Fit Lines

Write an equation of the regression line for the data in each table below. Then find the correlation coefficient.

1. **TURTLES** The table shows the number of turtles hatched at a zoo each year since 2006.

Year	2006	2007	2008	2009	2010
Turtles Hatched	21	17	16	16	14

$y = –1.5x + 19.8$; $r = –0.916$

2. **SCHOOL LUNCHES** The table shows the percentage of students receiving free or reduced price school lunches at a certain school each year since 2006.

Year	2006	2007	2008	2009	2010
Percentage	14.4%	15.8%	18.3%	18.6%	20.9%

Source: KidsData

$y = 1.58x + 14.44$; $r = 0.983$

3. **SPORTS** Below is a table showing the number of students signed up to play lacrosse after school in each age group. **$y = –1.5x + 34.1$; $r = –0.554$**

Age	13	14	15	16	17
Lacrosse Players	17	14	6	9	12

4. **LANGUAGE** The State of California keeps track of how many millions of students are learning English as a second language each year.

Year	2003	2004	2005	2006	2007
English Learners	1.600	1.599	1.592	1.570	1.569

Source: California Department of Education

a. Find an equation for the median-fit line. **$y = –0.01x + 1.607$**

b. Predict the number of students who were learning English in California in 2001.
about 1,627,000 students

c. Predict the number of students who were learning English in California in 2010.
about 1,537,000 students

5. **POPULATION** Detroit, Michigan, like a number of large cities, is losing population every year. Below is a table showing the population of Detroit each decade.

Year	1960	1970	1980	1990	2000
Population (millions)	1.67	1.51	1.20	1.03	0.95

Source: U.S. Census Bureau

a. Find an equation for the regression line. **$y = –0.019x + 1.656$**

b. Find the correlation coefficient and explain the meaning of its sign.
$r = –0.982$; The sign is negative, meaning that there is a negative correlation to the data.

c. Estimate the population of Detroit in 2008. **about 734,000 people**

Copyright © Glencoe/McGraw-Hill, a division of The McGraw-Hill Companies, Inc.

NAME ______________________ DATE ____________ PERIOD ____________

Lesson 6 Practice

Inverse Linear Functions

Find the inverse of each relation.

1. {(−2, 1), (−5, 0), (−8, −1), (−11, 2)}

{(1, –2), (0, –5), (–1, –8), (2, –11)}

2. {(3, 5), (4, 8), (5, 11), (6, 14)}

{(5, 3), (8, 4), (11, 5), (14, 6)}

3. {(5, 11), (1, 6), (−3, 1), (−7, −4)}

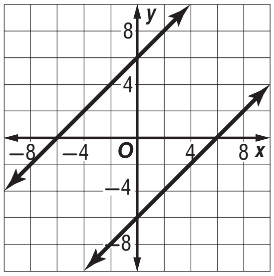
{(11, 5), (6, 1), (1, –3), (–4, –7)}

4. {(0, 3), (2, 3), (4, 3), (6, 3)}

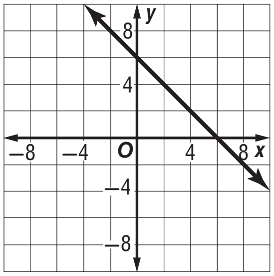
{(3, 0), (3, 2), (3, 4), (3, 6)}

Graph the inverse of each function.

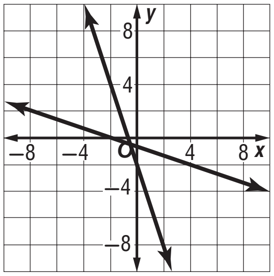
5.



6.



7.



Find the inverse of each function.

8. $f(x) = \frac{6}{5}x - 3$

$f^{-1}(x) = \frac{5}{6}(x + 3)$

9. $f(x) = \frac{4x + 2}{3}$

$f^{-1}(x) = \frac{3x - 2}{4}$

10. $f(x) = \frac{3x - 1}{6}$

$f^{-1}(x) = \frac{6x + 1}{3}$

11. $f(x) = 3(3x + 4)$

$f^{-1}(x) = \frac{\frac{x}{3} - 4}{3}$

12. $f(x) = -5(-x - 6)$

$f^{-1}(x) = \frac{x}{5} - 6$

13. $f(x) = \frac{2x - 3}{7}$

$f^{-1}(x) = \frac{7x + 3}{4}$

Write the inverse of each equation in $f^{-1}(x)$ notation.

13. $4x + 6y = 24$

$f^{-1}(x) = \frac{24 - 6x}{4}$

14. $-3y + 5x = 18$

$f^{-1}(x) = \frac{3x + 18}{5}$

15. $x + 5y = 12$

$f^{-1}(x) = -5x + 12$

16. $5x + 8y = 40$

$f^{-1}(x) = \frac{40 - 8x}{5}$

17. $-4y - 3x = 15 + 2y$

$f^{-1}(x) = -2x - 5$

18. $2x - 3 = 4x + 5y$

$f^{-1}(x) = \frac{-5x - 3}{2}$

19. CHARITY Jenny is running in a charity event. One donor is paying an initial amount of \$20.00 plus an extra \$5.00 for every mile that Jenny runs.

a. Write a function $D(x)$ for the total donation for x miles run. $D(x) = 5x + 20$

b. Find the inverse function, $D^{-1}(x)$. $D^{-1}(x) = \frac{x - 20}{5}$

c. What do x and $D^{-1}(x)$ represent in the context of the inverse function? **x represents the total donation and $P^{-1}(x)$ represents the number miles run.**

Copyright © Glencoe/McGraw-Hill, a division of The McGraw-Hill Companies, Inc.

NAME ______________________________ DATE ____________ PERIOD ____________

Lesson 8 Practice

Rational Exponents

Write each expression in radical form, or write each radical in exponential form.

1. $\sqrt{13}$ **$13^{\frac{1}{2}}$**

2. $\sqrt{37}$ **$37^{\frac{1}{2}}$**

3. $\sqrt{17x}$ **$(17x)^{\frac{1}{2}}$**

4. $(7ab)^{\frac{1}{2}}$ **$\sqrt{7ab}$**

5. $21z^{\frac{1}{2}}$ **$21\sqrt{z}$**

6. $13(ab)^{\frac{1}{2}}$ **$13\sqrt{ab}$**

Simplify.

7. $\left(\frac{1}{81}\right)^{\frac{1}{4}}$ **$\frac{1}{3}$**

8. $\sqrt[5]{1024}$ **4**

9. $512^{\frac{1}{3}}$ **8**

10. $\left(\frac{32}{1024}\right)^{\frac{1}{5}}$ **$\frac{1}{2}$**

11. $\sqrt[4]{1296}$ **6**

12. $3125^{\frac{1}{5}}$ **5**

Solve each equation.

13. $3^x = 729$ **6**

14. $4^x = 4096$ **6**

15. $5^x = 15{,}625$ **6**

16. $6^{x+3} = 7776$ **2**

17. $3^{x-3} = 2187$ **10**

18. $4^{3x+4} = 16{,}384$ **1**

19. WATER The flow of water F in cubic feet per second over a wier, a small overflow dam, can be represented by $F = 1.26H^{\frac{3}{2}}$, where H is the height of the water in meters above the crest of the wier. Find the height of the water if the flow of the water is 10.08 cubic feet per second. **4 ft**

Copyright © Glencoe/McGraw-Hill, a division of The McGraw-Hill Companies, Inc.

NAME ______________________ DATE ____________ PERIOD ____________

Lesson 10 Practice

Transformations of Quadratic Functions

Describe how the graph of each function is related to the graph of $f(x) = x^2$.

1. $g(x) = (10 + x)^2$

Translation of $f(x) = x^2$ to the left 10 units.

2. $g(x) = -\frac{2}{5} + x^2$

Translation of $f(x) = x^2$ down $\frac{2}{5}$ unit.

3. $g(x) = 9 - x^2$

Reflection of $f(x) = x^2$ across the x-axis translated up 9 units.

4. $g(x) = 2x^2 + 2$

Stretch of $f(x) = x^2$ narrower than the graph of $f(x) = x^2$ translated up 2 units.

5. $g(x) = -\frac{3}{4}x^2 - \frac{1}{2}$

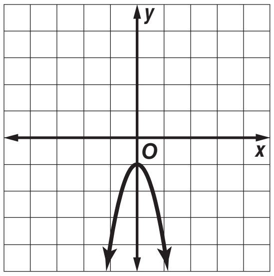
Compression of $f(x) = x^2$ wider than the graph of $f(x) = x^2$, reflected over the x-axis, translated down $\frac{1}{2}$ unit.

6. $g(x) = -3(x + 4)^2$

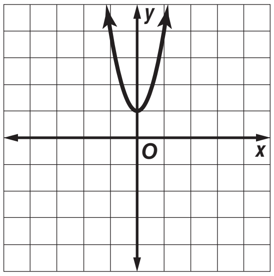
Stretch of $f(x) = x^2$ narrower than the graph of $f(x) = x^2$, reflected over the x-axis translated to the left 4 units.

Match each equation to its graph.

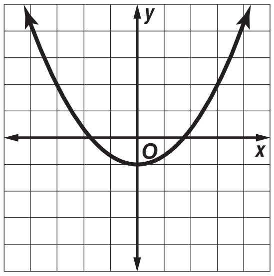
A.



B.



C.



7. $y = -3x^2 - 1$ **A**

8. $y = \frac{1}{3}x^2 - 1$ **C**

9. $y = 3x^2 + 1$ **B**

List the functions in order from the most vertically stretched to the least vertically stretched graph.

10. $f(x) = 3x^2, g(x) = \frac{1}{2}x^2, h(x) = -2x^2$

$f(x)$, $h(x)$, $g(x)$

11. $f(x) = \frac{1}{2}x^2, g(x) = -\frac{1}{6}x^2, h(x) = 4x^2$

$h(x)$, $f(x)$, $g(x)$

12. **PARACHUTING** Two parachutists jump at the same time from two different planes as part of an aerial show. The height h_1 of the first parachutist in feet after t seconds is modeled by the function $h_1 = -16t^2 + 5000$. The height h_2 of the second parachutist in feet after t seconds is modeled by the function $h_2 = -16t^2 + 4000$.

   a. What is the parent function of the two functions given? **$h = t^2$**

   b. Describe the transformations needed to obtain the graph of h_1 from the parent function. **Stretch of $y = x^2$ narrower than the graph of $f(x) = x^2$, reflected over the x-axis, translated up 5000 units.**

   c. Which parachutist will reach the ground first? **the second parachutist**

Copyright © Glencoe/McGraw-Hill, a division of The McGraw-Hill Companies, Inc.

NAME ______________________ DATE ____________ PERIOD ____________

Lesson 16 Practice

Recursive Formulas

Find the first five terms of each sequence.

1. $a_1 = 25, a_n = a_{n-1} - 12, n \geq 2$
25, 13, 1, −11, −23

2. $a_1 = -101, a_n = a_{n-1} + 38, n \geq 2$
−101, −63, −25, 13, 51

3. $a_1 = 3.3, a_n = a_{n-1} + 2.7, n \geq 2$
3.3, 6, 8.7, 11.4, 14.1

4. $a_1 = 7, a_n = -3a_{n-1} + 20, n \geq 2$
7, −1, 23, −49, 167

5. $a_1 = 20, a_n = \frac{1}{5}a_{n-1}, n \geq 2$
$20, 4, \frac{4}{5}, \frac{4}{25}, \frac{4}{125}$

6. $a_1 = \frac{2}{3}, a_n = \frac{1}{3}a_{n-1} - \frac{2}{9}, n \geq 2$
$\frac{2}{3}, 0, -\frac{2}{9}, -\frac{8}{27}, -\frac{26}{81}$

Write a recursive formula for each sequence.

7. 80, −40, 20, −10, …
$a_1 = 80, a_n = -0.5a_{n-1}, n \geq 2$

8. 87, 52, 17, −18, …
$a_1 = 87, a_n = a_{n-1} - 35, n \geq 2$

9. $\frac{1}{3}, \frac{4}{15}, \frac{16}{75}, \frac{64}{375}, \ldots$
$a_1 = \frac{1}{3}, a_n = \frac{4}{5}a_{n-1}, n \geq 2$

10. $\frac{4}{5}, \frac{3}{10}, -\frac{1}{5}, -\frac{7}{10}, \ldots$
$a_1 = \frac{4}{5}, a_n = a_{n-1} - \frac{1}{2}, n \geq 2$

11. 2.6, 5.2, 7.8, 10.4, …
$a_1 = 2.6, a_n = a_{n-1} + 2.6, n \geq 2$

12. 100, 120, 144, 172.8, …
$a_1 = 100, a_n = 1.2a_{n-1}, n \geq 2$

13. PIZZA The total costs for ordering one to five cheese pizzas from Luigi's Pizza Palace are shown.

Total Number of Pizzas Ordered	Cost
1	$7.00
2	$12.50
3	$18.00
4	$23.50
5	$29.00

a. Write a recursive formula for the sequence.
$a_1 = 7, a_n = a_{n-1} + 5.5, n \geq 2$

b. Write an explicit formula for the sequence.
$a_n = 5.5n + 1.5$

Copyright © Glencoe/McGraw-Hill, a division of The McGraw-Hill Companies, Inc.

NAME _______________ DATE _______ PERIOD _______

Lesson 21 Practice

Distributions of Data

Use a graphing calculator to construct a histogram for the data, and use it to describe the shape of the distribution. **1–3. See students' graphs.**

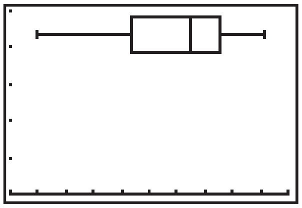
1. 52, 42, 46, 53, 22, 36, 49, 23, 50, 44, 25, 28, 48, 45, 54, 50, 18, 38
40, 34, 53, 42, 16, 44, 50, 42, 45, 50, 25, 47, 33, 48, 49, 36, 49, 39
negatively skewed

2. 51, 44, 54, 48, 63, 57, 58, 46, 55, 51, 63, 52, 46, 56, 57, 48, 52, 49
50, 56, 61, 51, 45, 52, 53, 55, 62, 55, 50, 53, 60, 56, 57, 59, 54, 45
symmetric

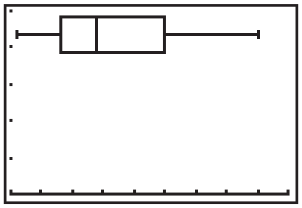
3. 42, 19, 24, 14, 55, 21, 51, 36, 22, 16, 32, 18, 46, 49, 64, 12, 19, 39
17, 20, 35, 52, 23, 17, 25, 33, 18, 26, 17, 24, 13, 27, 37, 29, 30, 19
positively skewed

Describe the center and spread of the data using either the mean and standard deviation or the five-number summary. Justify your choice by constructing a box-and-whisker plot for the data.

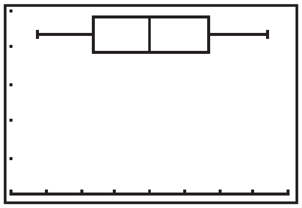
4. 78, 82, 76, 48, 71, 78, 65, 78, 81, 76, 53, 63, 79, 60, 78, 59, 78, 61 70, 68, 70, 58, 45, 72, 78, 86, 73, 77, 80, 60, 75, 84, 67, 79, 70, 75
Sample answer: The distribution is skewed, so use the five-number summary. The minimum 45, the maximum is 84, the median is 72.5, and half of the data are between 78 and 62.


[40, 90] scl: 5 by [0, 5] scl: 1

5. 63, 46, 48, 41, 72, 54, 48, 57, 53, 80, 52, 64, 55, 44, 67, 45, 71, 48 61, 45, 74, 49, 69, 54, 50, 72, 66, 50, 44, 58, 60, 54, 48, 59, 43, 70
Sample answer: The distribution is skewed, so use the five-number summary. The minimum is 41, the maximum is 80, the median is 54, and half of the data are between 48 and 65.


[40, 85] scl: 5 by [0, 5] scl: 1

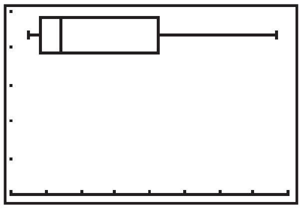
6. 33, 25, 18, 46, 35, 25, 18, 39, 33, 44, 20, 31, 39, 24, 24, 26, 15, 28 23, 29, 40, 19, 20, 31, 45, 37, 30, 17, 38, 21, 43, 14, 30, 47, 42, 34
Sample answer: The distribution is symmetric, so use the mean and standard deviation. The mean is about 30.1 with standard deviation of about 9.5.


[10, 50] scl: 5 by [0, 5] scl: 1

7. GASOLINE The average prices per gallon of gasoline during the first week of August on the east coast for the past 18 years are shown. Describe the center and spread of the data using either the mean and standard deviation or the five-number summary. Justify your choice by creating a box-and-whisker plot for the data.

Price per Gallon (dollars)
1.05, 1.09, 1.13, 1.17, 1.15, 0.99, 1.12, 1.44, 1.28, 1.34, 1.49, 1.85, 2.26, 3.00, 2.81, 3.87, 2.51, 2.66

Sample answer: The distribution is skewed, so use the five-number summary. The range is \$3.87 – \$0.99 or \$2.88. The median is \$1.39, and half of the data are between \$1.13 and \$2.51.


[0.8, 4] scl: 0.4 by [0, 5] scl: 1

Copyright © Glencoe/McGraw-Hill, a division of The McGraw-Hill Companies, Inc.

NAME ______________________ DATE __________ PERIOD __________

Lesson 22 Practice

Comparing Sets of Data

Find the mean, median, mode, range, and standard deviation of each data set that is obtained after adding the given constant to each value.

1. 62, 58, 57, 65, 68, 71, 49, 48, 52, 47; + 5.8

63.5, 63.3, no mode, 24, 8.2

2. 2, 8, 1, 5, 1, 3, 1, 7, 5, 4, 3, 1; + (−0.3)

3.1, 2.7, 0.7, 7, 2.3

3. 4.3, 3.8, 3.1, 4.5, 3.7, 4.4, 4.9, 3.9; + (−2.4)

1.7, 1.7, no mode, 1.8, 0.5

4. 17, 21, 18, 32, 29, 24, 19, 32; + 7.6

31.6, 30.1, 39.6, 15, 5.8

Find the mean, median, mode, range, and standard deviation of each data set that is obtained after multiplying each value by the given constant.

5. 94, 90, 88, 92, 85, 92, 86, 98, 92, 90; × 0.8

72.6, 72.8, 73.6, 10.4, 2.9

6. 41, 44, 47, 40, 43, 41, 42, 48; × 2.3

99.5, 97.8, 94.3, 18.4, 6.3

7. 63, 62, 59, 68, 67, 72, 70, 75, 64, 61; $\times \frac{1}{3}$

22, 21.8, no mode, 5.3, 1.6

8. 9, 7, 5, 2, 8, 4, 5, 6, 9, 5, 2, 1; $\times \frac{4}{9}$

2.3, 2.2, 2.2, 3.6, 1.1

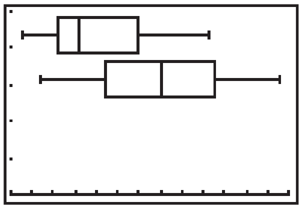
9. RECYCLING The weekly totals of recycled paper in pounds for two neighboring high schools are shown below.

Highland Heights High School
86, 57, 52, 43, 48, 55, 47, 64, 51, 77, 50, 62, 74, 70, 68, 53, 81, 53

Valley Forge High School
68, 79, 58, 101, 83, 65, 47, 73, 62, 77, 49, 84, 103, 70, 54, 97, 88, 94

a. Use a graphing calculator to construct a box-and-whisker plot for each set of data. Then describe the shape of each distribution.

Highland Heights, positively skewed;
Valley Forge, symmetric



[40, 105] scl: 5 by [0, 5] scl: 1

b. Compare the data sets using either the means and standard deviations or the five-number summaries. Justify your choice.

Sample answer: One distribution is symmetric and the other is skewed, so use the five-number summaries. The maximum for Highland Heights is 86, while the upper quartile for Valley Forge is 88. This means that 25% of the data for Valley Forge is greater than all of the data from Highland Heights. The median for Highland Heights is 56, while the median for Valley Forge is 75. Overall, we can conclude that the weekly totals for Valley Forge are greater than the weekly totals for Highland Heights.

Copyright © Glencoe/McGraw-Hill, a division of The McGraw-Hill Companies, Inc.

Lesson 2 (Guided Practice)

1. Linear; a y-intercept of about -5 means that the temperature was about $-5°$ at the beginning of the sample time. An x-int. of about 1 means that after about 1 hour, the temperature was 0°.

3. The function is positive for $x > 1$ and negative for $x < 1$. This means that the temperatures were below zero before 1 hour of elapsed time, but after 1 hour had passed, all the temperatures were above zero. The function is increasing for all values of x, which means that the day gets warmer as time elapses. There are no relative maxima or minima since the function continues to increase for all values of x as the temperatures continue to increase over time. As x increases, the value of y increases. As x decreases, the value of y decreases. The end behavior of the graph indicates that as the day goes on, the temperatures increase.

Lesson 2

1. Nonlinear; the y-intercept is 0, so there is no change in the stock value at the opening bell. The x-intercepts are 0, about 3.2, and about 4.5, so there is no change in the stock value after 0 hours, after about 3.2 hours, and after about 4.5 hours after the opening bell. The graph has no line symmetry. The stock went up in value for the first 3.2 hours, then dropped below the starting value from about 3.2 hours until 4.5 hours, and finally went up again after 4.5 hours. The stock value starts the day increasing in value for the first 2 hours, then it goes down in value from 2 hours until 4 hours, and after 4 hours it goes up in value for the remainder of the day. The stock had a relative high value after 2 hours and then a relative low value after 4 hours. As the day goes on, the stock increases in value.

2. Nonlinear; the y-intercept is about 60, so there is an initial production cost of about \$60. There are no x-intercepts, so the cost per widget will never be \$0. The cost of producing 0 to 16 widgets is the same as the cost of producing 16 to 32 widgets. There is always a cost for producing any number of widgets. The average production cost decreases for making 0 to 16 widgets and then goes up for producing 16 to 32 widgets. The lowest production cost occurs when 16 widgets are produced. As greater numbers of widgets are produced, the average cost per widget will continue to increase.

3. Linear; the y-intercept is about 45, so the temperature was about 45°F when the measurement started. The x-intercept is about 5.5, so after about 5.5 hours, the temperature was 0°F. The graph has no line symmetry. The temperature is above zero for the first 5.5 hours, and then below zero after 5.5 hours. The temperature is going down for the entire time. There are no extrema. As the time increases, the temperature will continue to drop, which is not very likely.

4. Linear; the y-intercept is about -400, so the mowing service has a start-up cost of about \$400. The x-intercept is about 4, so after about 4 weeks, the profit will be \$0. The graph has no line symmetry. The profits will be in the negative until after about 4 weeks, and then will be positive for all time afterwards. The profits are constantly increasing. There are no extrema. As the number of weeks increases, the profits will increase.

5. Nonlinear; the y-intercept is about 20, so the purchase price of the vehicle was about \$20,000. There is no x-intercept, so the value of the vehicle will never equal 0. The graph has no line symmetry. The value of the vehicle is always positive. The value of the vehicle is always decreasing. There are no extrema. As the number of years increase, the value of the vehicle decreases.

6. Nonlinear; the y-intercept is about 5000, so the company has a profit of about \$5000 without spending any money on advertising. The x-intercepts are about -1000 and about 21,000, so the company will make a profit of \$0 if they spend \$21,000 on advertising. Spending between \$0 to \$10,000 on advertising will produce the same profits as spending between \$10,000 to \$20,000. The company will make a profit if they spend between \$0 and \$210,000. If they spend more than \$210,000 on advertising, they will lose money. The profits will increase until the company spends about \$100,000, and then the profits will decrease for any amount greater than \$100,000. Spending about \$100,000 will produce the greatest profit. As more money is spent on advertising, the profits will decrease so that the company is losing money.

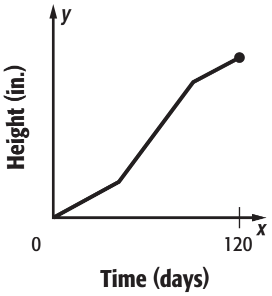
7. Nonlinear; the y-intercept is about 100. This means that the Web site had 100 hits before the time began. There is no x-intercept. The function is positive for all values of x. This means that the Web site has never experienced a time of inactivity. The function is increasing for all values of x, with no relative maxima or minima. As x-increases, y-increases, which means that the upward trend in the number of hits is expected to continue.

8. Nonlinear; the y-intercept is 0, which means that at the start, there was no medicine in the bloodstream. There appears to be no x-intercept, which means that the medicine does not ever fully leave the bloodstream for the time shown. The function is positive for all values of x, which means that after the medicine is taken, there is always some amount in the bloodstream. The function is increasing between about $x = 0$ and $x = 8$ and decreasing for $x > 8$, with a maximum value of about 1.5 at about $x = 8$. This means that the concentration of medicine increased over the first 8 hours to a maximum concentration of about 2.5 mg/mL, and then decreased. As x-increases, the value of y-decreases towards 0, which means that the concentration of medicine in the bloodstream becomes less and less, until there is practically none left.

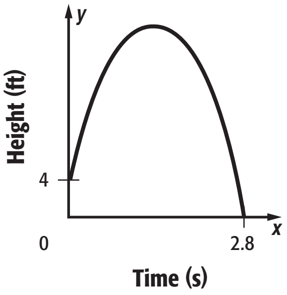
9. Nonlinear; the x- and y-intercept is 0, which means that a pendulum with no length cannot complete a swing. The function is positive and increasing for all values of x. Also, as x-increases, y-increases. The function has no relative minima or maxima. This means that as the pendulum gets longer, the time it takes for it to complete one full swing increases.

10. The graph is nonlinear with a *y*-intercept of 0, indicating that the cart started at the same height as the center of the wheel. The *x*-intercepts are 4, 8, 12, 16, 20, and 24, indicating that the ride returned to this same height 4, 8, 12, 16, and 20 seconds after the ride started. The function is positive between times 0 and 4, 8 and 12, and 16 and 20 seconds. During these times, the cart was higher than the center of the wheel. The function is negative between times 4 and 8, 12 and 16, and 20 and 24 seconds. During these times, the car was lower than the center of the wheel. The function is increasing between times 0 and 2, 6 and 10, 14 and 18, and 22 and 24 seconds. During these times, the wheel was rotating such that the cart was ascending. The function is decreasing between times 2 and 6, 10 and 14, 18 and 22 seconds. During these times, the wheel was rotating such that the cart was descending. The cart reached a maximum height of about 25 feet above the center of the wheel 2, 10, and 18 seconds after the ride started and a minimum height of about 25 feet below the center of the wheel 6, 14, and 22 seconds after the ride started. The up and down pattern in the graph suggests that if the ride continues for more than 24 seconds, the cart will continue to move back and forth between 25 feet above and 25 feet below the center of the wheel.

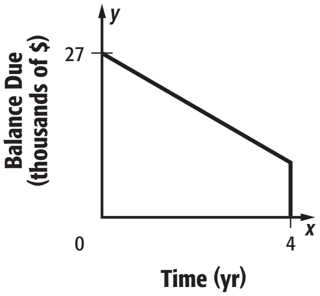
11. Sample answer: The function has a *y*-intercept of 0 and an *x*-intercept of 0, indicating that the plant started with no height as a seed in the ground. The function is increasing over its domain, so that plant was always getting taller. The function has no relative extrema.



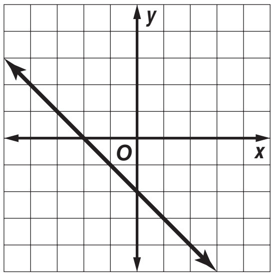
12. Sample answer: The function has a *y*-intercept of 4 and an *x*-intercept of 2.8, indicating that the ball started at a height of 4 feet and returned to ground level after 2.8 seconds. The function is increasing between approximately 0 and 1.5 seconds after the punt and decreasing between 1.5 and 2.8 seconds after the punt. The function has a relative maximum at about 1.5 seconds after the punt. At this time, the put reached its maximum height.



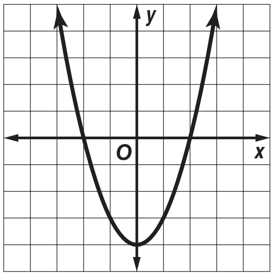
13. Sample answer: The function has a *y*-intercept of 27, indicating that the initial balance of the loan was $27,000. The *x*-intercept of 4 indicates that the loan was paid off after 4 years. The function is decreasing over its entire domain, indicting that the amount owed on the loan was always decreasing. The function has no relative extrema.



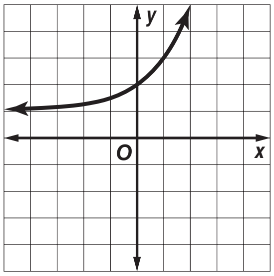
14. Sample graph:



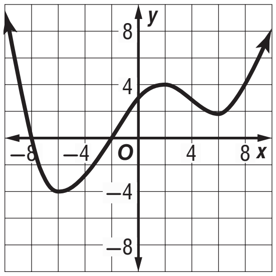
15. Sample graph:



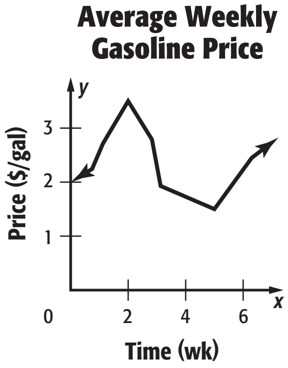
16. Sample graph:



17. Sample graph:



21. The graph has a relative maximum at about $x = 2$ and a relative minimum at about $x = 4.5$. This means that the weekly gasoline price spiked around week 2 at a high of about $3.50/gal and dipped around week 5 to a low of about $1.50/gal.



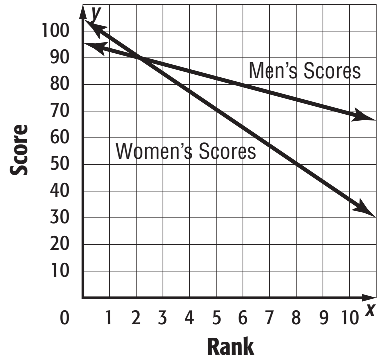
22. Sample answer: You could observe what the value of *y* is when *x* is zero to determine the *y*-intercept, and look for *x*-values that have a corresponding *y*-value of zero to determine the *x*-intercepts of the graph. The function is positive for *x*-values that have positive corresponding *y*-values and negative for *x*-values that have negative corresponding *y*-values. The function is increasing where as the *x*-values increase, the corresponding *y*-values increase and decreasing where as the *x*-values decrease, the corresponding *y*-values decrease. A relative maximum is located where the *y*-values change from increasing to decreasing. A relative minimum is located where the *y*-values change from decreasing to increasing. To describe the end behavior of the function, observe the value of *y* as *x* decreases and the value of *y* as *x*-increases, noticing whether it continues to increase, decrease, or approach a specific value.

Lab 4

1. Sample answer: For both lines that slant up and lines that slant down, the domain and range are all real numbers, there is one *x*-intercept and one *y*-intercept, there are no maximum or minimum points, and the graph has no symmetry. For lines that slant up, the function values are negative to the left of the *x*-intercept and positive to the right. For lines that slant down, the function values are positive to the left of the *x*-intercept and negative to the right. For lines that slant up, the function is increasing on the entire domain. For lines that slant down, the function is decreasing on the entire domain. For lines that slant up, as *x* decreases, *y* decreases and as *x* increases, *y* increases. For lines that slant down, as *x* decreases, *y* increases and as *x* increases, *y* decreases.
2. Sample answer: The domain is still all real numbers. The range is one negative value instead of a positive value. The end behavior is the same. There is still no *x*-intercept. The one *y*-intercept is a negative value instead of a positive value. There are still no maximum or minimum points. The function values are all negative instead of all positive. The function is constant on the entire domain. The graph is still symmetric about any vertical line.

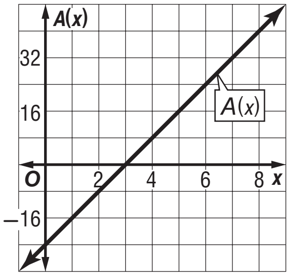
Lesson 5

15. Sample answer: Men: $y = -2.92x + 95.92$;
Women: $y = -7x + 106$; Women's scores have a steeper slope.

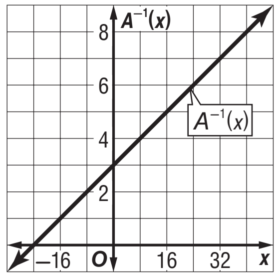


Lesson 6

37b. Sample answer: The domain represents possible values of *x*. The range represents the area of the rectangle and must be positive. This means that the domain of *A*(*x*) is all real numbers greater than 3, and the range of *A*(*x*) is all positive real numbers.



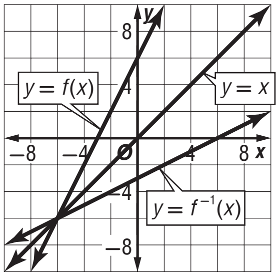
37d. Sample answer: The domain represents the area of the rectangle and must be positive. The range represents possible values for *x* in the expression $x - 3$. This means that the domain of *A*—1(*x*) is all positive real numbers, and the range of *A*—1(*x*) is all real numbers greater than 3.



41. Sometimes; sample answer: $f(x)$ and $g(x)$ do not need to be inverse functions for $f(a) = b$ and $g(b) = a$. For example, if $f(x) = 2x + 10$, then $f(2) = 14$ and if $g(x) = x - 12$, then $g(14) = 2$, but $f(x)$ and $g(x)$ are not inverse functions. However, if $f(x)$ and $g(x)$ are inverse functions, then $f(a) = b$ and $g(b) = a$.

42. Sample answer: $f(x) = 2x + 6$,

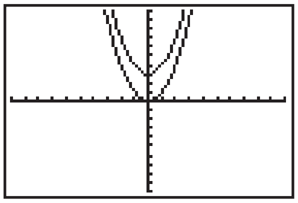
$f^1(x) = \frac{1}{2}x - 3$



43. Sample answer: A situation may require substituting values for the dependent variable into a function. By finding the inverse of the function, the dependent variable becomes the independent variable. This makes the substitution an easier process.

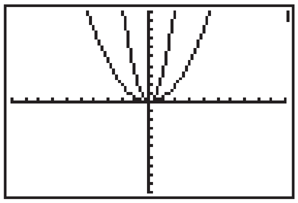
Lab 9

4. Both graphs have the same shape, but the graph of $y = x^2 + 3$ is 3 units above the graph of $y = x^2$.



[−10, 10] scl: 1 by [−10, 10] scl: 1

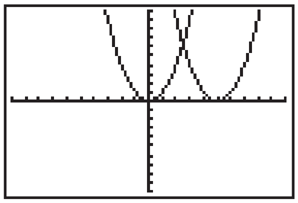
5.



[−10, 10] scl: 1 by [−10, 10] scl: 1

The graph of $y = 3x^2$ is narrower than the graph of $y = x^2$.

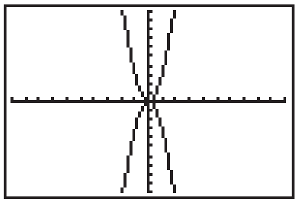
6.



[−10, 10] scl: 1 by [−10, 10] scl: 1

Both graphs have the same shape, but the graph of $y = (x - 5)^2$ is shifted to the right 5 units.

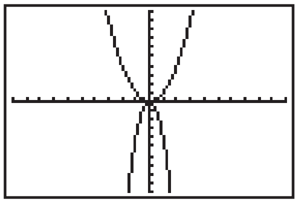
7.



[−10, 10] scl: 1 by [−10, 10] scl: 1

Both graphs have the same shape, but the graph of $y = -3x^2$ opens down while the graph of $y = 3x^2$ opens up.

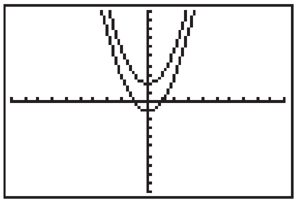
8.



[−10, 10] scl: 1 by [−10, 10] scl: 1

The graph of $y = -4x^2$ opens down and is narrower than the graph of $y = x^2$.

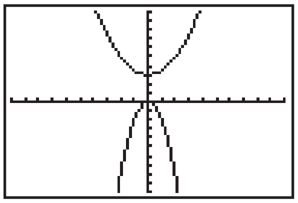
9.



[−10, 10] scl: 1 by [−10, 10] scl: 1

Both graphs have the same shape, but the graph of $y = x^2 + 2$ is 2 units above the x-axis and the graph of $y = x^2 - 1$ is 1 unit below the x-axis.

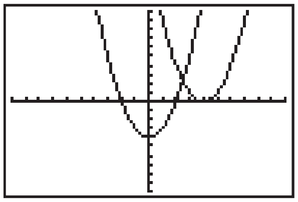
10.



[−10, 10] scl: 1 by [−10, 10] scl: 1

The graph of $y = -2x^2$ opens down and is narrower than the graph of $y = x^2 + 3$; also the graph of $y = x^2 + 3$ is 3 units above the x-axis.

11.

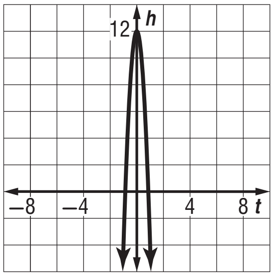


[−10, 10] scl: 1 by [−10, 10] scl: 1

Both graphs have the same shape, but the graph of $y = x^2 - 4$ is 4 units below the x-axis and the graph of $y = (x - 4)^2$ is 4 units to the right of the y-axis.

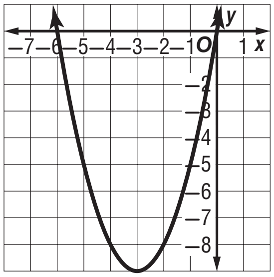
Lesson 10

24.



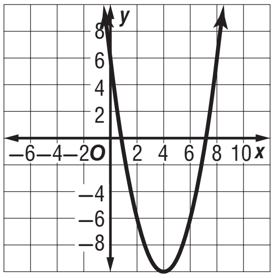
Lab 11

2.



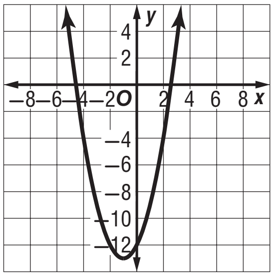
$y = (x + 3)^2 - 9$; $x = -3$; min. at $(-3, -9)$; $-6, 0$

3.



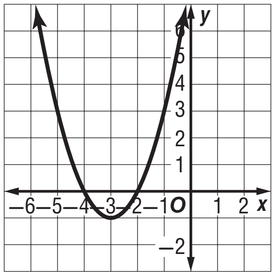
$y = (x - 4)^2 - 10$; $x = 4$; min. at $(4, -10)$; $1, 7$

4.



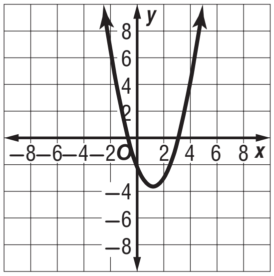
$y = (x + 1)^2 - 13$; $x = -1$; min. at $(-1, -13)$; $-4.61, 2.61$

5.



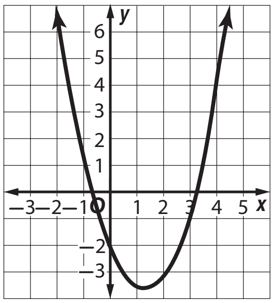
$y = (x + 3)^2 - 1$; $x = -3$; min. at $(-3, -1)$; $-4, -2$

6.



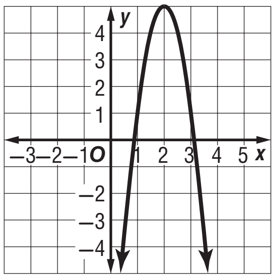
$y = (x - 2)^2 - 1$; $x = 2$; min. at $(2, -1)$; $1, 3$

7.



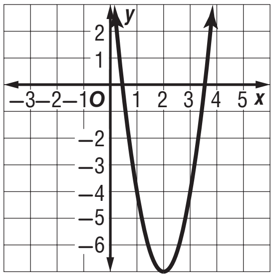
$y = (x - 1.2)^2 - 3.64$; $x = 1.2$; min. at $(1.2, -3.64)$; $-0.71, 3.11$

8.



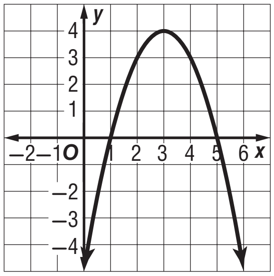
$y = -4(x - 2)^2 + 5$; $x = 2$; max. at $(2, 5)$; $0.88, 3.11$

9.



$y = 3(x - 2)^2 - 7$; $x = 2$; min. at $(2, -7)$; $0.5, 3.5$

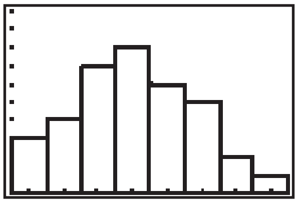
10.



$y = -(x - 3)^2 + 4$; $x = 3$; max. at $(3, 4)$; $1, 5$

Lesson 21 (Guided Practice)

1.

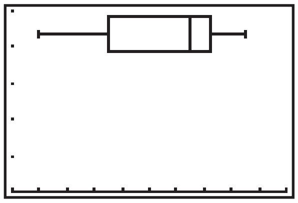


[6, 38] scl: 4 by [0, 10] scl: 1

symmetric

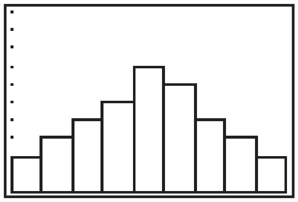
Lesson 21

4. Sample answer: The distribution is skewed, so use the five-number summary. The range is 21 − 6 or 15. The median time is 17, and half of the times are between 11 and 18.5 minutes.

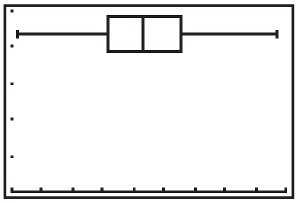


[4, 24] scl: 2 by [0, 5] scl: 1

5.



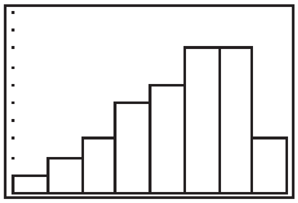
[24, 78] scl: 6 by [0, 10] scl: 1



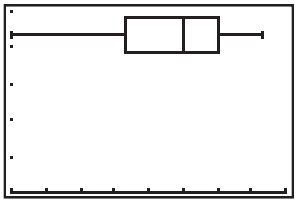
[24, 78] scl: 6 by [0, 5] scl: 1

symmetric

6.



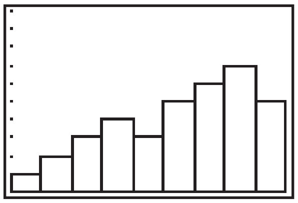
[33, 57] scl: 3 by [0, 10] scl: 1



[33, 57] scl: 3 by [0, 5] scl: 1

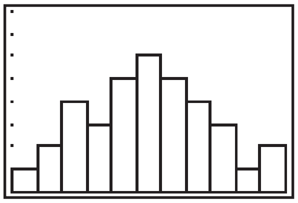
negatively skewed

7. Sample answer: The distribution is skewed, so use the five-number summary. The range is 53 − 12 or 41. The median is 39.5, and half of the data are between 28 and 48.



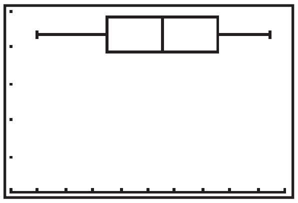
[10, 55] scl: 5 by [0, 10] scl: 1

8. Sample answer: The distribution is symmetric, so use the mean and standard deviation. The mean is 82 with a standard deviation of about 7.4.



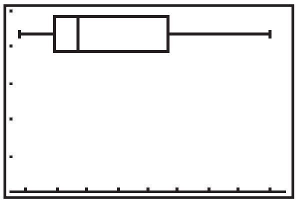
[66, 99] scl: 3 by [0, 8] scl: 1

9. Sample answer: The distribution is symmetric, so use the mean and standard deviation to describe the center and spread. The mean temperature is 52.8° with a standard deviation of about 4.22°.



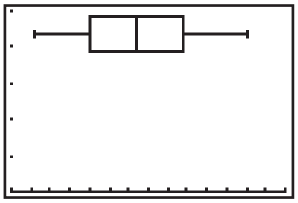
[42, 62] scl: 2 by [0, 5] scl: 1

10a. Sample answer: The distribution is skewed, so use the five-number summary. The range is 13.6 − 11.96 or 1.64. The median time is 12.34, and half of the times are between 12.18 and 12.93 seconds.



[11.9, 13.7] scl: 0.2 by [0, 5] scl: 1

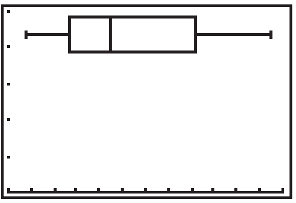
10b. Sample answer: The distribution is symmetric, so use the mean and standard deviation. The mean is about 12.22 seconds with a standard deviation of about 0.15 second.



[11.9, 12.6] scl: 0.05 by [0, 5] scl: 1

10c. Sample answer: Removing the times causes the shape of the distribution to go from being skewed to being symmetric. Therefore, the center and spread should be described using the mean and standard deviation.

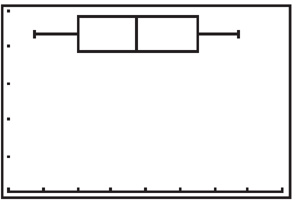
11a.



[7, 19] scl: 1 by [0, 5] scl: 1

Sample answer: The distribution is skewed, so use the five-number summary. The range is $18.50 – $7.75 or $10.75. The median price is $11.50, and half of the prices are between $9.63 and $15.13.

11b.



[7, 15] scl: 1 by [0, 5] scl: 1

Sample answer: The distribution is symmetric, so use the mean and standard deviation. The mean is about $10.67 with a standard deviation of about $1.84.

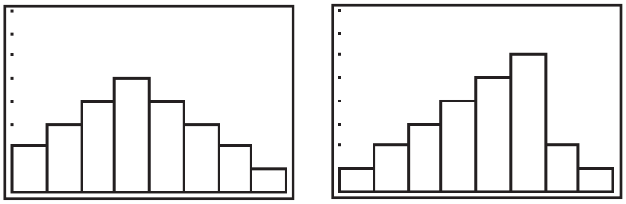
15. Sample answer: A bimodal distribution is a distribution of data that is characterized by having data divided into two clusters, thus producing two modes, and having two peaks. The distribution can be described by summarizing the center and spread of each cluster of data.

16. Sample answer: The average high temperature over the course of a year for a city may have a symmetrical distribution. The attendance at a baseball stadium over the course of a season for a team may be skewed.

17. Sample answer: In a symmetrical distribution, the majority of the data are located near the center of the distribution. The mean of the distribution is also located near the center of the distribution. Therefore, the mean and standard deviation should be used to describe the data. In a skewed distribution, the majority of the data lies either on the right or left side of the distribution. Since the distribution has a tail or may have outliers, the mean is pulled away from the majority of the data. The median is less affected. Therefore, the five-number summary should be used to describe the data.

Lesson 22

16a. The Electronics Superstore Game Central

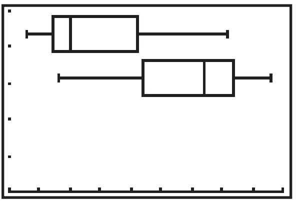


[25, 65] scl: 5 by [0, 8] scl: 1 [25, 65] scl: 5 by [0, 8] scl: 1

The Electronics Superstore, symmetric; Game Central, negatively skewed

16b. Sample answer: One distribution is symmetric and the other is skewed, so use the five-number summaries. The minimum and maximum for The Electronic Superstore are $25 and $62. The minimum and maximum for Game Stop Central are $26 and $61. Therefore, the ranges are approximately equal. The upper quartile for The Electronic Superstore is $49, while the median for Game Stop Central is $48. Since these two values are approximately equal, this means that about 50% of the data for Game Stop Central is greater than 75% of the data from The Electronic Superstore. Overall, while both stores have similar ranges, Game Stop Central has higher prices.

17a.

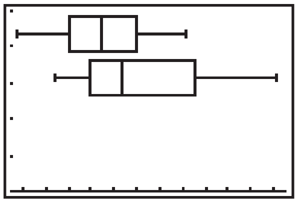


[1.5, 6] scl: 0.5 by [0, 5] scl: 1

Leon, positively skewed; Cassie, negatively skewed

17b. Sample answer: The distributions are skewed, so use the five-number summaries. The lower quartile for Leon’s times is 2.2 minutes, while the minimum for Cassie’s times is 2.3 minutes. This means that 25% of Leon’s times are less than all of Cassie’s times. The upper quartile for Leon’s times is 3.6 minutes, while the lower quartile for Cassie’s times is 3.7 minutes. This means that 75% of Leon’s times are less than 75% of Cassie’s time. Overall, we can conclude that Leon completed the brainteasers faster than Cassie.

18a.



[55, 175] scl: 10 by [0, 5] scl: 1

boys, symmetric; girls, positively skewed

18b. Sample answer: One distribution is symmetric and the other is skewed, so use the five-number summaries. The maximum for the boys is $131, while the upper quartile for the girls is $135.50. This means that 25% of the data from the girls is greater than all of the data from the boys. When listed from least to greatest, each statistic for the girls is greater than its corresponding statistic for the boys. We can conclude that in general, the girls spent more money on the dance than the boys.

22. Sample answer: The mean, median, and mode of the new data set can be found by multiplying each original statistic by *k*. The range and the standard deviation can be found by multiplying each original statistic by $|k|$.

23. Sample answer: Histograms show the frequency of values occurring within set intervals. This makes the shape of the distribution easy to recognize. However, no specific values of the data set can be identified from looking at a histogram, and the overall spread of the data can be difficult to determine. The box-and-whisker plots show the data divided into four sections. This aids when comparing the spread of one set of data to another. However, the box-and-whisker plots are limited because they cannot display the data any more specifically than showing it divided into four sections.

24. Sample answer: The mean, median, and mode of the new data set can be found by adding *k* to each original statistic, and then multiplying each resulting value by *m*. Since the range and the standard deviation are not affected when a constant is added to a set of data, they can be found by multiplying each original value by the constant *m*.

25. Sample answer: The mean and standard deviation are used to describe symmetric distributions. If both distributions are symmetric, then the mean and standard deviation will be used to compare the two distributions. If one of the distributions is skewed, the mean and standard deviation are no longer the best statistics to use to describe the distribution. Therefore, if one or both of the distributions is skewed, the five-number summaries should be used to compare the two distributions.

Lab 24

13c.

Conditional Relative Frequencies by Activities

Class	Potluck	Dinner	Gifts
Managers	9.1%	15.2%	13.6%
Staff	48.5%	54.5%	50%
Assistants	42.4%	30.3%	36.4%
Totals	100%	100%	100%

Conditional Relative Frequencies by Employees

Class	Potluck	Dinner	Gifts	Totals
Managers	27.3%	45.4%	27.3%	100%
Staff	35.6%	40%	24.4%	100%
Assistants	43.8%	31.2%	25%	100%

Additional Exercises, Lesson 1-1

1a. cost of extra minutes at $0.25 per minute

1b. cost of extra data used at $2 per megabyte

Additional Exercises, Lesson 1-2

2a. the cost of 45 blue zone tickets; 45(80) = $3600

2b. the cost of 15 preferred and 35 general tickets; 15(100) + 35(70) = $3950

2c. the total cost of 6 preferred, 11 blue zone, and 22 general admission tickets; 6(100) + 11(80) + 22(70) = $3020

Additional Exercises, Lesson 1-3

3a. the total cost for 5 shirts, 2 tank tops, and 7 shorts

3b. Sample answer: 8(7.99) + 4(4.99) + 4(6.99)
= $111.84; 8(5.99) + 4(8.99) + 4(2.99)
= $95.84; 4(7.99) + 4(5.99) + 4(4.99) + 4(2.99)
= $87.84

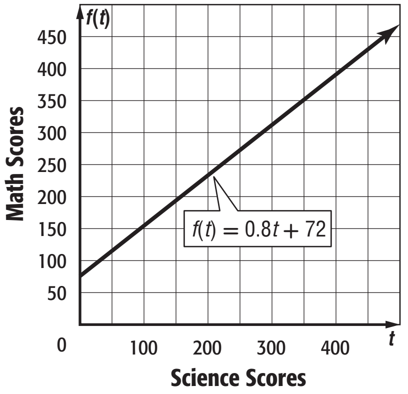
Additional Exercises, Lesson 1-7

5a. If x represents the school year and $f(x)$ is the enrollment then $f(x) = 0.25x$; $1.25, $3.00

5b. $f(5) = 1.25$, $f(12) = 3$; It costs $1.25 to send 5 photos and $3.00 to send 12 photos.

5c. The domain is the number of pictures sent and the range is the cost.

6a.



When the science score is 0, the math score is 72. For each point the science score increases, the math score increases by 0.8 points.

Additional Exercises, Lesson 2-4

10.

$5x + 2 = ax - 1$	Original equation
$5x + 2 - ax = -1$	Subtract *ax* from each side.
$5x - ax = -3$	Subtract 2 from each side.
$(5 - a)x = -3$	Distributive Property
$x = \frac{-3}{5 - a}$	Divide each side by 5 – a.

Additional Exercises, Lesson 4-4

18. Sample answer: Parallel lines: similarities: The domain and range are all real numbers, the functions are both either increasing or decreasing on the entire domain, the end behavior is the same; differences: *x*- and *y*-intercepts are different. Perpendicular lines: similarities: The domain and range are all real numbers; differences: One function is increasing and the other is decreasing on the entire domain, as *x* decreases, *y* increases for one function and decreases for the other and as *x* increases, *y* increases for one function and decreases for the other.

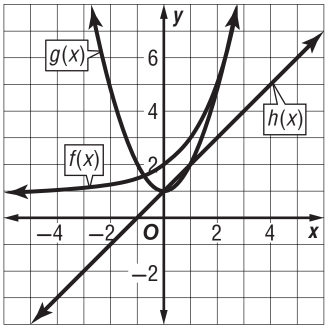
Additional Exercises, Lesson 7-3

24. $\frac{1.41 \times 10^6}{7.95 \times 10^5} \approx 1.774$; The phone from company B is about 1.774 times as fast as the phone from company A.

Additional Exercises, Lesson 9-1

27. Sample answer: Suppose $a = 1$, $b = 2$, and $c = 1$.

x	$f(x) = 2^x + 1$	$g(x) = x^2 + 1$	$h(x) = x + 1$
−10	1.00098	101	−9
−8	1.00391	65	−7
−6	1.01563	37	−5
−4	1.0625	17	−3
−2	1.25	5	−1
0	2	1	1
2	5	5	3
4	17	17	5
6	65	37	7
8	257	65	9
10	1205	101	11



Intercepts: $f(x)$ and $g(x)$ have no *x*-intercepts, $h(x)$ has one at -1 because $c = 1$. $g(x)$ and $h(x)$ have one *y*-intercept at 1 and $f(x)$ has one *y*-intercept at 2. The graphs are all shifted up 1 unit from the graph of the parent functions because $c = 1$.

Increasing/Decreasing: $f(x)$ and $h(x)$ are increasing on the entire domain. $g(x)$ is increasing to the right of the vertex and decreasing to the left.

Positive/Negative: The function values for $f(x)$ and $g(x)$ are all positive. The function values of $h(x)$ are negative for $x < -1$ and positive for $x > -1$.

Maxima/Minima: $f(x)$ and $h(x)$ have no maxima or minima. $g(x)$ has a minimum at (0, 1).

Symmetry: $f(x)$ and $h(x)$ have no symmetry. $g(x)$ is symmetric about the y-axis.

End behavior: For $f(x)$ and $h(x)$, as x increases, y increases and as x decreases, y decreases. For $g(x)$, as x increases, y increases and as x decreases, y increases.

The exponential function $f(x)$ eventually exceeds the others.

Additional Exercises, Lesson 9-5

28c. Sample answer: Multiply three factors written from the roots.

$(x - 1)(x - 2)(x - 3) = 0$

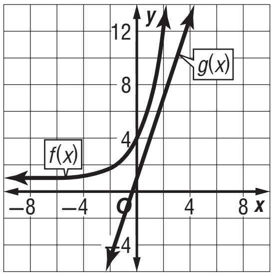
$x^3 - 6x^2 + 11x - 6 = 0$

The equation is not quadratic because its degree is 3 not 2.

Additional Exercises, Lesson 9-6

29c. Sample answer: Since Daniel's investment is the product of d and a factor not depending on d, it varies directly as the original deposit.

30. Sample answer: Let $a = 3$, $b = 2$ and $c = 1$.



x	$f(x) = 3.2^x + 1$	$g(x) = 3x + 1$
−5	1.09375	−14
−4	1.1875	−11
−3	1.375	−8
−2	1.75	−5
−1	2.5	−2
0	4	1
1	7	4
2	13	7
3	25	10
4	49	13
5	97	16

The y-intercept of $f(x)$ is 4 and the y-intercept of $g(x)$ is 1. Both $f(x)$ and $g(x)$ increase as x increases. All function values for $f(x)$ are positive, while $g(x)$ has both positive and negative values. Neither $f(x)$ nor $g(x)$ have maximum or minimum points, and neither has symmetry.

31. Sample answer: The domain of absolute value, step, quadratic, and exponential functions is all real numbers, while some piecewise functions may not be defined for all real numbers. The range of absolute value, step, quadratic, and exponential function is limited to a portion of the real numbers, but the range of a piecewise-defined function can be all real numbers. The graphs of absolute value and quadratic functions have either one maximum and no minima or one minimum and no maxima, and both have symmetry with respect to a vertical line through the point where this maximum or minimum occurs. The graphs of absolute value, quadratic, and exponential functions have no breaks or jumps, while graphs of step functions always do and graphs of piecewise-defined functions sometimes do.

Additional Exercises, Lesson 9-7

32d. $C(t) = 3000t + 20{,}500(0.995)^t$, The function represents the number of gallons of water remaining during the times in which more water is being pumped into the pool.

Additional Exercises, Lesson 9-9

33. If one linear term is ax, the next term is $a(x + 1)$, and the difference between the terms is $a(x + 1) - ax = ax + a - ax$ or a. If one exponential term is a^x, the next term is a^{x+1}, and the ratio of the terms is $\frac{a^{x+1}}{a^x}$ or a.

Additional Exercises, Lesson 10-3

34. Irrational; irrational; no rational number could be added to or multiplied by an irrational number so that the result is rational.