Fundamental Identities

Reciprocal Identities

$$\sec\theta = \frac{1}{\cos\theta}$$

$$\csc\theta = \frac{1}{\sin\theta}$$

$$\cot\theta = \frac{1}{\tan\theta}$$

Ratio Identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta}$$

Pythagorean Identities

$$\sin^2\theta + \cos^2\theta = 1$$

$$\tan^2\theta + 1 = \sec^2\theta$$

$$1 + \cot^2\theta = \csc^2\theta$$

Identities due to Symmetry

$$\sin(-\theta) = -\sin\theta$$

$$\cos(-\theta) = \cos\theta$$

$$\tan(-\theta) = -\tan\theta$$

Cofunction Identities

$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta \qquad \cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$$

$$\tan\left(\frac{\pi}{2} - \theta\right) = \cot\theta \qquad \cot\left(\frac{\pi}{2} - \theta\right) = \tan\theta$$

$$\sec\left(\frac{\pi}{2} - \theta\right) = \csc\theta \qquad \csc\left(\frac{\pi}{2} - \theta\right) = \sec\theta$$

Sum and Difference Identities

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$

$$\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}$$

Double-Angle Identities

$$\sin(2\alpha) = 2\sin\alpha\cos\alpha$$

$$\cos(2\alpha) = \cos^2\alpha - \sin^2\alpha$$

$$= 2\cos^2\alpha - 1$$

$$= 1 - 2\sin^2\alpha$$

Half-Angle Identities

$$\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1 - \cos\theta}{2}}$$

$$\cos\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1 + \cos\theta}{2}}$$

$$\tan\left(\frac{\theta}{2}\right) = \frac{1 - \cos\theta}{\sin\theta}$$

$$= \frac{\sin\theta}{1 + \cos\theta}$$

Power Reduction Identities

$$\sin^2\theta = \frac{1 - \cos 2\theta}{2}$$

$$\cos^2\theta = \frac{1 + \cos 2\theta}{2}$$

$$\tan^2\theta = \frac{1 - \cos 2\theta}{1 + \cos 2\theta}$$

Product-to-Sum Identities

$$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha + \beta) + \sin(\alpha - \beta)]$$

$$\cos\alpha\sin\beta = \frac{1}{2}[\sin(\alpha + \beta) - \sin(\alpha - \beta)]$$

$$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha + \beta) + \cos(\alpha - \beta)]$$

$$\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha - \beta) - \cos(\alpha + \beta)]$$

Sum-to-Product Identities

$$\sin\alpha + \sin\beta = 2\sin\left(\frac{\alpha + \beta}{2}\right)\cos\left(\frac{\alpha - \beta}{2}\right)$$

$$\sin\alpha - \sin\beta = 2\cos\left(\frac{\alpha + \beta}{2}\right)\sin\left(\frac{\alpha - \beta}{2}\right)$$

$$\cos\alpha + \cos\beta = 2\cos\left(\frac{\alpha + \beta}{2}\right)\cos\left(\frac{\alpha - \beta}{2}\right)$$

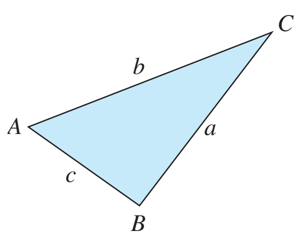
$$\cos\alpha - \cos\beta = -2\sin\left(\frac{\alpha + \beta}{2}\right)\sin\left(\frac{\alpha - \beta}{2}\right)$$

Law of Sines

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

Area of a Triangle

$$\text{Area} = \frac{1}{2}bc\sin A$$



Law of Cosines

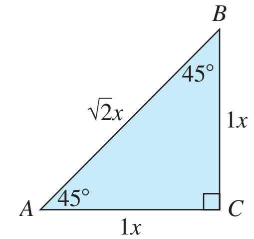
$$a^2 = b^2 + c^2 - 2bc\cos A$$

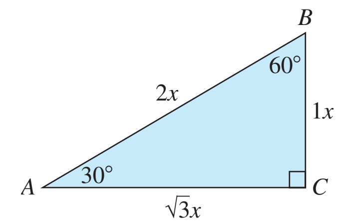
$$b^2 = a^2 + c^2 - 2ac\cos B$$

$$c^2 = a^2 + b^2 - 2ab\cos C$$

Special Triangles and Special Angles

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\csc\theta$	$\sec\theta$	$\cot\theta$
$0° = 0$	0	1	0	—	1	—
$30° = \frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$	2	$\frac{2}{\sqrt{3}}$	$\sqrt{3}$
$45° = \frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	$\sqrt{2}$	$\sqrt{2}$	1
$60° = \frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{2}{\sqrt{3}}$	2	$\frac{1}{\sqrt{3}}$
$90° = \frac{\pi}{2}$	1	0	—	1	—	0





Degree and Radian Conversions

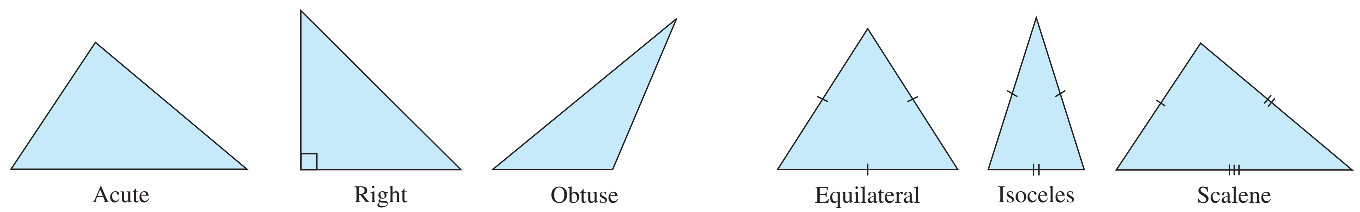
degrees to radians: multiply by $\frac{\pi}{180°}$ (degrees cancel)

radians to degrees: multiply by $\frac{180°}{\pi}$ (radians cancel)

Triangle Classifications

By Angle Measure

By Side Length

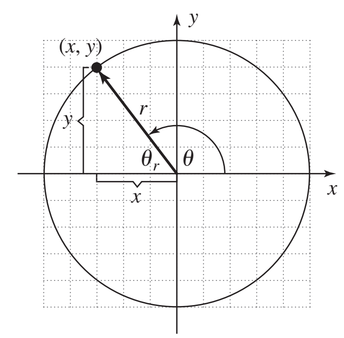


Trigonometry and the Coordinate Plane

For $P(x, y)$ a point on the terminal side of an angle θ in standard position:

$$\cos\theta = \frac{x}{r} \qquad \sin\theta = \frac{y}{r} \qquad \tan\theta = \frac{y}{x},\ x \neq 0$$

$$\sec\theta = \frac{r}{x},\ x \neq 0 \qquad \csc\theta = \frac{r}{y},\ y \neq 0 \qquad \cot\theta = \frac{x}{y},\ y \neq 0$$

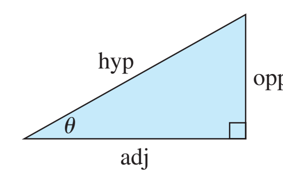


Right Triangle Trigonometry

For right $\triangle ABC$ with indicated sides **adj**acent and **opp**osite to acute angle θ:

$$\cos\theta = \frac{\text{adj}}{\text{hyp}} \qquad \sin\theta = \frac{\text{opp}}{\text{hyp}} \qquad \tan\theta = \frac{\text{opp}}{\text{adj}}$$

$$\sec\theta = \frac{\text{hyp}}{\text{adj}} \qquad \csc\theta = \frac{\text{hyp}}{\text{opp}} \qquad \cot\theta = \frac{\text{adj}}{\text{opp}}$$

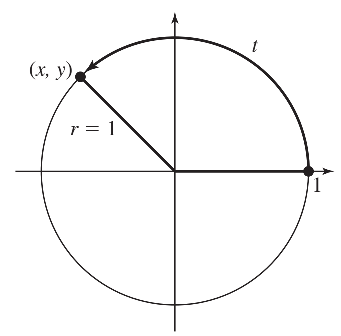


Trigonometric Functions of a Real Number

For any real number t and point $P(x, y)$ on the unit circle associated with t:

$$\cos t = x \qquad \sin t = y \qquad \tan t = \frac{y}{x};\ x \neq 0$$

$$\sec t = \frac{1}{x};\ x \neq 0 \qquad \csc t = \frac{1}{y};\ y \neq 0 \qquad \cot t = \frac{x}{y};\ y \neq 0$$



Special Constants

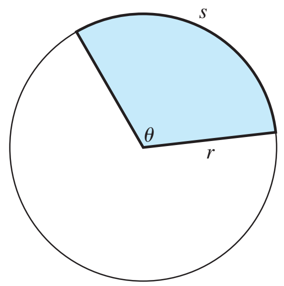
$\pi \approx 3.1416$	$\frac{\pi}{2} \approx 1.5708$	$\frac{\pi}{3} \approx 1.0472$	$\frac{\pi}{4} \approx 0.7854$	$\frac{\pi}{6} \approx 0.5236$	$\frac{\pi}{12} \approx 0.2618$
$e \approx 2.7183$	$\sqrt{2} \approx 1.4142$	$\sqrt{3} \approx 1.7321$	$\frac{\sqrt{2}}{2} \approx 0.7071$	$\frac{\sqrt{3}}{2} \approx 0.8660$	$\frac{\sqrt{3}}{3} \approx 0.5774$

Arcs and Sectors

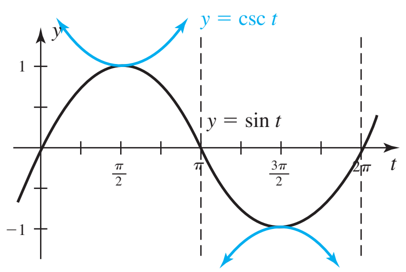
For a circle of radius r and angle θ in radians:

arc length: $s = r\theta$

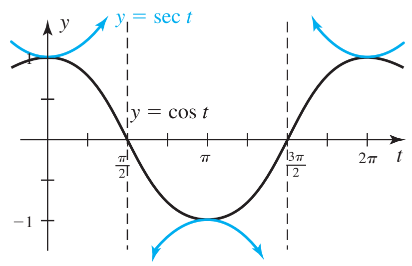
area of sector: $A = \frac{1}{2}r^2\theta$



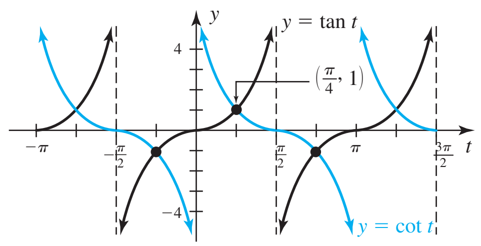
Graphs of the Trigonometric Functions



Domain: $t \in (-\infty, \infty)$
Range: $\sin t \in [-1, 1]$



Domain: $t \in (-\infty, \infty)$
Range: $\cos t \in [-1, 1]$



Domain: $t \neq \frac{\pi}{2}(2k + 1);\ k \in \mathbb{Z}$
Range: $\tan t \in \mathbb{R}$

Transformations of Basic Trig Graphs

Given Function

$y = f(x)$

Transformation of $y = f(x)$

$$y = Af\left[B\left(x \pm \frac{C}{B}\right)\right] \pm D$$

- A: north/south reflections; vertical stretches and compressions
- $\frac{C}{B}$: horizontal shift, opposite direction of sign
- D: vertical shift, same direction as sign

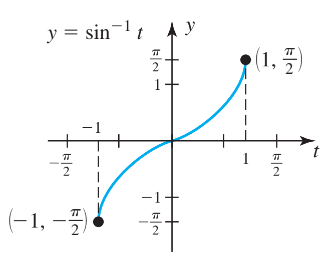
For $y = A\sin\left[B\left(x \pm \frac{C}{B}\right)\right] \pm D$ we have: amplitude: $|A|$, period: $\frac{2\pi}{B}$, horizontal shift: $\frac{C}{B}$, vertical shift: D

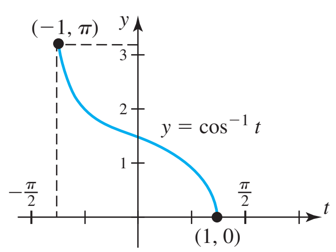
The Inverse Trigonometric Functions

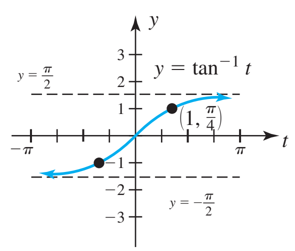
For $y = \sin t$ with $t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $y \in [1,\ 1]$, the inverse function is $y = \sin^{-1}t$, where $t \in [1, 1]$ and $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

For $y = \cos t$ with $t \in [0,\ \pi]$ and $y \in [1,\ 1]$, the inverse function is $y = \cos^{-1}t$, where $t \in [1, 1]$ and $y \in [0,\ \pi]$.

For $y = \tan t$ with $t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $y \in \mathbb{R}$, the inverse function is $y = \tan^{-1}t$, where $t \in \mathbb{R}$ and $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.







Commonly used, small case Greek letters

α	alpha	β	beta	γ	gamma	δ	delta	ϵ	epsilon
ζ	zeta	θ	theta	λ	lambda	μ	mu	π	pi
ρ	rho	σ	sigma	ϕ	phi	ψ	psi	ω	omega

Complex Numbers $z = a + bi$

Absolute Value

$$|z| = \sqrt{a^2 + b^2}$$

distance from (0, 0) to (a, b)

Trigonometric Form

$$z = r(\cos\theta + i\sin\theta)$$

where $r = |z|$

Products and Quotients

$$z_1z_2 = r_1r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]$$

Powers and DeMoivres Theorem

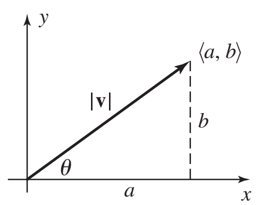
$z^n = r^n(\cos n\theta + i\sin n\theta)$ for positive integers n

Roots and the *n*th Roots Theorem

$\sqrt[n]{z} = \sqrt[n]{r}\left(\cos\frac{\theta + 2\pi k}{n} + i\sin\frac{\theta + 2\pi k}{n}\right)$ for $k = 0, 1, 2, \ldots, n - 1$

Vectors and the Dot Product

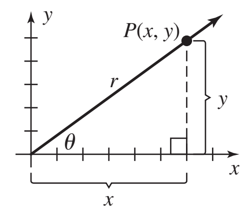
- For a position vector, $\mathbf{v} = \langle a, b\rangle$ and angle θ as shown, $a = |\mathbf{v}|\cos\theta$ and $b = |\mathbf{v}|\sin\theta$, where $\theta_r = \tan^{-1}\left|\frac{b}{a}\right|$ and $|\mathbf{v}| = \sqrt{a^2 + b^2}$.
- For any nonzero vector $\mathbf{v} = \langle a, b\rangle = a\mathbf{i} + b\mathbf{j}$, the vector $\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|}$ is a unit vector in the same direction as $\mathbf{v}$.
- Given the vectors $\mathbf{u} = \langle a, b\rangle$ and $\mathbf{v} = \langle c, d\rangle$, their dot product is denoted $\mathbf{u}\cdot\mathbf{v}$ and is defined as: $\mathbf{u}\cdot\mathbf{v} = \langle a, b\rangle\cdot\langle c, d\rangle = ac + bd$.
- Given the nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ and angle θ between them, $\cos\theta = \frac{\mathbf{u}}{|\mathbf{u}|}\cdot\frac{\mathbf{v}}{|\mathbf{v}|}$.



Polar Coordinates

$P(x, y)$ in rectangular coordinates can be represented as $P(r, \theta)$ in polar coordinates:

$x = r\cos\theta$ $\quad$ $y = r\sin\theta$ $\quad$ $r = \sqrt{x^2 + y^2}$ $\quad$ $\theta_r = \tan^{-1}\left|\frac{y}{x}\right|, x \neq 0$



Logarithms and Logarithmic Properties

$y = \log_b x \Leftrightarrow b^y = x$	$\log_b b = 1$	$\log_b 1 = 0$
$\log_b b^x = x$	$b^{\log_b x} = x$	$\log_c x = \frac{\log_b x}{\log_b c}$
$\log_b MN = \log_b M + \log_b N$	$\log_b\frac{M}{N} = \log_b M - \log_b N$	$\log_b M^P = P\cdot\log_b M$

Applications of Exponentials and Logarithms

$A \rightarrow$ amount accumulated $\quad$ $P \rightarrow$ initial deposit, $\mathcal{P} \rightarrow$ periodic payment $\quad$ $n \rightarrow$ compounding periods/year

$r \rightarrow$ interest rate per year $\quad$ $R \rightarrow$ interest rate per time period $\left(\frac{r}{n}\right)$ $\quad$ $t \rightarrow$ time in years

Interest Compounded *n* Times per Year

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

Interest Compounded Continuously

$$A = Pe^{rt}$$

Accumulated Value of an Annuity

$$A = \frac{\mathcal{P}}{R}[(1 + R)^{nt} - 1]$$

Payments Required to Accumulate Amount *A*

$$\mathcal{P} = \frac{AR}{(1 + R)^{nt} - 1}$$

Topics from Algebra

Special Products

$$(a + b)^2 = a^2 + 2ab + b^2 \qquad (a - b)^2 = a^2 - 2ab + b^2$$

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3 \qquad (a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$$

$$(x + c)(x + d) = x^2 + (c + d)x + cd \qquad (ax + c)(bx + d) = abx^2 + (ad + bc)x + cd$$

Special Factorizations

$$a^2 + 2ab + b^2 = (a + b)^2 \qquad a^2 - 2ab + b^2 = (a - b)^2$$

$$x^2 + (c + d)x + cd = (x + c)(x + d) \qquad abx^2 + (ad + bc)x + cd = (ax + c)(bx + d)$$

$a^2 - b^2 = (a + b)(a - b)$ $\qquad$ $a^2 + b^2$ is prime over the real numbers

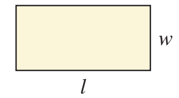
$$a^3 - b^3 = (a - b)(a^2 + ab + b^2) \qquad a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$

Formulas from Plane Geometry: $P \longrightarrow$ perimeter, $C \longrightarrow$ circumference, $A \longrightarrow$ area

Rectangle

$P = 2l + 2w$

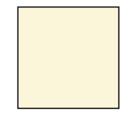
$A = lw$



Square

$P = 4s$

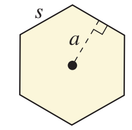
$A = s^2$



Regular Polygon

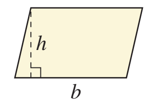
$P = ns$

$A = \frac{a}{2}P$



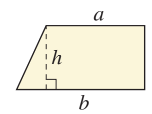
Parallelogram

$A = bh$



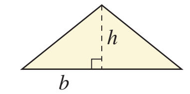
Trapezoid

$A = \frac{h}{2}(a + b)$



Triangle

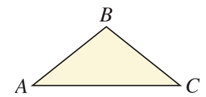
$A = \frac{1}{2}bh$



Triangle

Sum of angles

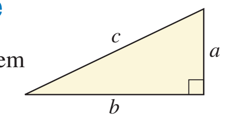
$A + B + C = 180°$



Right Triangle

Pythagorean Theorem

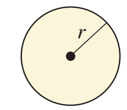
$a^2 + b^2 = c^2$



Circle

$A = \pi r^2$

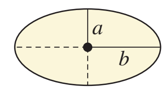
$C = 2\pi r = \pi d$



Ellipse

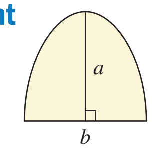
$A = \pi ab$

$C \approx \sqrt{2(a^2 + b^2)}$



Right Parabolic Segment

$A = \frac{2}{3}ab$



Formulas from Solid Geometry: $S \longrightarrow$ surface area, $V \longrightarrow$ volume

Rectangular Solid

$V = lwh$

$S = lw + lh + wh$

Cube

$V = s^3$

$S = 6s^2$

Right Circular Cylinder

$V = \pi r^2h$

$S = 2\pi r(r + h)$

Right Circular Cone

$V = \frac{1}{3}\pi r^2h$

$S = \pi r(r + \sqrt{r^2 + h^2})$

Right Square Pyramid

$V = \frac{1}{3}s^2h$

$S = s^2 + s\sqrt{s^2 + 4h^2}$

Sphere

$V = \frac{4}{3}\pi r^3$

$S = 4\pi r^2$

Formulas from Analytical Geometry: $P_1 \longrightarrow (x_1, y_1)$, $P_2 \longrightarrow (x_2, y_2)$

Distance between P_1 and P_2

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Slope of Line Containing P_1 and P_2

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

Equation of Line Containing P_1 and P_2

Point-Slope Form

$$y - y_1 = m(x - x_1)$$

Equation of Line Containing P_1 and P_2

Slope-Intercept Form (slope m, y-intercept b)

$y = mx + b$, where $b = y_1 - mx_1$

Parallel Lines

Slopes Are Equal: $m_1 = m_2$

Perpendicular Lines

Slopes Have a Product of -1: $m_1 = -\frac{1}{m_2}$ or $m_1m_2 = -1$

Intersecting Lines

Slopes Are Unequal: $m_1 \neq m_2$

Dependent (Coincident) Lines

Slopes and y-Intercepts Are Equal: $m_1 = m_2$, $b_1 = b_2$