Polynomial and Rational Functions

CHAPTER OUTLINE

CHAPTER CONNECTIONS

In a study of demographics, the population density of a city and its surrounding area is measured using a unit called *people per square mile*. The population density is much greater near the city's center, and tends to decrease as you move out into suburban and rural areas. The density can be modeled using the formula $D(x) = \frac{ax}{x^2 + b}$, where $D(x)$ represents the density at a distance of x mi from the center of a city, and a and b are constants related to a particular city and its sprawl. Using this equation, city planners can determine how far from the city's center the population drops below a certain level, and answer other important questions to help plan for future growth. This application appears as Exercise 71 in Section 3.5.

Check out these other real-world connections:

- Modeling the height of a rocket (Section 3.1, Exercise 46)
- Tourist Population of a Resort Town (Section 3.2, Exercise 81)
- County Deficits (Section 3.3, Exercise 107)
- Volume of Traffic (Section 3.4, Exercise 85)

3.1 Quadratic Functions and Applications

Learning Objectives

In Section 3.1 you will learn how to:

- ❑ **A.** Graph quadratic functions by completing the square
- ❑ **B.** Graph quadratic functions using the vertex formula
- ❑ **C.** Find the equation of a quadratic function from its graph
- ❑ **D.** Solve applications involving extreme values

As our knowledge of functions grows, our ability to apply mathematics in new ways likewise grows. In this section, we'll build on the foundation laid in Chapter 2, as we introduce additional function families and the tools needed to apply them effectively. We begin with the family of quadratic functions.

A. Graphing Quadratic Functions by Completing the Square

The squaring function $f(x) = x^2$ is actually a member of the family of **quadratic functions,** defined as follows.

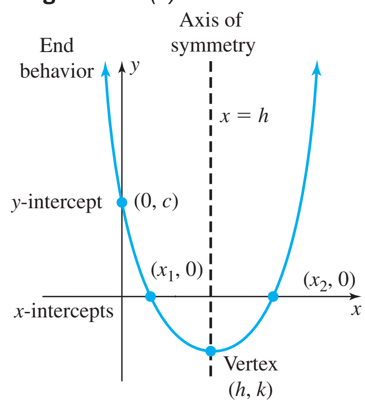
Quadratic Functions

A quadratic function is one of the form

$$f(x) = ax^2 + bx + c,$$

where a, b, and c are real numbers and $a \neq 0$.

Figure 3.1 $f(x) = ax^2 + bx + c$



As shown in Figure 3.1, the function is written in **standard form.** For $f(x) = x^2$, $a = 1$ with b and c equal to 0. The function $f(x) = 2x^2 + x - 3$ is also quadratic, with $a = 2$, $b = 1$ and $c = -3$. Our earlier work suggests the graph of *any* quadratic function will be a parabola. Figure 3.1 provides a summary of the characteristic features of this graph. As pictured, the parabola opens upward with the vertex at (h, k), so k is a global minimum. Since the vertex is below the x-axis, the graph has two x-intercepts. The axis of symmetry goes through the vertex, and has equation $x = h$. The y-intercept is $(0, c)$, since $f(0) = c$.

In Section 2.6, we graphed transformations of $f(x) = x^2$, using $y = a(x \pm h)^2 \pm k$. Here, we'll show that by completing the square, we can graph *any* quadratic function as a transformation of this basic graph.

When completing the square on a quadratic *equation* (Section 1.5), we applied the standard properties of equality to both sides of the equation. When completing the square on a ***quadratic function,*** the process is altered slightly in that we operate on only one side.

EXAMPLE 1 ▶ Graphing a Quadratic Function by Completing the Square

Given $g(x) = x^2 - 6x + 5$, complete the square to rewrite g as a transformation of $f(x) = x^2$, then graph the function.

Solution ▶ To begin we note the leading coefficient is $a = 1$.

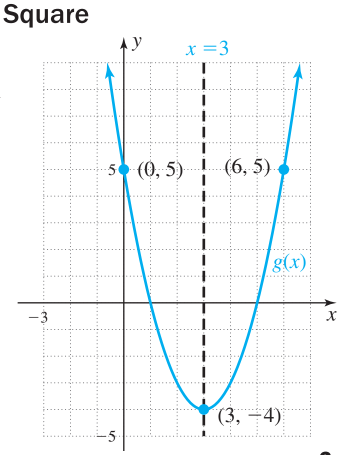
$g(x) = x^2 - 6x + 5$ — given function

$= 1(x^2 - 6x + \_\_\_) + 5$ — group variable terms, note $a = 1$

$= 1(x^2 - 6x + 9) - 9 + 5$ — $\left[\left(\frac{1}{2}\right)(-6)\right]^2 = 9$

(adds $1 \cdot 9 = 9$; subtract 9)

$= (x - 3)^2 - 4$ — factor and simplify



The graph of g is the graph of f shifted 3 units right, and 4 units down. The graph opens upward ($a > 0$) with the vertex at $(3, -4)$, and axis of symmetry $x = 3$. From the original equation we find $g(0) = 5$, giving a y-intercept of $(0, 5)$. The point $(6, 5)$ was obtained using the axis of symmetry. The graph is shown in the figure.

Now try Exercises 7 through 10 ►

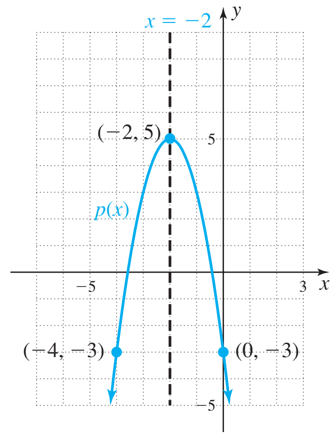
Note that by **adding 9** and simultaneously **subtracting 9** (essentially adding "0"), we changed only the *form* of the function, not its value. In other words, the resulting expression is equivalent to the original. If the leading coefficient is not 1, we factor it out from the variable terms, but take it into account when we add the constant needed to maintain an equivalent expression.

EXAMPLE 2 ► Graphing a Quadratic Function by Completing the Square

Given $p(x) = -2x^2 - 8x - 3$, complete the square to rewrite p as a transformation of $f(x) = x^2$, then graph the function.

Solution ►

$$p(x) = -2x^2 - 8x - 3 \quad \text{given function}$$
$$= (-2x^2 - 8x + \_\_\_) - 3 \quad \text{group variable terms}$$
$$= -2(x^2 + 4x + \_\_\_) - 3 \quad \text{factor out } a = -2 \text{ (notice sign change)}$$
$$= -2(x^2 + 4x + 4) - (-8) - 3 \quad \left[\left(\frac{1}{2}\right)(4)\right]^2 = 4$$
$$\text{adds } -2 \cdot 4 = -8 \qquad \text{subtract } -8$$
$$= -2(x + 2)^2 + 8 - 3 \quad \text{factor trinomial, simplify}$$
$$= -2(x + 2)^2 + 5 \quad \text{result}$$



The graph of p is a parabola, shifted 2 units left, stretched by a factor of 2, reflected across the x-axis (opens downward), and shifted up 5 units. The vertex is $(-2, 5)$, and the axis of symmetry is $x = -2$. From the original function, the y-intercept is $(0, -3)$. The point $(-4, -3)$ was obtained using the axis of symmetry. The graph is shown in the figure.

Now try Exercises 11 through 14 ►

WORTHY OF NOTE

In cases like $f(x) = 3x^2 - 10x + 5$, where the linear coefficient has no integer factors of a, we factor out 3 and *simultaneously divide the linear coefficient by 3*. This yields $h(x) = 3\left(x^2 - \frac{10}{3}x + \_\_\_\right) + 5$, and the process continues as before: $[(\frac{1}{2})(\frac{10}{3})]^2 = (\frac{5}{3})^2 = \frac{25}{9}$, and so on. For more on this idea, **see Exercises 15 through 20.**

By adding 4 to the variable terms within parentheses, we actually added $-2 \cdot 4 = -8$ to the value of the function. To adjust for this we subtracted -8. The basic ideas are summarized here.

Graphing $f(x) = ax^2 + bx + c$ by Completing the Square

1. Group the variable terms apart from the constant c.
2. Factor out the leading coefficient a.
3. Compute $[\frac{1}{2}(\frac{b}{a})]^2$ and add the result to the grouped terms, then subtract $a \cdot [\frac{1}{2}(\frac{b}{a})]^2$ to maintain an equivalent expression.
4. Factor the grouped terms as a binomial square and simplify.
5. Graph using transformations of $f(x) = x^2$.

☑ **A.** You've just learned how to graph quadratic functions by completing the square

B. Graphing Quadratic Functions Using the Vertex Formula

When the process of completing the square is applied to $f(x) = ax^2 + bx + c$, we obtain a very useful result. Notice the close similarities to Example 2.

$$f(x) = ax^2 + bx + c \quad \text{quadratic function}$$
$$= (ax^2 + bx + \underline{\quad}) + c \quad \text{group variable terms apart from the constant } c$$
$$= a\left(x^2 + \frac{b}{a}x + \underline{\quad}\right) + c \quad \text{factor out } a$$
$$= a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) - a\left(\frac{b^2}{4a^2}\right) + c \quad \left[\left(\frac{1}{2}\right)\left(\frac{b}{a}\right)\right]^2 = \frac{b^2}{4a^2}$$
$$= a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c \quad \text{factor the trinomial, simplify}$$
$$= a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a} \quad \text{result}$$

By comparing this result with previous transformations, we note the x-coordinate of the vertex is $h = \frac{-b}{2a}$ (since the graph shifts horizontally "opposite the sign"). While we could use the expression $\frac{4ac - b^2}{4a}$ to find k, we find it easier to substitute $\frac{-b}{2a}$ back into the function: $k = f\left(\frac{-b}{2a}\right)$. The result is called the **vertex formula.**

Vertex Formula

For the quadratic function $f(x) = ax^2 + bx + c$, the coordinates of the vertex are

$$(h, k) = \left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right)$$

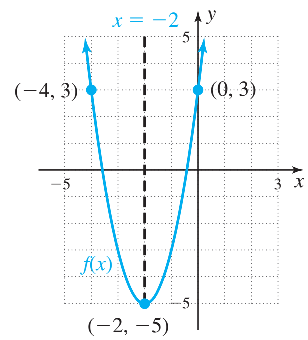
Since all characteristic features of the graph (end-behavior, vertex, axis of symmetry, x-intercepts, and y-intercept) can now be determined using the original equation, we'll rely on these features to sketch quadratic graphs, rather than having to complete the square.

EXAMPLE 3 ▶ **Graphing a Quadratic Function Using the Vertex Formula**

Graph $f(x) = 2x^2 + 8x + 3$ using the vertex formula and other features of a quadratic graph.

Solution ▶ The graph will open upward since $a > 0$.
The y-intercept is $(0, 3)$.
The vertex formula gives

$$h = \frac{-b}{2a} \quad x\text{-coordinate of vertex}$$
$$= \frac{-8}{2(2)} \quad \text{substitute 2 for } a \text{ and 8 for } b$$
$$= -2 \quad \text{simplify}$$



Computing $f(-2)$ to find the y-coordinate of the vertex yields

$$\begin{aligned} f(-2) &= 2(-2)^2 + 8(-2) + 3 && \text{substitute } -2 \text{ for } x \\ &= 2(4) - 16 + 3 && \text{multiply} \\ &= 8 - 13 && \text{simplify} \\ &= -5 && \text{result} \end{aligned}$$

The vertex is $(-2, -5)$. The graph is shown in the figure, with the point $(-4, 3)$ obtained using symmetry.

Now try Exercises 21 through 32 ▶

B. You've just learned how to graph quadratic functions using the vertex formula

C. Finding the Equation of a Quadratic Function from Its Graph

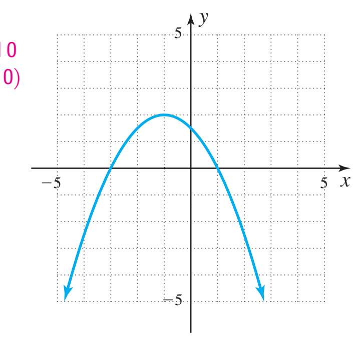
While most of our emphasis so far has centered on graphing quadratic functions, it would be hard to overstate the importance of the reverse process—determining the equation of the function from its graph (as in Section 2.6). This reverse process, which began with our study of lines, will be a continuing theme each time we consider a new function.

EXAMPLE 4 ▶ **Finding the Equation of a Quadratic Function**

The graph shown is a transformation of $f(x) = x^2$. What function defines this graph?

Solution ▶ Compared to the graph of $f(x) = x^2$, the vertex has been shifted left 1 and up 2, so the function will have the form $F(x) = a(x + 1)^2 + 2$. Since the graph opens downward, we know a will be negative. As before, we select one additional point on the graph and substitute to find the value of a. Using $(x, y) \rightarrow (1, 0)$ we obtain

$$\begin{aligned} F(x) &= a(x + 1)^2 + 2 && \text{transformation} \\ 0 &= a(1 + 1)^2 + 2 && \text{substitute 1 for } x \text{ and 0 for } F(x)\text{: } (x, y) \rightarrow (1, 0) \\ 0 &= 4a + 2 && \text{simplify} \\ -2 &= 4a && \text{subtract 2} \\ -\frac{1}{2} &= a && \text{solve for } a \end{aligned}$$



The equation of this function is

$$F(x) = -\frac{1}{2}(x + 1)^2 + 2.$$

Now try Exercises 33 through 38 ▶

WORTHY OF NOTE

It helps to remember that any point (x, y) on the parabola can be used. To verify this, try the calculation again using $(-3, 0)$.

C. You've just learned how to find the equation of a quadratic function from its graph

D. Quadratic Functions and Extreme Values

If $a > 0$, the parabola opens upward, and the y-coordinate of the vertex is a global minimum, the smallest value attained by the function anywhere in its domain. Conversely, if $a < 0$ the parabola opens downward and the vertex yields a global maximum. These greatest and least points are known as **extreme values** and have a number of significant applications.

EXAMPLE 5 ▶ **Applying a Quadratic Model to Manufacturing**

An airplane manufacturer can produce up to 15 planes per month. The profit made from the sale of these planes is modeled by $P(x) = -0.2x^2 + 4x - 3$, where $P(x)$ is the profit in hundred-thousands of dollars per month, and x is the number of planes sold. Based on this model,

a. Find the y-intercept and explain what it means in this context.

b. How many planes should be made and sold to maximize profit?

c. What is the maximum profit?

Solution ▶ **a.** $P(0) = -3$, which means the manufacturer loses \$300,000 each month if the company produces no planes.

b. Since $a < 0$, we know the graph opens downward and has a maximum value. To find the required number of sales needed to "maximize profit," we use the vertex formula with $a = -0.2$ and $b = 4$:

$$x = \frac{-b}{2a} \quad \text{vertex formula}$$

$$= \frac{-4}{2(-0.2)} \quad \text{substitute } -0.2 \text{ for } a \text{ and } 4 \text{ for } b$$

$$= 10 \quad \text{result}$$

The result shows 10 planes should be sold each month for maximum profit.

c. Evaluating $P(10)$ we find that a maximum profit of 17 "hundred thousand dollars" will be earned (\$1,700,000).

Now try Exercises 41 through 45 ▶

Note that if the leading coefficient is positive and the vertex is below the x-axis ($k < 0$), the graph will have two x-intercepts (see Figure 3.2). If $a > 0$ and the vertex is above the x-axis ($k > 0$), the graph will not cross the x-axis (Figure 3.3). Similar statements can be made for the case where a is negative.

Figure 3.2

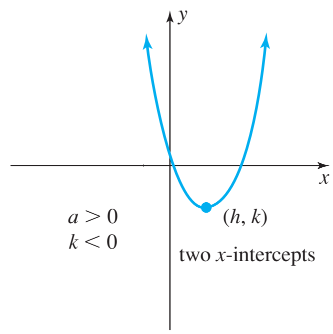
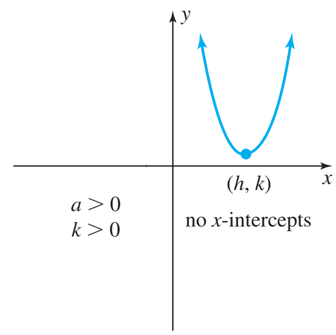


Figure 3.3



In some applications of quadratic functions, our interest includes the x-intercepts of the graph. Drawing on our previous work, we note that the following statements are equivalent, meaning if any one statement is true, then all four statements are true.

- $(r, 0)$ is an ***x*-intercept** of the graph of $f(x)$.
- $x = r$ is a **solution** or **root** of the equation $f(x) = 0$.
- $(x - r)$ is a **factor** of $f(x)$.
- r is a **zero** of $f(x)$.

When the quadratic function is in standard form, our primary tool for finding the zeroes is the quadratic formula. If the function is expressed as a transformation, we will often solve for x using inverse operations.

EXAMPLE 6 ▶ Modeling the Height of a Projectile

In the 1976 Pro Bowl, NFL punter Ray Guy of the Oakland Raiders kicked the ball so high it hit the scoreboard hanging from the roof of the New Orleans SuperDome. If we let $h(t)$ represent the height of the football (in feet) after t sec, the function $h(t) = -22t^2 + 132t + 1$ models the relationship (time, height of ball).

a. What does the y-intercept of this function represent?
b. After how many seconds did the football reach its maximum height?
c. What was the maximum height of this kick?
d. To the nearest hundredth of a second, how long until the ball returns to the ground (what was the hang time)?

Solution ▶ **a.** $h(0) = 1$, meaning the ball was 1 ft off the ground when Ray Guy kicked it.

b. Since $a < 0$, we know the graph opens downward and has a maximum value. To find the time needed to reach the maximum height, we use the vertex formula with $a = -22$ and $b = 132$:

$$t = \frac{-b}{2a} \quad \text{vertex formula}$$
$$= \frac{-132}{2(-22)} \quad \text{substitute } -22 \text{ for } a \text{ and } 132 \text{ for } b$$
$$= 3 \quad \text{result}$$

The ball reached its maximum height after 3 sec.

c. To find the maximum height, we substitute 3 for t [evaluate $h(3)$]:

$$h(t) = -22t^2 + 132t + 1 \quad \text{given function}$$
$$h(3) = -22(3)^2 + 132(3) + 1 \quad \text{substitute 3 for } t$$
$$= 199 \quad \text{result}$$

The ball reached a maximum height of 199 ft.

d. When the ball returns to the ground it has a height of 0 ft. Substituting 0 for $h(t)$ gives $0 = -22t^2 + 132t + 1$, which we solve using the quadratic formula.

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \text{quadratic formula}$$
$$= \frac{-132 \pm \sqrt{132^2 - 4(-22)(1)}}{2(-22)} \quad \text{substitute } -22 \text{ for } a,\ 132 \text{ for } b, \text{ and } 1 \text{ for } c$$
$$= \frac{-132 \pm \sqrt{17512}}{-44} \quad \text{simplify}$$
$$t \approx -0.01 \quad \text{or} \quad t \approx 6.01$$

The punt had a hang time of just over 6 sec.

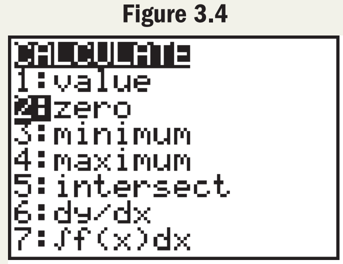
D. You've just learned how to solve applications involving extreme values

Now try Exercises 46 through 49 ▶

TECHNOLOGY HIGHLIGHT

Estimating Irrational Zeroes

Figure 3.4



Once a function is entered into a graphing calculator, an estimate for irrational zeroes can easily be found. Enter the function $y = x^2 - 8x + 9$ on the Y= screen and graph using the standard window (ZOOM 6). Pressing 2nd TRACE (CALC) displays the screen in Figure 3.4. Pressing the number "2" selects the **2:zero** option and returns you to the graph, where you are asked to enter a "Left Bound." The calculator is asking you to narrow down the area it has to search for the x-intercept. Select any number that is conveniently to the left of the x-intercept you're interested in. For this graph, we entered a left bound of "0" (press ENTER), which the calculator indicates with a "▶" marker. It then asks you to enter a "Right Bound."

—continued

Figure 3.5

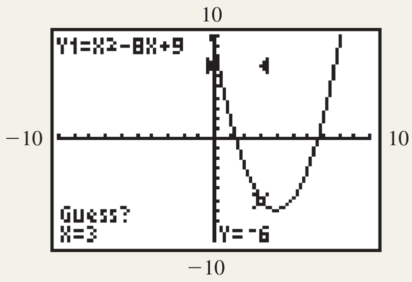
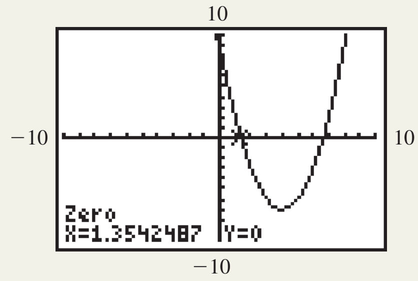


Figure 3.6



Select any value to the right of this x-intercept, but be sure the value *bounds only one intercept* (see Figure 3.5). For this graph, a choice of 10 would include both x-intercepts, while a choice of 3 would bound only the x-intercept on the left. After entering 3, the calculator asks for a "guess." This option is used only when there are many different zeroes close by or if you entered a large interval. Most of the time we'll simply bypass this option by pressing ENTER. The cursor will be located at the zero you chose, with the coordinates displayed at the bottom of the screen (see Figure 3.6). The x-value is an approximation of the irrational zero. Find the zeroes of these functions using the 2nd TRACE (CALC) **2:Zero** feature.

Exercise 1: $y = x^2 - 8x + 9$

Exercise 2: $y = 3a^2 - 5a - 6$

Exercise 3: $y = 2x^2 + 4x - 5$

Exercise 4: $y = 9w^2 + 6w - 1$

3.1 EXERCISES

▶ CONCEPTS AND VOCABULARY

Fill in each blank with the appropriate word or phrase. Carefully reread the section if needed.

1. Fill in the blank to complete the square, given $f(x) = -2x^2 - 10x - 7$: $f(x) = -2(x^2 + 5x + \frac{25}{4}) - 7 +$ ______.

2. The maximum and minimum values are called ______ values and can be found using the ______ formula.

3. To find the zeroes of $f(x) = ax^2 + bx + c$, we substitute ______ for ______ and solve.

4. If the leading coefficient is positive and the vertex (h, k) is in Quadrant IV, the graph will have ______ x-intercepts.

5. Compare/Contrast how to complete the square on an *equation,* versus how to complete the square on a function. Use the equation $2x^2 + 6x - 3 = 0$ and the function $f(x) = 2x^2 + 6x - 3 = 0$ to illustrate.

6. Discuss/Explain why the graph of a quadratic function has no x-intercepts if a and k [vertex (h, k)] have like signs. Under what conditions will the function have a single real root?

▶ DEVELOPING YOUR SKILLS

Graph each function using end behavior, intercepts, and completing the square to write the function in shifted form. Clearly state the transformations used to obtain the graph, and label the vertex and all intercepts (if they exist). Use the quadratic formula to find the x-intercepts.

7. $f(x) = x^2 + 4x - 5$

8. $g(x) = x^2 - 6x - 7$

9. $h(x) = -x^2 + 2x + 3$

10. $H(x) = -x^2 + 8x - 7$

11. $Y_1 = 3x^2 + 6x - 5$

12. $Y_2 = 4x^2 - 24x + 15$

13. $f(x) = -2x^2 + 8x + 7$

14. $g(x) = -3x^2 + 12x - 7$

15. $p(x) = 2x^2 - 7x + 3$

16. $q(x) = 4x^2 - 9x + 2$

17. $f(x) = -3x^2 - 7x + 6$

18. $g(x) = -2x^2 + 9x - 7$

19. $p(x) = x^2 - 5x + 2$

20. $q(x) = x^2 + 7x + 4$

Graph each function using the vertex formula and other features of a quadratic graph. Label all important features.

21. $f(x) = x^2 + 2x - 6$

22. $g(x) = x^2 + 8x + 11$

23. $h(x) = -x^2 + 4x + 2$

24. $H(x) = -x^2 + 10x - 19$

25. $Y_1 = 0.5x^2 + 3x + 7$

26. $Y_2 = 0.2x^2 - 2x + 8$

27. $Y_1 = -2x^2 + 10x - 7$

28. $Y_2 = -2x^2 + 8x - 3$

29. $f(x) = 4x^2 - 12x + 3$

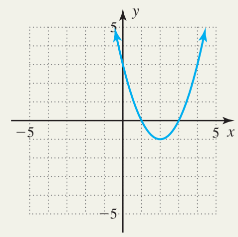
30. $g(x) = 3x^2 + 12x + 5$

31. $p(x) = \frac{1}{2}x^2 + 3x - 5$

32. $q(x) = \frac{1}{3}x^2 - 2x - 4$

State the equation of the function whose graph is shown.

33.



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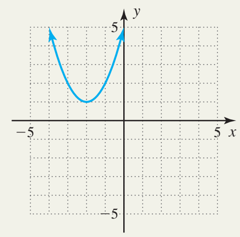

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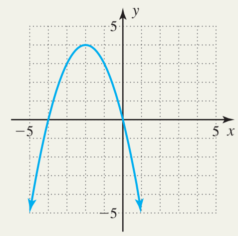

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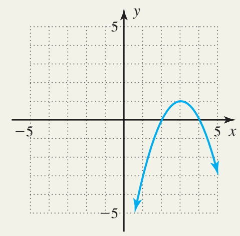

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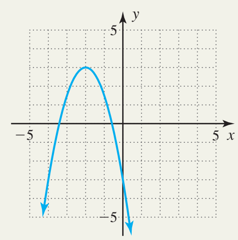

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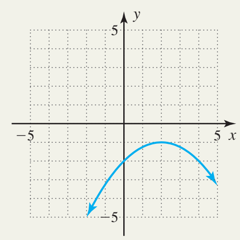


▶ WORKING WITH FORMULAS

39. **Vertex/intercept formula:** $x = h \pm \sqrt{-\frac{k}{a}}$

As an alternative to using the quadratic formula *prior* to completing the square, the x-intercepts can more easily be found using the vertex/intercept formula *after* completing the square, when the coordinates of the vertex are known. (a) Beginning with the shifted form $y = a(x - h)^2 + k$, substitute 0 for y and solve for x to derive the formula, and (b) use the formula to find zeroes, real or complex, of the following functions.

i. $y = (x + 3)^2 - 5$ **ii.** $y = -(x - 4)^2 + 3$

iii. $y = 2(x + 4)^2 - 7$ **iv.** $y = -3(x - 2)^2 + 6$

v. $s(t) = 0.2(t + 0.7)^2 - 0.8$

vi. $r(t) = -0.5(t - 0.6)^2 + 2$

40. **Surface area of a rectangular box with square ends:** $S = 2h^2 + 4Lh$

The surface area of a rectangular box with square ends is given by the formula shown, where h is the height and width of the square ends, and L is the length of the box. (a) If L is 3 ft and the box must have a surface area of 32 ft^2, find the dimensions of the square ends. (b) Solve for L, then find the length if the height is 1.5 ft and surface area is 22.5 ft^2.

▶ APPLICATIONS

41. **Maximum profit:** An automobile manufacturer can produce up to 300 cars per day. The profit made from the sale of these vehicles can be modeled by the function $P(x) = -10x^2 + 3500x - 66{,}000$, where $P(x)$ is the profit in dollars and x is the number of automobiles made and sold. Based on this model:

a. Find the y-intercept and explain what it means in this context.

b. Find the x-intercepts and explain what they mean in this context.

c. How many cars should be made and sold to maximize profit?

d. What is the maximum profit?

42. Maximum profit: The profit for a manufacturer of collectible grandfather clocks is given by the function shown here, where $P(x)$ is the profit in dollars and x is the number of clocks made and sold. Answer the following questions based on this model: $P(x) = -1.6x^2 + 240x - 375$.

a. Find the y-intercept and explain what it means in this context.

b. Find the x-intercepts and explain what they mean in this context.

c. How many clocks should be made and sold to maximize profit?

d. What is the maximum profit?

43. Depth of a dive: As it leaves its support harness, a minisub takes a deep dive toward an underwater exploration site. The dive path is modeled by the function $d(x) = x^2 - 12x$, where $d(x)$ represents the depth of the minisub in hundreds of feet at a distance of x mi from the surface ship.

a. How far from the mother ship did the minisub reach its deepest point?

b. How far underwater was the submarine at its deepest point?

c. At $x = 4$ mi, how deep was the minisub explorer?

d. How far from its entry point did the minisub resurface?

44. Optimal pricing strategy: The director of the Ferguson Valley drama club must decide what to charge for a ticket to the club's performance of *The Music Man.* If the price is set too low, the club will lose money; and if the price is too high, people won't come. From past experience she estimates that the profit P from sales (in hundreds) can be approximated by $P(x) = -x^2 + 46x - 88$, where x is the cost of a ticket and $0 \le x \le 50$.

a. Find the lowest cost of a ticket that would allow the club to break even.

b. What is the highest cost that the club can charge to break even?

c. If the theater were to close down before any tickets are sold, how much money would the club lose?

d. How much should the club charge to maximize their profits? What is the maximum profit?

45. Maximum profit: A kitchen appliance manufacturer can produce up to 200 appliances per day. The profit made from the sale of these machines can be modeled by the function $P(x) = -0.5x^2 + 175x - 3300$, where $P(x)$ is the profit in dollars, and x is the number of appliances made and sold. Based on this model,

a. Find the y-intercept and explain what it means in this context.

b. Find the x-intercepts and explain what they mean in this context.

c. Determine the domain of the function and explain its significance.

d. How many should be sold to maximize profit? What is the maximum profit?

The projectile function: $h(t) = -16t^2 + vt + k$ applies to any object projected upward with an initial velocity v, from a height k but not to objects under propulsion (such as a rocket). Consider this situation and answer the questions that follow.

46. Model rocketry: A member of the local rocketry club launches her latest rocket from a large field. At the moment its fuel is exhausted, the rocket has a velocity of 240 ft/sec and an altitude of 544 ft (t is in seconds).

a. Write the function that models the height of the rocket.

b. How high is the rocket at $t = 0$? If it took off from the ground, why is it this high at $t = 0$?

c. How high is the rocket 5 sec after the fuel is exhausted?

d. How high is the rocket 10 sec after the fuel is exhausted?

e. How could the rocket be at the same height at $t = 5$ and at $t = 10$?

f. What is the maximum height attained by the rocket?

g. How many seconds was the rocket airborne *after* its fuel was exhausted?

47. Height of a projectile: A projectile is thrown upward with an initial velocity of 176 ft/sec. After t sec, its height $h(t)$ above the ground is given by the function $h(t) = -16t^2 + 176t$.

a. Find the projectile's height above the ground after 2 sec.

b. Sketch the graph modeling the projectile's height.

c. What is the projectile's maximum height? What is the value of t at this height?

d. How many seconds after it is thrown will the projectile strike the ground?

48. Height of a projectile: In the movie *The Court Jester* (1956; Danny Kaye, Basil Rathbone, Angela Lansbury, and Glynis Johns), a catapult is used to toss the nefarious adviser to the king into a river. Suppose the path flown by the king's adviser is modeled by the function $h(d) = -0.02d^2 + 1.64d + 14.4$, where $h(d)$ is the height of the adviser in feet at a distance of d ft from the base of the catapult.

a. How high was the release point of this catapult?

b. How far from the catapult did the adviser reach a maximum altitude?

c. What was this maximum altitude attained by the adviser?

d. How far from the catapult did the adviser splash into the river?

49. Blanket toss competition: The Fraternities at Steele Head University are participating in a blanket toss competition, an activity borrowed from the whaling villages of the Inuit Eskimos. If the person being tossed is traveling at 32 ft/sec as he is projected into the air, and the Frat members are holding the canvas blanket at a height of 5 ft,

a. Write the function that models the height at time t of the person being tossed.

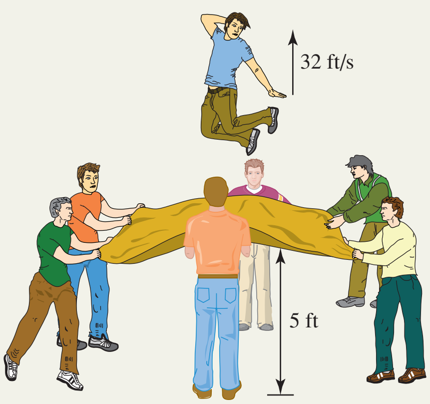
b. How high is the person when (i) $t = 0.5$, (ii) $t = 1.5$?

c. From part (b) what do you know about *when* the maximum height is reached?

d. To the nearest tenth of a second, when is the maximum height reached?

e. To the nearest one-half foot, what was the maximum height?

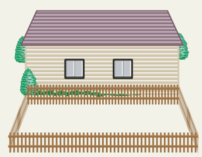
f. To the nearest tenth of a second, how long was this person airborne?



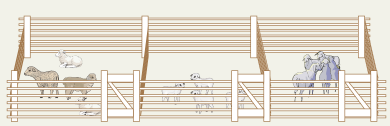
50. Cost of production: The cost of producing a plastic toy is given by the function $C(x) = 2x + 35$, where x is the number of hundreds of toys. The revenue from toy sales is given by $R(x) = -x^2 + 122x - 365$. Since profit = revenue − cost, the profit function must be $P(x) = -x^2 + 120x - 400$ (verify). How many toys sold will produce the maximum profit? What is the maximum profit?

51. Cost of production: The cost to produce bottled spring water is given by $C(x) = 16x - 63$, where x is the number of thousands of bottles. The total income (revenue) from the sale of these bottles is given by the function $R(x) = -x^2 + 326x - 7463$. Since profit = revenue − cost, the profit function must be $P(x) = -x^2 + 310x - 7400$ (verify). How many bottles sold will produce the maximum profit? What is the maximum profit?

52. Fencing a backyard: Tina and Imai have just purchased a purebred German Shepherd, and need to fence in their backyard so the dog can run. What is the maximum rectangular area they can enclose with 200 ft of fencing, if (a) they use fencing material along all four sides? What are the dimensions of the rectangle? (b) What is the maximum area if they use the house as one of the sides? What are the dimensions of *this* rectangle?



53. Building sheep pens: It's time to drench the sheep again, so Chance and Chelsea-Lou are fencing off a large rectangular area to build some temporary holding pens. To prep the males, females, and kids, they are separated into three smaller and equal-size pens partitioned within the large rectangle. If 384 ft of fencing is available and the maximum area is desired, what will be (a) the dimensions of the larger, outer rectangle? (b) the dimensions of the smaller holding pens?



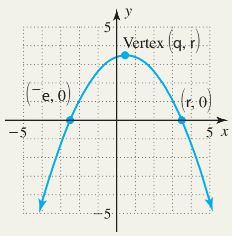
▶ EXTENDING THE CONCEPT

54. Use the general solutions from the quadratic formula to show that the average value of the x-intercepts is $\frac{-b}{2a}$. Explain/Discuss why the result is valid even if the roots are complex.

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \qquad x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

55. Write the equation of a quadratic function whose x-intercepts are given by $x = 2 \pm 3i$.

56. Write the equation for the parabola given.



57. Referring to Exercise 39, discuss the nature (real or complex, rational or irrational) and number of zeroes (0, 1, or 2) given by the vertex/intercept formula if (a) a and k have like signs, (b) a and k have unlike signs, (c) k is zero, (d) the ratio $-\frac{k}{a}$ is positive and a perfect square, and (e) the ratio $-\frac{k}{a}$ is positive and not a perfect square.

▶ MAINTAINING YOUR SKILLS

58. **(2.3)** Identify the slope and y-intercept for $-4x + 3y = 9$. Do not graph.

59. **(R.5)** Multiply: $\frac{x^2 - 4x + 4}{x^2 + 3x - 10} \cdot \frac{x^2 - 25}{x^2 - 10x + 25}$

60. **(2.8)** Given $f(x) = \sqrt[3]{x + 3}$ and $g(x) = x^3 - 3$, find $(f \circ g)(x)$ and $(g \circ f)(x)$.

61. **(2.7)** Given $f(x) = 3x^2 + 7x - 6$, solve $f(x) \leq 0$ using the x-intercepts and concavity of f.

3.2 Synthetic Division; the Remainder and Factor Theorems

Learning Objectives

In Section 3.2 you will learn how to:

- ☐ **A.** Divide polynomials using long division and synthetic division
- ☐ **B.** Use the remainder theorem to evaluate polynomials
- ☐ **C.** Use the factor theorem to factor and build polynomials
- ☐ **D.** Solve applications using the remainder theorem

To find the zero of a linear function, we can use properties of equality to isolate x. To find the zeroes of a quadratic function, we can factor or use the quadratic formula. To find the zeroes of higher degree polynomials, we must first develop additional tools, including synthetic division and the remainder and factor theorems. These will help us to write a higher degree polynomial in terms of linear and quadratic polynomials, whose zeroes can easily be found.

A. Long Division and Synthetic Division

To help understand **synthetic division** and its use as a mathematical tool, we first review the process of **long division.**

Long Division

Polynomial long division closely resembles the division of whole numbers, with the main difference being that *we group each partial product* in parentheses to prevent errors in subtraction.

EXAMPLE 1 ▶ Dividing Polynomials Using Long Division

Divide $x^3 - 4x^2 + x + 6$ by $x - 1$.

Solution ▶ The divisor is $(x - 1)$ and the dividend is $(x^3 - 4x^2 + x + 6)$. To find the first multiplier, we compute *the ratio of leading terms* from each expression. Here the ratio $\frac{x^3 \text{ (from dividend)}}{x \text{ (from divisor)}}$ shows our first multiplier will be "x^2," with $x^2(x - 1) = x^3 - x^2$.

$$\begin{array}{r} x^2 \phantom{{}-4x^2+x+6} \\ x - 1 \overline{) x^3 - 4x^2 + x + 6} \\ -(x^3 - x^2) \phantom{{}+x+6} \end{array} \text{ subtraction} \longrightarrow \begin{array}{r} x^2 \phantom{{}-4x^2+x+6} \\ x - 1 \overline{) x^3 - 4x^2 + x + 6} \\ -x^3 + x^2 \phantom{{}+x+6} \\ \hline -3x^2 + x \phantom{{}+6} \end{array} \text{ algebraic addition}$$

At each stage, after writing the subtraction as algebraic addition (distributing the negative) we compute the sum in each column and "bring down" the next term. Each following multiplier is found as before, using the ratio $\frac{ax^k \text{ next leading term}}{x \text{ from divisor}}$.

$$\begin{array}{r} x^2 - 3x - 2 \\ x - 1 \overline{) x^3 - 4x^2 + x + 6} \\ -(x^3 - x^2) \phantom{{}+x+6} \\ \hline -3x^2 + x \phantom{{}+6} \\ -(-3x^2 + 3x) \phantom{{}+6} \\ \hline -2x + 6 \\ -(-2x + 2) \\ \hline 4 \end{array}$$

next multiplier: $\frac{-3x^2}{x} = -3x$ (ratio of leading terms)

subtract $-3x(x - 1) = -3x^2 + 3x$

next multiplier: $\frac{-2x}{x} = -2$

algebraic addition, bring down next term

subtract $-2(x - 1) = -2x + 2$

algebraic addition, remainder is 4

The result shows $\frac{x^3 - 4x^2 + x + 6}{x - 1} = x^2 - 3x - 2 + \frac{4}{x - 1}$, or after multiplying both sides by $x - 1$, $x^3 - 4x^2 + x + 6 = (x - 1)(x^2 - 3x - 2) + 4$.

Now try Exercises 7 through 12 ▶

The process illustrated is called the **division algorithm,** and like the division of whole numbers, the final result can be checked by multiplication.

check: (dividend, divisor, quotient, remainder)

$$\begin{aligned} x^3 - 4x^2 + x + 6 &= (x - 1)(x^2 - 3x - 2) + 4 \\ &= (x^3 - 3x^2 - 2x - x^2 + 3x + 2) + 4 && \text{divisor} \cdot \text{quotient} \\ &= (x^3 - 4x^2 + x + 2) + 4 && \text{combine like terms} \\ &= x^3 - 4x^2 + x + 6 \checkmark && \text{add remainder} \end{aligned}$$

In general, the division algorithm for polynomials says

Division of Polynomials

Given polynomials $p(x)$ and $d(x) \neq 0$, there exists unique polynomials $q(x)$ and $r(x)$ such that

$$p(x) = d(x)q(x) + r(x),$$

where $r(x) = 0$ or the degree of $r(x)$ is less than the degree of $d(x)$. Here, $d(x)$ is called the *divisor,* $q(x)$ is the *quotient,* and $r(x)$ is the *remainder.*

In other words, "a polynomial of greater degree can be divided by a polynomial of equal or lesser degree to obtain a quotient and a remainder." As with whole numbers, if the remainder is zero, the divisor is a factor of the dividend.

Synthetic Division

As the word "synthetic" implies, synthetic division *simulates* the long division process, but condenses it and makes it more efficient when the divisor is linear. The process works by capitalizing on the repetition found in the division algorithm. First, the polynomials involved are written in decreasing order of degree, so the variable part of each term is unnecessary as we can let the *position of each coefficient* indicate the degree of the term. For the dividend from Example 1, 1 −4 1 6 would represent the polynomial $1x^3 - 4x^2 + 1x + 6$. Also, each stage of the algorithm involves a product of the divisor with the next multiplier, followed by a subtraction. These can likewise be computed using the coefficients only, as the degree of each term is still determined by its position. Here is the division from Example 1 in the synthetic division format. Note that we must use the *zero of the divisor* (as in $x = \frac{3}{2}$ for a divisor of $2x - 3$, or in this case, "1" from $x - 1 = 0$) and the coefficients of the dividend in the following format:

WORTHY OF NOTE

The process of synthetic division is only summarized here. For a complete discussion, see Appendix II.

zero of the divisor — coefficients of the dividend

zero of the divisor	coefficients of the dividend			
1	1	−4	1	6
	↓			← partial products
	1			(remainder)

coefficients of the quotient — remainder

As this template indicates, the quotient and remainder will be read from the last row.

The arrow indicates we begin by "dropping the leading coefficient into place." We then multiply this coefficient by the "divisor," and place the result in the next column and add. Note that using the zero of the divisor enables us to *add in each column directly,* rather than subtracting then changing to algebraic addition as in long division.

		add		
1	1	−4	1	6
		1		
	1	−3		

multiply 1 · 1

multiply divisor · coefficient, place result in next column and add

In a sense, we "multiply in the diagonal direction," and "add in the vertical direction." Repeat the process until the division is complete.

			add	
1	1	−4	1	6
		1	−3	
	1	−3	−2	

multiply $1 \cdot (-3) = -3$, place result in next column and add

				add
1	1	−4	1	6
		1	−3	−2
	1	−3	−2	4

multiply $1 \cdot (-2) = -2$, place result in next column and add

The quotient is read from the last row by noting the remainder is 4, leaving the coefficients 1 −3 −2, which translates back into the polynomial $x^2 - 3x - 2$. The final result is identical to that in Example 1, but the new process is more efficient, since all stages are actually computed on a single template as shown here:

zero of the divisor	coefficients of the dividend			
1	1	−4	1	6
	↓	1	−3	−2
	1	−3	−2	4
	coefficients of the quotient			remainder

EXAMPLE 2 ▶ **Dividing Polynomials Using Synthetic Division**

Compute the quotient of $(x^3 + 3x^2 - 4x - 12)$ and $(x + 2)$, then check your answer.

Solution ▶ Using -2 as our "divisor" (from $x + 2 = 0$), we set up the synthetic division template and begin.

use -2 as a "divisor"	$-2\rfloor$	1	3	-4	-12	drop lead coefficient into place;
		↓	-2	-2	12	multiply by divisor, place result
		1	1	-6	0	in next column and add

The result shows $\dfrac{x^3 + 3x^2 - 4x - 12}{x + 2} = x^2 + x - 6$, with no remainder.

Check ▶

$$\begin{aligned} x^3 + 3x^2 - 4x - 12 &= (x + 2)(x^2 + x - 6) \\ &= (x^3 + x^2 - 6x + 2x^2 + 2x - 12) \\ &= x^3 + 3x^2 - 4x - 12 \checkmark \end{aligned}$$

Now try Exercises 13 through 20 ▶

Since the division process is so dependent on the place value (degree) of each term, polynomials such as $2x^3 + 3x + 7$, which has no term of degree 2, must be written using a zero *placeholder:* $2x^3 + \mathbf{0x^2} + 3x + 7$. This ensures that like place values "line up" as we carry out the division.

EXAMPLE 3 ▶ **Dividing Polynomials Using a Zero Placeholder**

Compute the quotient $\dfrac{2x^3 + 3x + 7}{x - 3}$ and check your answer.

Solution ▶

use 3 as a "divisor"	$3\rfloor$	2	0	3	7	note place holder $\mathbf{0}x^2$ for "x^2" term
		↓	6	18	63	
		2	6	21	70	

The result shows $\dfrac{2x^3 + 3x + 7}{x - 3} = 2x^2 + 6x + 21 + \dfrac{70}{x - 3}$. Multiplying by $x - 3$ gives

$$2x^3 + 3x + 7 = (2x^2 + 6x + 21)(x - 3) + 70$$

Check ▶

$$\begin{aligned} 2x^3 + 3x + 7 &= (x - 3)(2x^2 + 6x + 21) + 70 \\ &= (2x^3 + 6x^2 + 21x - 6x^2 - 18x - 63) + 70 \\ &= 2x^3 + 3x + 7 \checkmark \end{aligned}$$

Now try Exercises 21 through 30 ▶

WORTHY OF NOTE

Many corporations now pay their employees monthly to save on payroll costs. If your monthly salary was \$2037/mo, but you received a check for only \$237, would you complain? Just as placeholder zeroes ensure the correct value of each digit, they also ensure the correct valuation of each term in the division process.

As noted earlier, for synthetic division the divisor must be a linear polynomial and the zero of this divisor is used. This means for the quotient $\dfrac{2x^3 - 3x^2 - 8x + 12}{2x - 3}$, $\dfrac{3}{2}$ would be used for synthetic division **(see Exercises 43 and 44).** If the divisor is nonlinear, long division must be used.

EXAMPLE 4 ▶ Division with a Nonlinear Divisor

Compute the quotient: $\dfrac{2x^4 + x^3 - 7x^2 + 3}{x^2 - 2}$.

Solution ▶ Write the dividend as $2x^4 + x^3 - 7x^2 + \mathbf{0x} + 3$, and the divisor as $x^2 + \mathbf{0x} - 2$.

The quotient of leading terms gives $\dfrac{2x^4 \text{ (from dividend)}}{x^2 \text{ (from divisor)}} = 2x^2$ as our first multiplier.

$$
\begin{array}{lrl}
 & & 2x^2 + x - 3 \\
\text{divisor} \rightarrow & x^2 + 0x - 2 & \overline{)\,2x^4 + x^3 - 7x^2 + 0x + 3} \\
\text{Multiply } 2x^2(x^2 + 0x - 2) & & -(2x^4 + 0x^3 - 4x^2) \quad \text{subtract (algebraic addition)} \\
 & & \qquad x^3 - 3x^2 + 0x \quad \text{bring down next term} \\
\text{Multiply } x(x^2 + 0x - 2) & & \qquad -(x^3 + 0x^2 - 2x) \quad \text{subtract (algebraic addition)} \\
 & & \qquad\quad -3x^2 + 2x + 3 \quad \text{bring down next term} \\
\text{Multiply } -3(x^2 + 0x - 2) & & \qquad\quad -(\mathbf{-3x^2 + 0x + 6}) \quad \text{subtract} \\
 & & \qquad\qquad 2x - 3 \quad \text{remainder is } 2x - 3
\end{array}
$$

☑ **A.** You've just learned how to divide polynomials using long division and synthetic division

Since the degree of $2x - 3$ (degree 1) is less than the degree of the divisor (degree 2), the process is complete.

$$\frac{2x^4 + x^3 - 7x^2 + 3}{x^2 - 2} = (2x^2 + x - 3) + \frac{(2x - 3)}{x^2 - 2}$$

Now try Exercises 31 through 34 ▶

B. The Remainder Theorem

In Example 2, we saw that $(x^3 + 3x^2 - 4x - 12) \div (x + 2) = x^2 + x - 6$, with no remainder. Similar to whole number division, this means $x + 2$ must be a factor of $x^3 + 3x^2 - 4x - 12$, a fact made clear as we checked our answer: $x^3 + 3x^2 - 4x - 12 = (x + 2)(x^2 + x - 6)$. To help us find the factors of higher degree polynomials, we combine synthetic division with a relationship known as the **remainder theorem.** Consider the functions $p(x) = x^3 + 5x^2 + 2x - 8$, $d(x) = x + 3$, and their quotient $\dfrac{p(x)}{d(x)} = \dfrac{x^3 + 5x^2 + 2x - 8}{x + 3}$. Using -3 as the divisor in synthetic division gives

$$
\begin{array}{l|rrrr}
\text{use } -3 \text{ as a “divisor”} \quad -3 & 1 & 5 & 2 & -8 \\
 & \downarrow & -3 & -6 & 12 \\
\hline
 & 1 & 2 & -4 & \boxed{4}
\end{array}
$$

This shows $x + 3$ is *not* a factor of $P(x)$, since it didn't "divide evenly." However, from the result $p(x) = (x + 3)(x^2 + 2x - 4) + 4$, we make a remarkable observation—if we evaluate $p(-3)$, *the quotient portion becomes zero,* showing $p(-3) = 4$—*which is the remainder*.

$$
\begin{aligned}
p(-3) &= (-3 + 3)[(-3)^2 + 2(-3) - 4] + 4 \\
&= (\mathbf{0})(-1) + 4 \\
&= 4
\end{aligned}
$$

This can also be seen by evaluating $p(-3)$ in its original form:

$$
\begin{aligned}
p(x) &= x^3 + 5x^2 + 2x - 8 \\
p(-3) &= (-3)^3 + 5(-3)^2 + 2(-3) - 8 \\
&= -27 + 45 + (-6) - 8 \\
&= 4
\end{aligned}
$$

The result is no coincidence, and illustrates the conclusion of the remainder theorem.

The Remainder Theorem

If a polynomial $p(x)$ is divided by $(x - c)$ using synthetic division, the remainder is equal to $p(c)$.

This gives us a powerful tool for evaluating polynomials. Where a direct evaluation involves powers of numbers and a long series of calculations, synthetic division reduces the process to simple products and sums.

EXAMPLE 5 ▶ **Using the Remainder Theorem to Evaluate Polynomials**

Use the remainder theorem to find $p(-5)$ for $p(x) = x^4 + 3x^3 - 8x^2 + 5x - 6$. Verify the result using a substitution.

Solution ▶

use −5 as a "divisor"

-5	1	3	-8	5	-6
		-5	10	-10	25
	1	-2	2	-5	19

The result shows $p(-5) = 19$, which we verify directly:

$$\begin{aligned} p(-5) &= (-5)^4 + 3(-5)^3 - 8(-5)^2 + 5(-5) - 6 \\ &= 625 - 375 - 200 - 25 - 6 \\ &= 19 \checkmark \end{aligned}$$

☑ **B.** You've just learned how to use the Remainder Theorem to evaluate polynomials

Now try Exercises 35 through 44 ▶

WORTHY OF NOTE

Since $p(-5) = 19$, we know $(-5, 19)$ must be a point of the graph of $p(x)$. The ability to quickly evaluate polynomial functions using the remainder theorem will be used extensively in the sections that follow.

C. The Factor Theorem

As a consequence of the remainder theorem, when $p(x)$ is divided by $x - c$ and the remainder is 0, $p(c) = 0$, and c is a zero of the polynomial. The relationship between $x - c$, c, and $p(c) = 0$ are summarized into the **factor theorem.**

The Factor Theorem

For a polynomial $p(x)$,

1. If $p(c) = 0$, then $x - c$ is a factor of $p(x)$.

2. If $x - c$ is a factor of $p(x)$, then $p(c) = 0$.

The remainder and factor theorems often work together to help us find factors of higher degree polynomials.

EXAMPLE 6 ▶ **Using the Factor Theorem to Find Factors of a Polynomial**

Use the factor theorem to determine if

a. $x - 2$ **b.** $x + 1$

are factors of $p(x) = x^4 + x^3 - 10x^2 - 4x + 24$.

Solution ▶ **a.** If $x - 2$ is a factor, then $p(2)$ must be 0. Using the remainder theorem we have

2	1	1	-10	-4	24
	↓	2	6	-8	-24
	1	3	-4	-12	0

Since the remainder is zero, we know $p(2) = 0$ (remainder theorem) and $(x - 2)$ is a factor (factor theorem).

b. Similarly, if $x + 1$ is a factor, then $p(-1)$ must be 0.

-1	1	1	-10	-4	24
	↓	-1	0	10	-6
	1	0	-10	6	18

Since the remainder is not zero, $(x + 1)$ is not a factor of p.

Now try Exercises 45 through 56 ▶

EXAMPLE 7 ▶ Building a Polynomial Using the Factor Theorem

A polynomial $p(x)$ has the zeroes 3, $\sqrt{2}$, and $-\sqrt{2}$. Use the factor theorem to find the polynomial.

Solution ▶ Using the factor theorem, the factors of $p(x)$ must be $(x - 3)$, $(x - \sqrt{2})$, and $(x + \sqrt{2})$. Computing the product will yield the polynomial.

$$\begin{aligned} p(x) &= (x - 3)(x - \sqrt{2})(x + \sqrt{2}) \\ &= (x - 3)(x^2 - 2) \\ &= x^3 - 3x^2 - 2x + 6 \end{aligned}$$

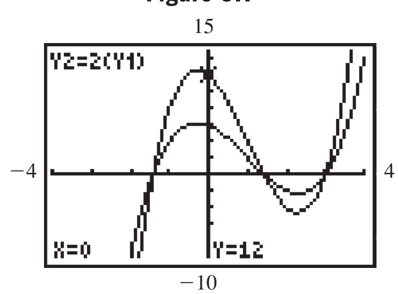
Now try Exercises 57 through 64 ▶

As the following *Graphical Support* feature shows, the result obtained in Example 7 is not unique, since any polynomial of the form $a(x^3 - 3x^2 - 2x + 6)$ will also have the same three roots for $a \in \mathbb{R}$.

GRAPHICAL SUPPORT

A graphing calculator helps to illustrate there are actually many different polynomials that have the three roots required by Example 7. Figure 3.7 shows the graph of $Y_1 = p(x)$, as well as graph of $Y_2 = 2p(x)$. The only difference is $2p(x)$ has been vertically stretched. Likewise, the graph of $-1p(x)$ would be a vertical reflection, *but still with the same zeroes.*

Figure 3.7



EXAMPLE 8 ▶ Finding Zeroes Using the Factor Theorem

Given that 2 is a zero of $p(x) = x^4 + x^3 - 10x^2 - 4x + 24$, use the factor theorem to help find all other zeroes.

Solution ▶ Using synthetic division gives:

use 2 as a "divisor"

2	1	1	-10	-4	24
	↓	2	6	-8	-24
	1	3	-4	-12	0

Since the remainder is zero, $(x - 2)$ is a factor and p can be written:

$$x^4 + x^3 - 10x^2 - 4x + 24 = (x - 2)(x^3 + 3x^2 - 4x - 12)$$

WORTHY OF NOTE

In Section R.4 we noted a third degree polynomial $ax^3 + bx^2 + cx + d$ is factorable if $ad = bc$. In Example 8, $1(-12) = 3(-4)$ and the polynomial is factorable.

Note the quotient polynomial can be factored by grouping to find the remaining factors of p.

$$\begin{aligned} x^4 + x^3 - 10x^2 - 4x + 24 &= (x - 2)(x^3 + 3x^2 - 4x - 12) && \text{group terms (in color)} \\ &= (x - 2)[x^2(x + 3) - 4(x + 3)] && \text{remove common factors from each group} \\ &= (x - 2)[(x + 3)(x^2 - 4)] && \text{factor common binomial} \\ &= (x - 2)(x + 3)(x + 2)(x - 2) && \text{factor difference of squares} \\ &= (x + 3)(x + 2)(x - 2)^2 && \text{completely factored form} \end{aligned}$$

The final result shows $(x - 2)$ is actually a repeated factor, and the remaining zeroes of p are -3 and -2.

C. You've just learned how to use the factor theorem to factor and build polynomials

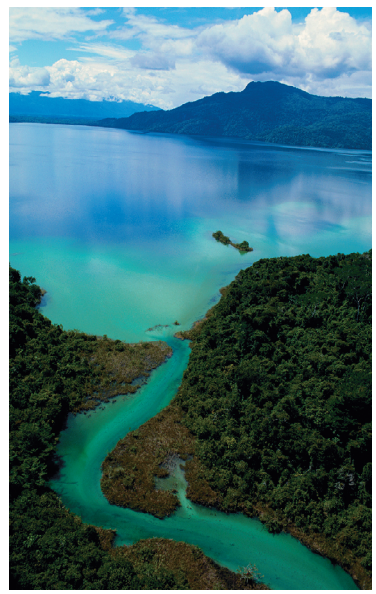
Now try Exercises 65 through 78 ▶

D. Applications

While the factor and remainder theorems are valuable tools for factoring higher degree polynomials, each has applications that extend beyond this use.

EXAMPLE 9 ▶ Using the Remainder Theorem to Solve a Discharge Rate Application

The *discharge rate* of a river is a measure of the river's water flow as it empties into a lake, sea, or ocean. The rate depends on many factors, but is primarily influenced by the precipitation in the surrounding area and is often seasonal. Suppose the discharge rate of the Shimote River was modeled by $D(m) = -m^4 + 22m^3 - 147m^2 + 317m + 150$, where $D(m)$ represents the discharge rate in thousands of cubic meters of water per second in month m ($m = 1 \rightarrow$ Jan).



a. What was the discharge rate in June (summer heat)?

b. Is the discharge rate higher in February (winter runoff) or October (fall rains)?

Solution ▶

a. To find the discharge rate in June, we evaluate D at $m = 6$.
Using the remainder theorem gives

$$\begin{array}{r|rrrrr} 6 & -1 & 22 & -147 & 317 & 150 \\ & \downarrow & -6 & 96 & -306 & 66 \\ \hline & -1 & 16 & -51 & 11 & \boxed{216} \end{array}$$

In June, the discharge rate is 216,000 m^3/sec.

b. For the discharge rates in February ($m = 2$) and October ($m = 10$), we have

$$\begin{array}{r|rrrrr} 2 & -1 & 22 & -147 & 317 & 150 \\ & \downarrow & -2 & 40 & -214 & 206 \\ \hline & -1 & 20 & -107 & 103 & \boxed{356} \end{array} \qquad \begin{array}{r|rrrrr} 10 & -1 & 22 & -147 & 317 & 150 \\ & \downarrow & -10 & 120 & -270 & 470 \\ \hline & -1 & 12 & -27 & 47 & \boxed{620} \end{array}$$

The discharge rate during the fall rains in October is much higher.

D. You've just learned how to solve applications using the remainder theorem

Now try Exercises 81 through 84 ▶

3.2 EXERCISES

CONCEPTS AND VOCABULARY

Fill in each blank with the appropriate word or phrase. Carefully reread the section if needed.

1. For ________ division, we use the ________ of the divisor to begin.
2. If the ________ is zero after division, then the ________ is a factor of the dividend.
3. If polynomial $P(x)$ is divided by a linear divisor of the form $x - c$, the remainder is identical to ________. This is a statement of the ________ theorem.
4. If $P(c) = 0$, then ________ must be a factor of $P(x)$. Conversely, if ________ is a factor of $P(x)$, then $P(c) = 0$. These are statements from the ________ theorem.
5. Discuss/Explain how to write the quotient and remainder using the last line from a synthetic division.
6. Discuss/Explain why (a, b) is a point on the graph of P, given b was the remainder after P was divided by a using synthetic division.

Divide using long division. Write the result as dividend = (divisor)(quotient) + remainder.

7. $\dfrac{x^3 - 5x^2 - 4x + 23}{x - 2}$
8. $\dfrac{x^3 + 5x^2 - 17x - 26}{x + 7}$
9. $(2x^3 + 5x^2 + 4x + 17) \div (x + 3)$
10. $(3x^3 + 14x^2 - 2x - 37) \div (x + 4)$
11. $(x^3 - 8x^2 + 11x + 20) + (x - 5)$
12. $(x^3 - 5x^2 - 22x - 16) + (x + 2)$

Divide using synthetic division. Write answers in two ways: (a) $\frac{\text{dividend}}{\text{divisor}}$ = quotient + $\frac{\text{remainder}}{\text{divisor}}$, and (b) dividend = (divisor)(quotient) + remainder. For Exercises 13–18, check answers using multiplication.

13. $\dfrac{2x^2 - 5x - 3}{x - 3}$
14. $\dfrac{3x^2 + 13x - 10}{x + 5}$
15. $(x^3 - 3x^2 - 14x - 8) \div (x + 2)$
16. $(x^3 - 6x^2 - 25x - 17) \div (x + 1)$
17. $\dfrac{x^3 - 5x^2 - 4x + 23}{x - 2}$
18. $\dfrac{x^3 + 12x^2 + 34x - 7}{x + 7}$
19. $(2x^3 - 5x^2 - 11x - 17) \div (x - 4)$
20. $(3x^3 - x^2 - 7x + 27) \div (x - 1)$

Divide using synthetic division. Note that some terms of a polynomial may be "missing." Write answers as dividend = (divisor)(quotient) + remainder.

21. $(x^3 + 5x^2 + 7) \div (x + 1)$
22. $(x^3 - 3x^2 - 37) \div (x - 5)$
23. $(x^3 - 13x - 12) \div (x - 4)$
24. $(x^3 - 7x + 6) \div (x + 3)$
25. $\dfrac{3x^3 - 8x + 12}{x - 1}$
26. $\dfrac{2x^3 + 7x - 81}{x - 3}$
27. $(n^3 + 27) \div (n + 3)$
28. $(m^3 - 8) \div (m - 2)$
29. $(x^4 + 3x^3 - 16x - 8) \div (x - 2)$
30. $(x^4 + 3x^2 + 29x - 21) \div (x + 3)$

Compute each indicated quotient. Write answers in the form $\frac{\text{dividend}}{\text{divisor}}$ = quotient + $\frac{\text{remainder}}{\text{divisor}}$.

31. $\dfrac{2x^3 + 7x^2 - x + 26}{x^2 + 3}$
32. $\dfrac{x^4 + 3x^3 + 2x^2 - x - 5}{x^2 - 2}$
33. $\dfrac{x^4 - 5x^2 - 4x + 7}{x^2 - 1}$
34. $\dfrac{x^4 + 2x^3 - 8x - 16}{x^2 + 5}$

DEVELOPING YOUR SKILLS

Use the remainder theorem to evaluate $P(x)$ as given.

35. $P(x) = x^3 - 6x^2 + 5x + 12$
    a. $P(-2)$ b. $P(5)$
36. $P(x) = x^3 + 4x^2 - 8x - 15$
    a. $P(-2)$ b. $P(3)$
37. $P(x) = 2x^3 - x^2 - 19x + 4$
    a. $P(-3)$ b. $P(2)$

38. $P(x) = 3x^3 - 8x^2 - 14x + 9$
a. $P(-2)$ **b.** $P(4)$

39. $P(x) = x^4 - 4x^2 + x + 1$
a. $P(-2)$ **b.** $P(2)$

40. $P(x) = x^4 + 3x^3 - 2x - 4$
a. $P(-2)$ **b.** $P(2)$

41. $P(x) = 2x^3 - 7x + 33$
a. $P(-2)$ **b.** $P(-3)$

42. $P(x) = -2x^3 + 9x^2 - 11$
a. $P(-2)$ **b.** $P(-1)$

43. $P(x) = 2x^3 + 3x^2 - 9x - 10$
a. $P(\frac{3}{2})$ **b.** $P(-\frac{5}{2})$

44. $P(x) = 3x^3 + 11x^2 + 2x - 16$
a. $P(\frac{1}{3})$ **b.** $P(-\frac{8}{3})$

Use the factor theorem to determine if the factors given are factors of $f(x)$.

45. $f(x) = x^3 - 3x^2 - 13x + 15$
a. $(x + 3)$ **b.** $(x - 5)$

46. $f(x) = x^3 + 2x^2 - 11x - 12$
a. $(x + 4)$ **b.** $(x - 3)$

47. $f(x) = x^3 - 6x^2 + 3x + 10$
a. $(x + 2)$ **b.** $(x - 5)$

48. $f(x) = x^3 + 2x^2 - 5x - 6$
a. $(x - 2)$ **b.** $(x + 4)$

49. $f(x) = -x^3 + 7x - 6$
a. $(x + 3)$ **b.** $(x - 2)$

50. $f(x) = -x^3 + 13x - 12$
a. $(x + 4)$ **b.** $(x - 3)$

Use the factor theorem to show the given value is a zero of $P(x)$.

51. $P(x) = x^3 + 2x^2 - 5x - 6$
$x = -3$

52. $P(x) = x^3 + 3x^2 - 16x + 12$
$x = -6$

53. $P(x) = x^3 - 7x + 6$
$x = 2$

54. $P(x) = x^3 - 13x + 12$
$x = -4$

55. $P(x) = 9x^3 + 18x^2 - 4x - 8$
$x = \dfrac{2}{3}$

56. $P(x) = 5x^3 + 13x^2 - 9x - 9$
$x = -\dfrac{3}{5}$

A polynomial P with integer coefficients has the zeroes and degree indicated. Use the factor theorem to write the function in factored form and standard form.

57. $-2, 3, -5$; degree 3

58. $1, -4, 2$; degree 3

59. $-2, \sqrt{3}, -\sqrt{3}$; degree 3

60. $\sqrt{5}, -\sqrt{5}, 4$; degree 3

61. $-5, 2\sqrt{3}, -2\sqrt{3}$; degree 3

62. $4, 3\sqrt{2}, -3\sqrt{2}$; degree 3

63. $1, -2, \sqrt{10}, -\sqrt{10}$; degree 4

64. $\sqrt{7}, -\sqrt{7}, 3, -1$; degree 4

In Exercises 65 through 70, a known zero of the polynomial is given. Use the factor theorem to write the polynomial in completely factored form.

65. $P(x) = x^3 - 5x^2 - 2x + 24$; $x = -2$

66. $Q(x) = x^3 - 7x^2 + 7x + 15$; $x = 3$

67. $p(x) = x^4 + 2x^3 - 12x^2 - 18x + 27$; $x = -3$

68. $q(x) = x^4 + 4x^3 - 6x^2 - 4x + 5$; $x = 1$

69. $f(x) = 2x^3 + 11x^2 - x - 30$; $x = \frac{3}{2}$

70. $g(x) = 3x^3 + 2x^2 - 75x - 50$; $x = -\frac{2}{3}$

If $p(x)$ is a polynomial with rational coefficients and a leading coefficient of $a = 1$, the rational zeroes of p (if they exist) *must be factors of the constant term*. Use this property of polynomials with the factor and remainder theorems to factor each polynomial completely.

71. $p(x) = x^3 - 3x^2 - 9x + 27$

72. $p(x) = x^3 - 4x^2 - 16x + 64$

73. $p(x) = x^3 - 6x^2 + 12x - 8$

74. $p(x) = x^3 - 15x^2 + 75x - 125$

75. $p(x) = (x^2 - 6x + 9)(x^2 - 9)$

76. $p(x) = (x^2 - 1)(x^2 - 2x + 1)$

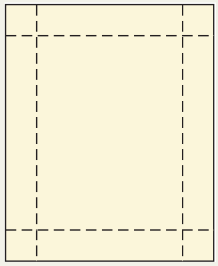
77. $p(x) = (x^3 + 4x^2 - 9x - 36)(x^2 + x - 12)$

78. $p(x) = (x^3 - 3x^2 + 3x - 1)(x^2 - 3x + 2)$

▶ WORKING WITH FORMULAS

Volume of an open box: $V(x) = 4x^3 - 84x^2 + 432x$

An open box is constructed by cutting square corners from a 24 in. by 18 in. sheet of cardboard and folding up the sides. Its volume is given by the formula shown, where x represents the size of the square cut.



79. Given a volume of 640 in^3, use synthetic division and the remainder theorem to determine if the squares were 2-, 3-, 4-, or 5-in. squares and state the dimensions of the box. (*Hint:* Write as a function $v(x)$ and use synthetic division.)

80. Given the volume is 357.5 in^3, use synthetic division and the remainder theorem to determine if the squares were 5.5-, 6.5-, or 7.5-in. squares and state the dimensions of the box. (*Hint:* Write as a function $v(x)$ and use synthetic division.)

▶ APPLICATIONS

81. Tourist population: During the 12 weeks of summer, the population of tourists at a popular beach resort is modeled by the polynomial $P(w) = -0.1w^4 + 2w^3 - 14w^2 + 52w + 5$, where $P(w)$ is the tourist population (in 1000s) during week w. Use the remainder theorem to help answer the following questions.

a. Were there more tourists at the resort in week 5 $(w = 5)$ or week 10? How many more tourists?

b. Were more tourists at the resort one week after opening $(w = 1)$ or one week before closing $(w = 11)$. How many more tourists?

c. The tourist population peaked (reached its highest) between weeks 7 and 10. Use the remainder theorem to determine the peak week.

82. Debt load: Due to a fluctuation in tax revenues, a county government is projecting a deficit for the next 12 months, followed by a quick recovery and the repayment of all debt near the end of this period. The projected debt can be modeled by the polynomial $D(m) = 0.1m^4 - 2m^3 + 15m^2 - 64m - 3$, where $D(m)$ represents the amount of debt (in millions of dollars) in month m. Use the remainder theorem to help answer the following questions.

a. Was the debt higher in month 5 $(m = 5)$ or month 10 of this period? How much higher?

b. Was the debt higher in the first month of this period (one month into the deficit) or after the eleventh month (one month before the expected recovery)? How much higher?

c. The total debt reached its maximum between months 7 and 10. Use the remainder theorem to determine which month.

83. Volume of water: The volume of water in a rectangular, in-ground, swimming pool is given by $V(x) = x^3 + 11x^2 + 24x$, where $v(x)$ is the volume in cubic feet when the water is x ft high. (a) Use the remainder theorem to find the volume when $x = 3$ ft. (b) If the volume is 100 ft^3 of water, what is the height x? (c) If the maximum capacity of the pool is 1000 ft^3, what is the maximum depth (to the nearest integer)?

84. Amusement park attendance: Attendance at an amusement park depends on the weather. After opening in spring, attendance rises quickly, slows during the summer, soars in the fall, then quickly falls with the approach of winter when the park closes. The model for attendance is given by $A(m) = -\frac{1}{4}m^4 + 6m^3 - 52m^2 + 196m - 260$, where $A(m)$ represents the number of people attending in month m (in thousands). (a) Did more people go to the park in April $(m = 4)$ or June $(m = 6)$? (b) In what month did maximum attendance occur? (c) When did the park close?

In these applications, synthetic division is applied in the usual way, treating k as an unknown constant.

85. Find a value of k that will make $x = -2$ a zero of $f(x) = x^3 - 3x^2 - 5x + k$.

86. Find a value of k that will make $x - 3$ a factor of $g(x) = x^3 + 2x^2 - 7x + k$.

87. For what value(s) of k will $x - 2$ be a factor of $p(x) = x^3 - 3x^2 + kx + 10$?

88. For what value(s) of k will $x + 5$ be a factor of $q(x) = x^3 + 6x^2 + kx + 50$?

▶ EXTENDING THE CONCEPT

89. To investigate whether the remainder and factor theorems can be applied when the coefficients or zeroes of a polynomial are complex, try using the factor theorem to find a polynomial with degree 3, whose zeroes are $x = 2i$, $x = -2i$, and $x = 3$. Then see if the result can be verified using the remainder theorem and these zeroes. What does the result suggest?

90. Since we use a base-10 number system, numbers like 1196 can be written in polynomial form as $p(x) = 1x^3 + 1x^2 + 9x + 6$, where $x = 10$. Divide $p(x)$ by $x + 3$ using synthetic division and write your answer as $\frac{x^3 + x^2 + 9x + 6}{x + 3}$ = quotient + $\frac{\text{remainder}}{\text{divisor}}$. For $x = 10$, what is the value of quotient + $\frac{\text{remainder}}{\text{divisor}}$? What is the result of dividing 1196 by $10 + 3 = 13$? What can you conclude?

91. The sum of the first n perfect cubes is given by the formula $S = \frac{1}{4}(n^4 + 2n^3 + n^2)$. Use the remainder theorem on S to find the sum of (a) the first three perfect cubes (divide by $n - 3$) and (b) the first five perfect cubes (divide by $n - 5$). Check results by adding the perfect cubes manually. To avoid working with fractions you can initially ignore the $\frac{1}{4}$ (use $n^4 + 2n^3 + n^2 + 0n + 0$), as long as you divide the remainder by 4.

92. Though not a direct focus of this course, the remainder and factor theorems, as well as synthetic division, *can also be applied using complex numbers*. Use the remainder theorem to show the value given is a zero of $P(x)$.

a. $P(x) = x^3 - 4x^2 + 9x - 36;\ x = 3i$

b. $P(x) = x^4 + x^3 + 2x^2 + 4x - 8;\ x = -2i$

c. $P(x) = -x^3 + x^2 - 3x - 5;\ x = 1 + 2i$

d. $P(x) = x^3 + 2x^2 + 16x + 32;\ x = -4i$

e. $P(x) = x^4 + x^3 - 5x^2 + x - 6;\ x = i$

f. $P(x) = -x^3 + x^2 - 8x - 10;\ x = 1 + 3i$

MAINTAINING YOUR SKILLS

93. **(1.1)** John and Rick are out orienteering. Rick finds the last marker first and is heading for the finish line, 1275 yd away. John is just seconds behind, and after locating the last marker tries to overtake Rick, who by now has a 250-yd lead. If Rick runs at 4 yd/sec and John runs at 5 yd/sec, will John catch Rick before they reach the finish line?

94. **(1.5)** Solve for w: $-2(3w^2 + 5) + 3 = -7w + w^2 - 7$

95. **(2.3)** The profit of a small business increased linearly from \$5000 in 2005 to \$12,000 in 2010. Find a linear function $G(t)$ modeling the growth of the company's profit (let $t = 0$ correspond to 2005).

96. **(2.7)** Given $f(x) = x^2 - 4x$, use the average rate of change formula to find $\frac{\Delta y}{\Delta x}$ in the interval $x \in [1.0, 1.1]$.

3.3 The Zeroes of Polynomial Functions

Learning Objectives

In Section 3.3 you will learn how to:

- ❑ **A.** Apply the fundamental theorem of algebra and the linear factorization theorem
- ❑ **B.** Locate zeroes of a polynomial using the intermediate value theorem
- ❑ **C.** Find rational zeroes of a polynomial using the rational zeroes theorem
- ❑ **D.** Use Descartes' rule of signs and the upper/lower bounds theorem
- ❑ **E.** Solve applications of polynomials

This section represents one of the highlights in the college algebra curriculum, because it offers a look at what many call *the big picture*. The ideas presented are the result of a cumulative knowledge base developed over a long period of time, and give a fairly comprehensive view of the study of polynomial functions.

A. The Fundamental Theorem of Algebra

From Section 1.4, we know that real numbers are a subset of the complex numbers: $\mathbb{R} \subset \mathbb{C}$. Because complex numbers are the "larger" set (containing all other number sets), properties and theorems about complex numbers are more powerful and far reaching than theorems about real numbers. In the same way, real polynomials are a subset of the complex polynomials, and the same principle applies.

WORTHY OF NOTE

Quadratic functions also belong to the larger family of **complex polynomial functions.** Since quadratics have a known number of terms, it is common to write the general form using the early letters of the alphabet: $P(x) = \mathbf{a}x^2 + \mathbf{b}x + \mathbf{c} = 0$. For higher degree polynomials, the number of terms is unknown or unspecified, and the general form is written using subscripts on a single letter.

Complex Polynomial Functions

A complex polynomial of degree n has the form

$$P(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x^1 + a_0,$$

where $a_n, a_{n-1}, \cdots, a_1, a_0$ are complex numbers and $a_n \neq 0$.

Notice that real polynomials have the same form, but here $a_n, a_{n-1}, \ldots, a_1, a_0$ *represent complex numbers*. In 1797, Carl Friedrich Gauss (1777–1855) proved that *all* polynomial functions have zeroes, and that the number of zeroes is equal to the degree of the polynomial. The proof of this statement is based on a theorem that is the bedrock for a complete study of polynomial functions, and has come to be known as the **fundamental theorem of algebra.**

The Fundamental Theorem of Algebra

Every complex polynomial of degree $n \geq 1$ has at least one complex zero.

Although the statement may seem trivial, it allows us to draw two important conclusions. The first is that our search for a solution will not be fruitless or wasted—zeroes for *all* polynomial equations exist. Second, the fundamental theorem combined with the factor theorem allows us to state the **linear factorization theorem.**

The Linear Factorization Theorem

If $p(x)$ is a polynomial function of degree $n \geq 1$, then p has exactly n linear factors and can be written in the form,

$$p(x) = a(x - c_1)(x - c_2) \cdot \cdots \cdot (x - c_n)$$

where $a \neq 0$ and $c_1, c_2, \ldots, c_n$ are (not necessarily distinct) complex numbers.

In other words, every complex polynomial of degree n can be rewritten as the product of a nonzero constant and exactly n linear factors (for a proof of this theorem, see Appendix II).

EXAMPLE 1 ▶ Writing Polynomials as a Product of Linear Factors

Rewrite $P(x) = x^4 - 8x^2 - 9$ as a product of linear factors, and find its zeroes.

Solution ▶ From its given form, we know $a = 1$. Since P has degree 4, the factored form must be $P(x) = (x - c_1)(x - c_2)(x - c_3)(x - c_4)$. Noting that P is in quadratic form, we substitute u for x^2 and u^2 for x^4 and attempt to factor:

$$x^4 - 8x^2 - 9 \rightarrow u^2 - 8u - 9 \qquad \text{substitute } u \text{ for } x^2\text{; } u^2 \text{ for } x^4$$
$$= (u - 9)(u + 1) \qquad \text{factor in terms of } u$$
$$= (x^2 - 9)(x^2 + 1) \qquad \text{rewrite in terms of } x \text{ (substitute } x^2 \text{ for } u\text{)}$$

We know $x^2 - 9$ will factor since it is a difference of squares. From our work with complex numbers (Section 1.4), we know $(a + bi)(a - bi) = a^2 + b^2$, and the factored form of $x^2 + 1$ must be $(x + i)(x - i)$. The completely factored form is

$$P(x) = (x + 3)(x - 3)(x + i)(x - i), \text{ and}$$

the zeroes of P are -3, 3, $-i$, and i.

Now try Exercises 7 through 10 ▶

WORTHY OF NOTE

While polynomials with complex coefficients are not the focus of this course, interested students can investigate the wider application of these theorems by completing **Exercise 115.**

EXAMPLE 2 ▶ Writing Polynomials as a Product of Linear Factors

Rewrite $P(x) = x^3 + 2x^2 - 4x - 8$ as a product of linear factors and find its zeroes.

Solution ▶ We observe that $a = 1$ and P has degree 3, so the factored form must be $P(x) = (x - c_1)(x - c_2)(x - c_3)$. Noting that $ad = bc$ (Section R.4), we start with factoring by grouping.

$$
\begin{aligned}
P(x) &= x^3 + 2x^2 - 4x - 8 && \text{group terms (in color)} \\
&= x^2(x + 2) - 4(x + 2) && \text{remove common factors (note sign change)} \\
&= (x + 2)(x^2 - 4) && \text{factor common binomial} \\
&= (x + 2)(x + 2)(x - 2) && \text{factor difference of squares}
\end{aligned}
$$

The zeroes of P are -2, -2, and 2.

Now try Exercises 11 through 14 ▶

Note the polynomial in Example 2 has three zeroes, but the zero -2 was repeated two times. In this case we say -2 is a zero of multiplicity two, and a zero of **even multiplicity.** It is also possible for a zero to be repeated three or more times, with those repeated an odd number of times called zeroes of **odd multiplicity** [the factor $(x - 2) = (x - 2)^1$ also gives a zero of odd multiplicity]. In general, repeated factors are written in exponential form and we have

Zeroes of Multiplicity

If p is a polynomial function with degree $n \geq 1$, and $(x - c)$ occurs as a factor of p exactly m times, then c is a zero of multiplicity m.

EXAMPLE 3 ▶ Identifying the Multiplicity of a Zero

Factor the given function completely, writing repeated factors in exponential form. Then state the multiplicity of each zero: $P(x) = (x^2 + 8x + 16)(x^2 - x - 20)(x - 5)$

Solution ▶

$$
\begin{aligned}
P(x) &= (x^2 + 8x + 16)(x^2 - x - 20)(x - 5) && \text{given polynomial} \\
&= (x + 4)(x + 4)(x - 5)(x + 4)(x - 5) && \text{trinomial factoring} \\
&= (x + 4)^3(x - 5)^2 && \text{exponential form}
\end{aligned}
$$

For function P, -4 is a zero of multiplicity 3 (odd multiplicity), and 5 is a zero of multiplicity 2 (even multiplicity).

Now try Exercises 15 through 18 ▶

These examples help illustrate three important consequences of the linear factorization theorem. From Example 1, if the coefficients of P are real, the polynomial can be factored into linear and quadratic factors using real numbers only $[(x + 3)(x - 3)(x^2 + 1)]$, where the quadratic factors have no real zeroes. Quadratic factors of this type are said to be **irreducible.**

WORTHY OF NOTE

When reconstructing a polynomial P having complex zeroes, it is often more efficient to determine the irreducible quadratic factors of P separately, as shown here. For the zeroes $2 \pm \sqrt{3}i$ we have

$$
\begin{aligned}
x &= 2 \pm i\sqrt{3} \\
x - 2 &= \pm i\sqrt{3} \\
(x - 2)^2 &= (\pm i\sqrt{3})^2 \\
x^2 - 4x + 4 &= -3 \\
x^2 - 4x + 7 &= 0.
\end{aligned}
$$

The quadratic factor is $(x^2 - 4x + 7)$.

Corollary I: Irreducible Quadratic Factors

If p is a polynomial with real coefficients, p can be factored into a product of linear factors (which are not necessarily distinct) and irreducible quadratic factors having real coefficients.

Closely related to this corollary and our previous study of quadratic functions, complex zeroes of the irreducible factors must occur in conjugate pairs.

Corollary II: Complex Conjugates

If p is a polynomial with real coefficients, complex zeroes must occur in conjugate pairs. If $a + bi$, $b \neq 0$ is a zero, then $a - bi$ will also be a zero.

Finally, the polynomial in Example 1 has degree 4 with 4 zeroes (two real, two complex), and the polynomial in Example 2 has degree 3 with 3 zeroes (three real, one repeated). While not shown explicitly, the polynomial in Example 3 has degree 5, and there were 5 zeroes (one repeated twice, one repeated three times). This suggests our final corollary.

Corollary III: Number of Zeroes

If p is a polynomial function with degree $n \geq 1$, then p has exactly n zeroes (real or complex), where zeroes of multiplicity m are counted m times.

These corollaries help us gain valuable information about a polynomial, when only partial information is given or known.

EXAMPLE 4 ▶ Constructing a Polynomial from Its Zeroes

A polynomial P of degree 3 with real coefficients has zeroes of -1 and $2 + i\sqrt{3}$. Find the polynomial (assume $a = 1$).

Solution ▶ Using the factor theorem, two of the factors are $(x + 1)$ and $x - (2 + i\sqrt{3})$. From Corollary II, $2 - i\sqrt{3}$ must also be a zero and $x - (2 - i\sqrt{3})$ is also a factor of P. This gives

$$\begin{aligned} P(x) &= (x + 1)[x - (2 + i\sqrt{3})][x - (2 - i\sqrt{3})] \\ &= (x + 1)[(x - 2) - i\sqrt{3}][(x - 2) + i\sqrt{3}] && \text{associative property} \\ &= (x + 1)[(x^2 - 4x + 4) + 3] && (a + bi)(a - bi) = a^2 + b^2 \\ &= (x + 1)(x^2 - 4x + 7) && \text{simplify} \\ &= x^3 - 3x^2 + 3x + 7 && \text{result} \end{aligned}$$

The polynomial is $P(x) = x^3 - 3x^2 + 3x + 7$, which can be verified using the remainder theorem and any of the original zeroes.

Now try Exercises 19 through 22 ▶

EXAMPLE 5 ▶ Building a Polynomial from Its Zeroes

Find a fourth degree polynomial P with real coefficients, if 3 is the only real zero and $2i$ is also a zero of P.

Solution ▶ Since complex zeroes must occur in conjugate pairs, $-2i$ is also a zero, but this accounts for only three zeroes. Since P has degree 4, 3 must be a *repeated* zero, and the factors of P are $(x - 3)(x - 3)(x - 2i)(x + 2i)$.

$$\begin{aligned} P(x) &= (x - 3)(x - 3)(x - 2i)(x + 2i) && \text{factored form} \\ &= (x^2 - 6x + 9)(x^2 + 4) && \text{multiply binomials, } (a + bi)(a - bi) = a^2 + b^2 \\ &= x^4 - 6x^3 + 13x^2 - 24x + 36 && \text{result} \end{aligned}$$

The polynomial is $P(x) = x^4 - 6x^3 + 13x^2 - 24x + 36$, which can be verified using the remainder theorem and any of the original zeroes.

Now try Exercises 23 through 28 ▶

☑ **A.** You've just learned how to apply the fundamental theorem of algebra and the linear factorization theorem

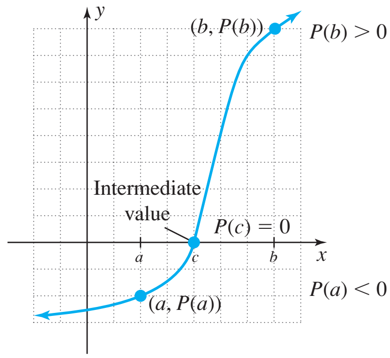
B. Real Polynomials and the Intermediate Value Theorem

The fundamental theorem of algebra is called an **existence theorem,** as it affirms the *existence* of the zeroes but does not tell us where or how to find them. Because polynomial graphs are continuous (there are no holes or breaks in the graph), the **intermediate value theorem (IVT)** can be used for this purpose.

The Intermediate Value Theorem

Given P is a polynomial with real coefficients, if $P(a)$ and $P(b)$ have opposite signs, there is *at least* one value c between a and b such that $P(c) = 0$.

Figure 3.8



WORTHY OF NOTE

You might recall a similar idea was used in Section 2.5, where we noted the graph of $P(x)$ crosses the x-axis at the zeroes determined by linear factors, with a corresponding change of sign in the function values.

EXAMPLE 6 ▶ **Finding Zeroes Using the Intermediate Value Theorem**

Use the intermediate value theorem to show $P(x) = x^3 - 9x + 6$ has at least one zero in the interval given:

a. $[-4, -3]$ **b.** $[0, 1]$

Solution ▶ **a.** Begin by evaluating P at $x = -4$ and $x = -3$.

$$P(-4) = (-4)^3 - 9(-4) + 6 = -64 + 36 + 6 = -22$$

$$P(-3) = x^3 - 9x + 6 = -27 + 27 + 6 = 6$$

Since $P(-4) < 0$ and $P(-3) > 0$, there must be at least one number c_1 between -4 and -3 where $P(c_1) = 0$. The graph must cross the x-axis in this interval.

b. Evaluate P at $x = 0$ and $x = 1$.

$$P(0) = (0)^3 - 9(0) + 6 = 0 - 0 + 6 = 6$$

$$P(1) = (1)^3 - 9(1) + 6 = 1 - 9 + 6 = -2$$

Since $P(0) > 0$ and $P(1) < 0$, there must be at least one number c_2 between 0 and 1 where $P(c_2) = 0$.

☑ **B.** You've just learned how to locate zeroes of a real polynomial function using the intermediate value theorem

Now try Exercises 29 through 32 ▶

C. The Rational Zeroes Theorem

The fundamental theorem of algebra tells us that zeroes of a polynomial function *exist*. The intermediate value theorem tells us how to *locate* zeroes within an interval. Our next theorem gives us the information we need to actually *find* certain zeroes of a polynomial. Recall that if c is a zero of P, then $P(c) = 0$, and when $P(x)$ is divided by $x - c$ using synthetic division, the remainder is zero (from the remainder and factor theorems).

To find *divisors that give a remainder of zero,* we make the following observations. To solve $3x^2 - 11x - 20 = 0$ by factoring, a beginner might write out all possible binomial pairs where the **F**irst term in the F-O-I-L process multiplies to $3x^2$ and the **L**ast term multiplies to 20. The six possibilities are shown here:

(3x 1)(x 20) (3x 20)(x 1) (3x 2)(x 10) (3x 10)(x 2)

(3x 4)(x 5) (3x 5)(x 4)

If $3x^2 - 11x - 20$ is factorable using integers, the factors *must be somewhere in this list*. Also, the first coefficient in each binomial must be a factor of the leading coefficient, and the second coefficient must be a factor of the constant term. This means that regardless of which factored form is correct, the solution will be a rational number whose numerator comes from the factors of 20, and whose denominator comes from the factors of 3. The correct factored form is shown here, along with the solution:

$$3x^2 - 11x - 20 = 0$$

$$(3x + 4)(x - 5) = 0$$

$$3x + 4 = 0 \qquad x - 5 = 0$$

$$x = \frac{-4}{3} \begin{array}{l} \leftarrow \text{from the factors of 20} \\ \leftarrow \text{from the factors of 3} \end{array} \qquad x = \frac{5}{1} \begin{array}{l} \leftarrow \text{from the factors of 20} \\ \leftarrow \text{from the factors of 3} \end{array}$$

This same principle also applies to polynomials of higher degree, and these observations suggest the following theorem.

The Rational Zeroes Theorem

Given polynomial P with integer coefficients, and $\frac{p}{q}$ a rational number in lowest terms, the rational zeroes of P (if they exist) must be of the form $\frac{p}{q}$, where p is a factor of the constant term, and q is a factor of the leading coefficient.

Note that if the leading coefficient is 1, the possible rational zeroes are limited to factors of the constant term: $\frac{p}{1} = p$. If the leading coefficient is not "1" and the constant term has a large number of factors, the set of possible rational zeroes becomes rather large. To list these possibilities, it helps to begin with all factor *pairs* of the constant a_0, then divide each of these by the factors of a_n as shown in Example 7.

EXAMPLE 7 ▶ **Identifying the Possible Rational Zeroes of a Polynomial**

List all possible rational zeroes for $3x^4 + 14x^3 - x^2 - 42x - 24 = 0$, but do not solve.

Solution ▶ All rational zeroes must be of the form $\frac{p}{q}$, where p is a factor of $a_0 = -24$ and q is a factor of $a_n = 3$. The factor pairs of -24 are: ± 1, ± 24, ± 2, ± 12, ± 3, ± 8, ± 4 and ± 6. Dividing each by ± 1 and ± 3 (the factor pairs of 3), we note division by ± 1 will not change any of the previous values, while division by ± 3 gives $\pm\frac{1}{3}, \pm\frac{2}{3}, \pm\frac{8}{3}, \pm\frac{4}{3}$ as additional possibilities. Any rational zeroes must be in the set $\{\pm 1, \pm 24, \pm 2, \pm 12, \pm 3, \pm 8, \pm 4, \pm 6, \pm\frac{1}{3}, \pm\frac{2}{3}, \pm\frac{8}{3}, \pm\frac{4}{3}\}$.

Now try Exercises 33 through 40 ▶

WORTHY OF NOTE

To test for -1, only the sign of terms with odd degree must be changed, since $(-1)^{\text{even\#}} = 1$, while $(-1)^{\text{odd\#}} = -1$. The method simply gives a shortcut for evaluating $P(1)$ and $P(-1)$, which often helps to break down a higher degree polynomial.

The actual solutions to the equation in Example 7 are $x = \sqrt{3}$, $x = -\sqrt{3}$, $x = -\frac{2}{3}$, and $x = -4$. Although the *rational* zeroes are indeed in the set noted, it's apparent we need a way to narrow down the number of possibilities (we don't want to try all 24 possible zeroes). If we're able to find even one factor easily, we can rewrite the polynomial using this factor and the quotient polynomial, with the hope of factoring further using trinomial factoring or factoring by grouping. Many times testing to see if 1 or -1 are zeroes will help.

Tests to Determine If 1 or −1 is a Zero of *P*

For any polynomial P with real coefficients,

1. *If the sum of all coefficients is zero, then 1 is a root and $(x - 1)$ is a factor.*
2. *After changing the sign of all terms with odd degree, if the sum of the coefficients is zero, then -1 is a root and $(x + 1)$ is a factor.*

EXAMPLE 8 ▶ **Finding the Rational Zeroes of a Polynomial**

Find all rational zeroes of $P(x) = 3x^4 - x^3 - 8x^2 + 2x + 4$, and use them to write the function in completely factored form. Then use the factored form to name all zeroes of P.

Solution ▶ Instead of listing all possibilities using the rational zeroes theorem, we first test for 1 and -1, then see if we're able to complete the factorization using other means. The sum of the coefficients is: $3 - 1 - 8 + 2 + 4 = 0$, which means 1 is a zero and $x - 1$ is a factor. By changing the sign on terms of odd degree, we have $3x^4 + x^3 - 8x^2 - 2x + 4$ and $3 + 1 - 8 - 2 + 4 = -2$, showing -1 is *not* a zero. Using $x = 1$ and the factor theorem, we have

use 1 as a "divisor"

$$\begin{array}{r|rrrrr} 1 & 3 & -1 & -8 & 2 & 4 \\ & & 3 & 2 & -6 & -4 \\ \hline & 3 & 2 & -6 & -4 & 0 \end{array}$$

and we write P as $P(x) = (x - 1)(\mathbf{3x^3} + \mathbf{2x^2} - \mathbf{6x} - \mathbf{4})$. Noting the quotient polynomial can be factored by grouping ($ad = bc$), we need not continue with synthetic division or the factor theorem.

$$\begin{aligned} P(x) &= (x - 1)(3x^3 + 2x^2 - 6x - 4) && \text{group terms} \\ &= (x - 1)[x^2(3x + 2) - 2(3x + 2)] && \text{factor common terms} \\ &= (x - 1)(3x + 2)(x^2 - 2) && \text{factor common binomial} \\ &= (x - 1)(3x + 2)(x + \sqrt{2})(x - \sqrt{2}) && \text{completely factored form} \end{aligned}$$

The zeroes of P are 1, $\frac{-2}{3}$, and $\pm\sqrt{2}$.

Now try Exercises 41 through 62 ▶

WORTHY OF NOTE

In the second to last line of Example 8, we factored $x^2 - 2$ as $(x + \sqrt{2})(x - \sqrt{2})$. As discussed in Section R.4, this is an application of factoring the difference of two squares: $a^2 - b^2 = (a + b)(a - b)$. By mentally rewriting $x^2 - 2$ as $x^2 - (\sqrt{2})^2$, we obtain the result shown. Also **see Exercise 113.**

In cases where the quotient polynomial is not easily factored, we continue with synthetic division and other possible zeroes, until the remaining zeroes can be determined.

EXAMPLE 9 ▶ **Finding the Zeroes of a Polynomial**

Find all zeroes of $P(x) = x^5 - 3x^4 + 3x^3 - 5x^2 + 12$.

Solution ▶ Using the rational zeroes theorem, the possibilities are: $\{\pm1, \pm12, \pm2, \pm6, \pm3, \pm4\}$. The test for 1 shows 1 is not a zero. After changing the signs of all terms with odd degree, we have $-1 - 3 - 3 - 5 + 12 = 0$, and find -1 *is* a zero. Using -1 with the factor theorem, we continue our search for additional factors. Noting that P is missing a linear term, we include a place-holder zero:

use -1 as a "divisor"

$$\begin{array}{r|rrrrrrl} -1 & 1 & -3 & 3 & -5 & 0 & 12 & \text{coefficients of } P \\ & & -1 & 4 & -7 & 12 & -12 & \\ \hline & 1 & -4 & 7 & -12 & 12 & 0 & \text{coefficients of } q_1(x) \end{array}$$

Here the quotient polynomial $q_1(x) = x^4 - 4x^3 + 7x^2 - 12x + 12$ is not easily factored, so we next try 2, *using the quotient polynomial:*

use 2 as a "divisor" on $q_1(x)$

$$\begin{array}{r|rrrrrl} 2 & 1 & -4 & 7 & -12 & 12 & \text{coefficients of } q_1(x) \\ & & 2 & -4 & 6 & -12 & \\ \hline & 1 & -2 & 3 & -6 & 0 & \text{coefficients of } q_2(x) \end{array}$$

If you miss the fact that $q_2(x)$ is actually factorable ($ad = bc$), the process would continue using -2 and the current quotient.

use -2 as a "divisor"

$$\begin{array}{r|rrrrl} -2 & 1 & -2 & 3 & -6 & \text{coefficients of } q_2(x) \\ & & -2 & 8 & -22 & \\ \hline & 1 & -4 & 11 & -28 & -2 \text{ is not a zero} \end{array}$$

We find -2 is not a zero, and in fact, trying *all other possible zeroes* will show that *none* of them are zeroes. As there must be five zeroes, we are reminded of three things:

1. This process can only find *rational zeros* (the remaining zeroes may be irrational or complex),
2. This process cannot find irreducible quadratic factors (unless they appear as the quotient polynomial), and
3. Some of the zeroes *may have multiplicities greater than 1!*

Testing the zero 2 for a second time using $q_2(x)$ gives

use 2 as a "divisor"	2	1	−2	3	−6	coefficients of $q_2(x)$
			2	0	6	
		1	0	3	0	2 is a *repeated* zero

✓ **C.** You've just learned how to find rational zeroes of a real polynomial function using the rational zeroes theorem

and we see that 2 is actually a zero of multiplicity two, and the final quotient is the irreducible quadratic factor $x^2 + 3$. Using this information produces the factored form $P(x) = (x + 1)(x - 2)^2(x^2 + 3) = (x + 1)(x - 2)^2(x + i\sqrt{3})(x - i\sqrt{3})$, and the zeroes of P are $-i\sqrt{3}$, $i\sqrt{3}$, -1, and 2 with multiplicity two.

Now try Exercises 63 through 82 ▶

D. Descartes' Rule of Signs and Upper/Lower Bounds

Testing $x = 1$ and $x = -1$ is one way to reduce the number of possible rational zeroes, but unless we're very lucky, factoring the polynomial can still be a challenge. **Descartes' rule of signs** and the **upper and lower bounds property** offer additional assistance.

Descartes' Rule of Signs

Given the real polynomial equation $P(x) = 0$,

1. The number of positive real zeroes is equal to the number of variations in sign for $\boldsymbol{P(x)}$, or an even number less.
2. The number of negative real zeroes is equal to the number of variations in sign for $\boldsymbol{P(-x)}$, or an even number less.

EXAMPLE 10 ▶ Finding the Zeroes of a Polynomial

For $P(x) = 2x^5 - 5x^4 + x^3 + x^2 - x + 6$,

a. Use the rational zeroes theorem to list all possible rational zeroes.

b. Apply Descartes' rule to count the number of possible positive, negative, and complex roots.

c. Use this information and the tools of this section to find all zeroes of P.

Solution ▶

a. The factors of 2 are $\{\pm 1, \pm 2\}$ and the factors of 6 are $\{\pm 1, \pm 6, \pm 2, \pm 3\}$. The possible rational zeroes for P are $\{\pm 1, \pm 6, \pm 2, \pm 3, \pm \frac{1}{2}, \pm \frac{3}{2}\}$.

b. For Descartes' rule, we organize our work in a table. Since P has degree 5, there must be a total of five zeroes. For this illustration, positive terms are in **blue** and negative terms in **red**: $P(x) = 2x^5 - 5x^4 + x^3 + x^2 - x + 6$. The terms change sign a total of four times, meaning there are four, two, or zero positive roots. For the negative roots, recall that $P(-x)$ will change the sign of *all odd-degree terms,* giving $P(-x) = -2x^5 - 5x^4 - x^3 + x^2 + x + 6$. This time there is only one sign change (from negative to positive) showing there is exactly one negative root, a fact that is highlighted in the following table.

possible positive zeroes	*known* negative zeroes	*possibilities for* complex roots	total number *must be 5*
4	**1**	0	5
2	**1**	2	5
0	**1**	4	5

WORTHY OF NOTE

As you recall from our study of quadratics, it's entirely possible for a polynomial function to have no real zeroes. Also, if the zeroes are irrational, complex, or a combination of these, they cannot be found using the rational zeroes theorem. For a look at ways to determine these zeroes, see the *Reinforcing Basic Skills* feature that follows Section 3.4.

c. Testing 1 and -1 shows $x = 1$ is not a root, but $x = -1$ *is*, and using -1 in synthetic division gives:

use -1 as a "divisor"

$$\begin{array}{r|rrrrrr} -1 & 2 & -5 & 1 & 1 & -1 & 6 \\ & & -2 & 7 & -8 & 7 & -6 \\ \hline & 2 & -7 & 8 & -7 & 6 & 0 \end{array}$$

coefficients of $P(x)$; $q_1(x)$ is not easily factored

Since there is *only one* negative root, we need only check the remaining positive zeroes. The quotient $q_1(x)$ is not easily factored, so we continue with synthetic division using the next larger positive root, $x = 2$.

use 2 as a "divisor"

$$\begin{array}{r|rrrrr} 2 & 2 & -7 & 8 & -7 & 6 \\ & & 4 & -6 & 4 & -6 \\ \hline & 2 & -3 & 2 & -3 & 0 \end{array}$$

coefficients of $q_1(x)$; $q_2(x)$ is easily factored

The partially factored form is $P(x) = (x + 1)(x - 2)(2x^3 - 3x^2 + 2x - 3)$, which we can complete using factoring by grouping. The factored form is

$$\begin{aligned} P(x) &= (x + 1)(x - 2)(2x^3 - 3x^2 + 2x - 3) && \text{group terms} \\ &= (x + 1)(x - 2)[x^2(2x - 3) + 1(2x - 3)] && \text{factor common terms} \\ &= (x + 1)(x - 2)(2x - 3)(x^2 + 1) && \text{factor out common binomial} \\ &= (x + 1)(x - 2)(2x - 3)(x + i)(x - i) && \text{completely factored form} \end{aligned}$$

The zeroes of P are $-1, 2, \frac{3}{2}, -i$ and i, with two positive, one negative, and two complex zeroes.

Now try Exercises 83 through 96 ▶

One final idea that helps reduce the number of possible zeroes is the **upper and lower bounds property.** A number b is an **upper bound** on the positive zeroes of a function if no positive zero is greater than b. In the same way, a number a is a **lower bound** on the negative zeroes if no negative zero is less than a.

Upper and Lower Bounds Property

Given $P(x)$ is a polynomial with real coefficients.

1. If $P(x)$ is divided by $x - b$ $(b > 0)$ using synthetic division and all coefficients in the quotient row are either positive or zero, then b is an upper bound on the zeroes of P.
2. If $P(x)$ is divided by $x - a$ $(a < 0)$ using synthetic division and all coefficients in the quotient row alternate in sign, then a is a lower bound on the zeroes of P.

For both 1 and 2, zero coefficients can be either positive or negative as needed.

☑ **D.** You just learned how to gain more information on the zeroes of real polynomials using Descartes' rule of signs and upper/lower bounds

While this test certainly helps narrow the possibilities, we gain the additional benefit of knowing the property actually places boundaries on *all* real zeroes of the polynomial, both rational and irrational. In Part (c) of Example 10, the quotient row of the first division alternates in sign, showing $x = -1$ is both a zero and a lower bound on the real zeroes of P. For more on the upper and lower bounds property, **see Exercise 111.**

E. Applications of Polynomial Functions

Polynomial functions can be very accurate models of real-world phenomena, though we often must restrict their domain, as illustrated in Example 11.

EXAMPLE 11 ▶ Using the Remainder Theorem to Solve an Oceanography Application

As part of an environmental study, scientists use radar to map the ocean floor from the coastline to a distance 12 mi from shore. In this study, ocean trenches appear as negative values and underwater mountains as positive values, as measured from the surrounding ocean floor. The terrain due west of a particular island can be modeled by $h(x) = x^4 - 25x^3 + 200x^2 - 560x + 384$, where $h(x)$ represents the height in feet, x mi from shore $(0 < x \le 12)$.

a. Use the remainder theorem to find the "height of the ocean floor" 10 mi out.

b. Use the tools developed in this section to find the number of times the ocean floor has height $h(x) = 0$ in this interval, given this occurs 12 mi out.

Solution ▶ **a.** For part (a) we simply evaluate $h(10)$ using the remainder theorem.

use 10 as a "divisor"	10	1	−25	200	−560	384	coefficients of $h(x)$
			10	−150	500	−600	
		1	−15	50	−60	−216	remainder is −216

Ten miles from shore, there is an ocean trench 216 ft deep.

b. For part (b), we know 12 is zero, so we again use the remainder theorem and work with the quotient polynomial.

use 12 as a "divisor"	12	1	−25	200	−560	384	coefficients of $h(x)$
			12	−156	528	−384	
		1	−13	44	−32	0	$q_1(x)$

The quotient is $q_1(x) = x^3 - 13x^2 + 44x - 32$. Since $a = 1$, we know the remaining zeroes must be factors of -32: $\{\pm 1, \pm 32, \pm 2, \pm 16, \pm 4, \pm 8\}$. Using $x = 1$ gives

use 1 as a "divisor"	1	1	−13	44	−32	coefficients of $q_1(x)$
			1	−12	32	
		1	−12	32	0	$q_2(x)$

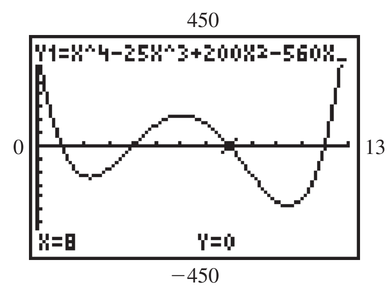
The function can now be written as $h(x) = (x - 12)(x - 1)(x^2 - 12x + 32)$ and in completely factored form $h(x) = (x - 12)(x - 1)(x - 4)(x - 8)$. The ocean floor has height zero at distances of 1, 4, 8, and 12 mi from shore.

E. You've just learned how to solve an application of polynomial functions

Now try Exercises 99 through 110 ▶

GRAPHICAL SUPPORT

The graph of $h(x)$ is shown here using a window size of $X \in [0, 13]$ and $Y \in [-450, 450]$. The graphs shows a great deal of variation in the ocean floor, but the zeroes occurring at 1, 4, 8, and 12 mi out are clearly evident.

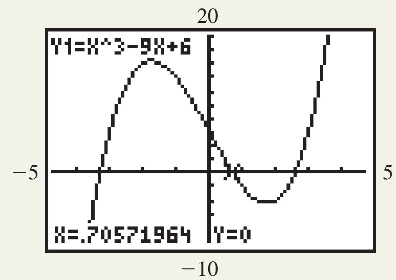


TECHNOLOGY HIGHLIGHT

The Intermediate Value Theorem and Split Screen Viewing

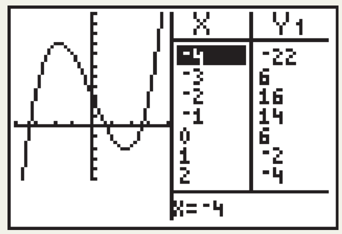
Graphical support for the results of Example 6 is shown in Figure 3.9 using the window $x \in [-5, 5]$ and $y \in [-10, 20]$. The zero of P between 0 and 1 is highlighted, and the zero between $x = -4$ and $x = -3$ is clearly seen. Note there is also a third zero between 2 and 3.

Figure 3.9



The TI 84 Plus (and other models) offer a useful feature called *split screen viewing,* that enables us to view a table of values and the graph of a function at the same time. To illustrate, enter the function $y = x^3 - 9x + 6$ for Y_1 on the `Y=` screen. Press the `ZOOM` **4:ZDecimal** keys to view the graph, then adjust the viewing window as needed to get a comprehensive view. Set up your table in **AUTO** mode with ΔTbl $= 1$ [use `2nd` `WINDOW` **(TBLSET)**]. Use the table of values (`2nd` `GRAPH`) to locate any real zeroes of f [look for where $f(x)$ changes in sign]. To support this concept we can view *both the graph and table at the same time.* Press the `MODE` key and notice the second-to-last entry on this screen reads: **Full** (for full screen viewing), **Horiz** for splitting the screen horizontally with the graph above a reduced home screen, and **G-T,** which represents **Graph-Table** and splits the screen vertically. In the **G-T** mode, the graph appears on the left and the table of values on the right. Navigate the cursor to the **G-T** mode and press `ENTER`. Pressing the `GRAPH` key at this point should give you a screen similar to Figure 3.10. Use this feature to complete the following exercises.

Figure 3.10



Exercise 1: What do the graph, table, and the IVT tell you about the zeroes of this function?

Exercise 2: Go to TBLSET and reset TblStart $= -4$ and ΔTbl $= 0.1$. Use `2nd` `GRAPH` to walk through the table values. Does this give you a better idea about where the zeroes are located?

Exercise 3: Press the `TRACE` key. What happens to the table as you trace through the points on Y_1?

3.3 EXERCISES

▶ CONCEPTS AND VOCABULARY

Fill in each blank with the appropriate word or phrase. Carefully reread the section if needed.

1. A complex polynomial is one where one or more ________ are complex numbers.

2. A polynomial function of degree n will have exactly ________ zeroes, real or ________, where zeroes of multiplicity m are counted m times.

3. If $a + bi$ is a complex zero of polynomial P with real coefficients, then ________ is also a zero.

4. According to Descartes' rule of signs, there are as many ________ real roots as changes in sign from term to term, or an ________ number less.

5. Which of the following values is *not* a possible root of $f(x) = 6x^3 - 2x^2 + 5x - 12$:

a. $x = \frac{4}{3}$ **b.** $x = \frac{3}{4}$ **c.** $x = \frac{1}{2}$

Discuss/Explain why.

6. Discuss/Explain each of the following: (a) irreducible quadratic factors, (b) factors that are complex conjugates, (c) zeroes of multiplicity m, and (d) upper bounds on the zeroes of a polynomial.

DEVELOPING YOUR SKILLS

Rewrite each polynomial as a product of linear factors, and find the zeroes of the polynomial.

7. $P(x) = x^4 + 5x^2 - 36$
8. $Q(x) = x^4 + 21x^2 - 100$
9. $Q(x) = x^4 - 16$
10. $P(x) = x^4 - 81$
11. $P(x) = x^3 + x^2 - x - 1$
12. $Q(x) = x^3 - 3x^2 - 9x + 27$
13. $Q(x) = x^3 - 5x^2 - 25x + 125$
14. $P(x) = x^3 + 4x^2 - 16x - 64$

Factor each polynomial completely. Write any repeated factors in exponential form, then name all zeroes and their multiplicity.

15. $p(x) = (x^2 - 10x + 25)(x^2 + 4x - 45)(x + 9)$
16. $q(x) = (x^2 + 12x + 36)(x^2 + 2x - 24)(x - 4)$
17. $P(x) = (x^2 - 5x - 14)(x^2 - 49)(x + 2)$
18. $Q(x) = (x^2 - 9x + 18)(x^2 - 36)(x - 3)$

Find a polynomial $P(x)$ having real coefficients, with the degree and zeroes indicated. Assume the lead coefficient is 1. Recall $(a + bi)(a - bi) = a^2 + b^2$.

19. degree 3, $x = 3$, $x = 2i$
20. degree 3, $x = -5$, $x = -3i$
21. degree 4, $x = -1$, $x = 2$, $x = i$
22. degree 4, $x = -1$, $x = 3$, $x = -2i$
23. degree 4, $x = 3$, $x = 2i$
24. degree 4, $x = -2$, $x = -3i$
25. degree 4, $x = -1$, $x = 1 + 2i$
26. degree 4, $x = -1$, $x = 1 - 3i$
27. degree 4, $x = -3$, $x = 1 + i\sqrt{2}$
28. degree 4, $x = -2$, $x = 1 + i\sqrt{3}$

Use the intermediate value theorem to verify the given polynomial has at least one zero "c_i" in the intervals specified. Do not find the zeroes.

29. $f(x) = x^3 + 2x^2 - 8x - 5$
    a. $[-4, -3]$ **b.** $[2, 3]$
30. $g(x) = x^4 - 2x^2 + 6x - 3$
    a. $[-3, -2]$ **b.** $[0, 1]$
31. $h(x) = 2x^3 + 13x^2 + 3x - 36$
    a. $[1, 2]$ **b.** $[-3, -2]$
32. $H(x) = 2x^4 + 3x^3 - 14x^2 - 9x + 8$
    a. $[-4, -3]$ **b.** $[-2, -1]$

List all possible rational zeroes for the polynomials given, but do not solve.

33. $f(x) = 4x^3 - 19x - 15$
34. $g(x) = 3x^3 - 2x + 20$
35. $h(x) = 2x^3 - 5x^2 - 28x + 15$
36. $H(x) = 2x^3 - 19x^2 + 37x - 14$
37. $p(x) = 6x^4 - 2x^3 + 5x^2 - 28$
38. $q(x) = 7x^4 + 6x^3 - 49x^2 + 36$
39. $Y_1 = 32t^3 - 52t^2 + 17t + 3$
40. $Y_2 = 24t^3 + 17t^2 - 13t - 6$

Use the rational zeroes theorem to write each function in factored form and find all zeroes. Note $a = 1$.

41. $f(x) = x^3 - 13x + 12$
42. $g(x) = x^3 - 21x + 20$
43. $h(x) = x^3 - 19x - 30$
44. $H(x) = x^3 - 28x - 48$
45. $p(x) = x^3 - 2x^2 - 11x + 12$
46. $q(x) = x^3 - 4x^2 - 7x + 10$
47. $Y_1 = x^3 - 6x^2 - x + 30$
48. $Y_2 = x^3 - 4x^2 - 20x + 48$
49. $Y_3 = x^4 - 15x^2 + 10x + 24$
50. $Y_4 = x^4 - 23x^2 - 18x + 40$
51. $f(x) = x^4 + 7x^3 - 7x^2 - 55x - 42$
52. $g(x) = x^4 + 4x^3 - 17x^2 - 24x + 36$

Find all rational zeroes of the functions given and use them to write the function in factored form. Use the factored form to state *all* zeroes of *f*. Begin by applying the tests for 1 and −1.

53. $f(x) = 4x^3 - 7x + 3$
54. $g(x) = 9x^3 - 7x - 2$
55. $h(x) = 4x^3 + 8x^2 - 3x - 9$
56. $H(x) = 9x^3 + 3x^2 - 8x - 4$

57. $Y_1 = 2x^3 - 3x^2 - 9x + 10$

58. $Y_2 = 3x^3 - 14x^2 + 17x - 6$

59. $p(x) = 2x^4 + 3x^3 - 9x^2 - 15x - 5$

60. $q(x) = 3x^4 + x^3 - 11x^2 - 3x + 6$

61. $r(x) = 3x^4 - 5x^3 + 14x^2 - 20x + 8$

62. $s(x) = 2x^4 - x^3 + 17x^2 - 9x - 9$

Find the zeroes of the polynomials given using any combination of the rational zeroes theorem, testing for 1 and −1, and/or the remainder and factor theorems.

63. $f(x) = 2x^4 - 9x^3 + 4x^2 + 21x - 18$

64. $g(x) = 3x^4 + 4x^3 - 21x^2 - 10x + 24$

65. $h(x) = 3x^4 + 2x^3 - 9x^2 + 4$

66. $H(x) = 7x^4 + 6x^3 - 49x^2 + 36$

67. $p(x) = 2x^4 + 3x^3 - 24x^2 - 68x - 48$

68. $q(x) = 3x^4 - 19x^3 + 6x^2 + 96x - 32$

69. $r(x) = 3x^4 - 20x^3 + 34x^2 + 12x - 45$

70. $s(x) = 4x^4 - 15x^3 + 9x^2 + 16x - 12$

71. $Y_1 = x^5 + 6x^2 - 49x + 42$

72. $Y_2 = x^5 + 2x^2 - 9x + 6$

73. $P(x) = 3x^5 + x^4 + x^3 + 7x^2 - 24x + 12$

74. $P(x) = 2x^5 - x^4 - 3x^3 + 4x^2 - 14x + 12$

75. $Y_1 = x^4 - 5x^3 + 20x - 16$

76. $Y_2 = x^4 - 10x^3 + 90x - 81$

77. $r(x) = x^4 + 2x^3 - 5x^2 - 4x + 6$

78. $s(x) = x^4 + x^3 - 5x^2 - 3x + 6$

79. $p(x) = 2x^4 - x^3 + 3x^2 - 3x - 9$

80. $q(x) = 3x^4 + x^3 + 13x^2 + 5x - 10$

81. $f(x) = 2x^5 - 7x^4 + 13x^3 - 23x^2 + 21x - 6$

82. $g(x) = 4x^5 + 3x^4 + 3x^3 + 11x^2 - 27x + 6$

Gather information on each polynomial using (a) the rational zeroes theorem, (b) testing for 1 and −1, (c) applying Descartes' rule of signs, and (d) using the upper and lower bounds property. Respond explicitly to each.

83. $f(x) = x^4 - 2x^3 + 4x - 8$

84. $g(x) = x^4 + 3x^3 - 7x - 6$

85. $h(x) = x^5 + x^4 - 3x^3 + 5x + 2$

86. $H(x) = x^5 + x^4 - 2x^3 + 4x - 4$

87. $p(x) = x^5 - 3x^4 + 3x^3 - 9x^2 - 4x + 12$

88. $q(x) = x^5 - 2x^4 - 8x^3 + 16x^2 + 7x - 14$

89. $r(x) = 2x^4 + 7x^2 + 11x - 20$

90. $s(x) = 3x^4 - 8x^3 - 13x - 24$

Use Descartes' rule of signs to determine the possible combinations of real and complex zeroes for each polynomial. Then graph the function on the standard window of a graphing calculator and adjust it as needed until you're certain all real zeroes are in clear view. Use this screen and a list of the possible rational zeroes to factor the polynomial and find all zeroes (real and complex).

91. $f(x) = 4x^3 - 16x^2 - 9x + 36$

92. $g(x) = 6x^3 - 41x^2 + 26x + 24$

93. $h(x) = 6x^3 - 73x^2 + 10x + 24$

94. $H(x) = 4x^3 + 60x^2 + 53x - 42$

95. $p(x) = 4x^4 + 40x^3 - 97x^2 - 10x + 24$

96. $q(x) = 4x^4 - 42x^3 - 70x^2 - 21x - 36$

▶ WORKING WITH FORMULAS

97. The absolute value of a complex number $z = a + bi$: $|z| = \sqrt{a^2 + b^2}$

The absolute value of a complex number z, denoted $|z|$, represents the distance between the origin and the point (a, b) in the complex plane. Use the formula to find $|z|$ for the complex numbers given (also see Section 1.4, Exercise 69): (a) $3 + 4i$, (b) $-5 + 12i$, and (c) $1 + \sqrt{3}\,i$.

98. The square root of $z = a + bi$:

$$\sqrt{z} = \frac{\sqrt{2}}{2}\left(\sqrt{|z| + a} \pm i\sqrt{|z| - a}\right)$$

The square roots of a complex number are given by the relations shown, where $|z|$ represents the absolute value of z and the sign is chosen to match the sign of b. Use the formula to find the square root of each complex number from Exercise 97, then check your answer by squaring the result (also see Section 1.4, Exercise 82).

► APPLICATIONS

99. Maximum and minimum values: To locate the maximum and minimum values of $F(x) = x^4 - 4x^3 - 12x^2 + 32x + 15$ requires finding the zeroes of $f(x) = 4x^3 - 12x^2 - 24x + 32$. Use the rational zeroes theorem and synthetic division to find the zeroes of f, then graph $F(x)$ on a calculator and see if the graph tends to support your calculations—do the maximum and minimum values occur at the zeroes of f?

100. Graphical analysis: Use the rational zeroes theorem and synthetic division to find the zeroes of $F(x) = x^4 - 4x^3 - 12x^2 + 32x + 15$ (see Exercise 99).

101. Maximum and minimum values: To locate the maximum and minimum values of $G(x) = x^4 - 6x^3 + x^2 + 24x - 20$ requires finding the zeroes of $g(x) = 4x^3 - 18x^2 + 2x + 24$. Use the rational zeroes theorem and synthetic division to find the zeroes of g, then graph $G(x)$ on a calculator and see if the graph tends to support your calculations—do the maximum and minimum values occur at the zeroes of g?

102. Graphical analysis: Use the rational zeroes theorem and synthetic division to find the zeroes of $G(x) = x^4 - 6x^3 + x^2 + 24x - 20$ (see Exercise 101).

Geometry: The volume of a cube is $V = x \cdot x \cdot x = x^3$, where x represents the length of the edges. If a slice 1 unit thick is removed from the cube, the remaining volume is $v = x \cdot x \cdot (x - 1) = x^3 - x^2$. Use this information for Exercises 103 and 104.

103. A slice 1 unit in thickness is removed from one side of a cube. Use the rational zeroes theorem and synthetic division to find the original dimensions of the cube, if the remaining volume is (a) 48 cm^3 and (b) 100 cm^3.

104. A slice 1 unit in thickness is removed from one side of a cube, then a second slice of the same thickness is removed from a different side (not the opposite side). Use the rational zeroes theorem and synthetic division to find the original dimensions of the cube, if the remaining volume is (a) 36 cm^3 and (b) 80 cm^3.

Geometry: The volume of a rectangular box is $V = LWH$. For the box to satisfy certain requirements, its length must be twice the width, and its height must be two inches less than the width. Use this information for Exercises 105 and 106.

105. Use the rational zeroes theorem and synthetic division to find the dimensions of the box if it must have a volume of 150 in^3.

106. Suppose the box must have a volume of 64 in^3. Use the rational zeroes theorem and synthetic division to find the dimensions required.

Government deficits: Over a 14-yr period, the balance of payments (deficit versus surplus) for a certain county government was modeled by the function $f(x) = \frac{1}{4}x^4 - 6x^3 + 42x^2 - 72x - 64$, where $x = 0$ corresponds to 1990 and $f(x)$ is the deficit or surplus in tens of thousands of dollars. Use this information for Exercises 107 and 108.

107. Use the rational zeroes theorem and synthetic division to find the years when the county "broke even" (debt = surplus = 0) from 1990 to 2004. How many years did the county run a surplus during this period?

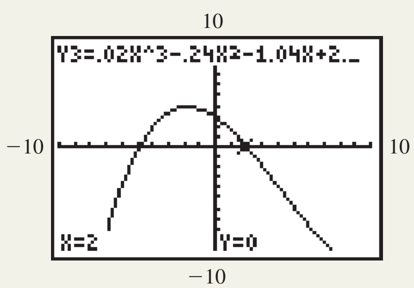
108. The deficit was at the $84,000 level $[f(x) = -84]$, four times from 1990 to 2004. Given this occurred in 1992 and 2000 ($x = 2$ and $x = 10$), use the rational zeroes theorem, synthetic division, and the remainder theorem to find the other two years the deficit was at $84,000.

109. Drag resistance on a boat: In a scientific study on the effects of drag against the hull of a sculling boat, some of the factors to consider are displacement, draft, speed, hull shape, and length, among others. If the first four are held constant and we assume a flat, calm water surface, length becomes the sole variable (as length changes, we adjust the beam by a uniform scaling to keep a constant displacement). For a fixed sculling speed of 5.5 knots, the relationship between drag and length can be modeled by $f(x) = -0.4192x^4 + 18.9663x^3 - 319.9714x^2 + 2384.2x - 6615.8$, where $f(x)$ is the efficiency rating of a boat with length x $(8.7 < x < 13.6)$. Here, $f(x) = 0$ represents an *average* efficiency rating. (a) Under these conditions, what lengths (to the nearest hundredth) will give the boat an average rating? (b) What length will maximize the efficiency of the boat? What is this rating?

110. Comparing densities: Why is it that when you throw a rock into a lake, it sinks, while a wooden ball will float half submerged, but the bobber on your fishing line floats on the surface? It all depends on the density of the object compared to the density of water ($d = 1$). For uniformity, we'll consider spherical objects of various densities, each with a radius of 5 cm. When placed into water, the depth that the sphere will sink beneath the surface (while still floating) is modeled by the polynomial $p(x) = \frac{\pi}{3}x^3 - 5\pi x^2 + \frac{500\pi}{3}d$, where d is the density of the object and the smallest positive zero of p is the depth of the sphere below the surface (in centimeters). How far submerged is the sphere if it's made of (a) balsa wood, $d = 0.17$; (b) pine wood, $d = 0.55$; (c) ebony wood, $d = 1.12$; (d) a large bobber made of lightweight plastic, $d = 0.05$?

▶ EXTENDING THE CONCEPT

111. In the figure, $P(x) = 0.02x^3 - 0.24x^2 - 1.04x + 2.68$ is graphed on the standard screen ($-10 \le x \le 10$), which shows two real zeroes. Since P has degree 3, there must be one more real zero but is it negative or positive? Use the upper/lower bounds property (a) to see if -10 is a lower bound and (b) to see if 10 is an upper bound. (c) Then use your calculator to find the remaining zero.



112. From Example 11, (a) what is the significance of the y-intercept? (b) If the domain were extended to include $0 < x \le 13$, what happens when x is approximately 12.8?

113A. It is often said that while the difference of two squares is factorable, $a^2 - b^2 = (a + b)(a - b)$, the sum of two squares is prime. To be 100% correct, we should say the sum of two squares cannot be factored *using real numbers*. If complex numbers are used, $(a^2 + b^2) = (a + bi)(a - bi)$. Use this idea to factor the following binomials.

a. $p(x) = x^2 + 25$ **b.** $q(x) = x^2 + 9$

c. $r(x) = x^2 + 7$

113B. It is often said that while $x^2 - 16$ is factorable as a difference of squares, $a^2 - b^2 = (a + b)(a - b)$, $x^2 - 17$ is not. To be 100% correct, we should say that $x^2 - 17$ is not factorable *using integers*. Since $(\sqrt{17})^2 = 17$, it can actually be factored in the same way: $x^2 - 17 = (x + \sqrt{17})(x - \sqrt{17})$. Use this idea to solve the following equations.

a. $x^2 - 7 = 0$ **b.** $x^2 - 12 = 0$ **c.** $x^2 - 18 = 0$

114. Every general cubic equation $aw^3 + bw^2 + cw + d = 0$ can be written in the form $x^3 + px + q = 0$ (where the squared term has been "depressed"), using the transformation $w = x - \frac{b}{3}$. Use this transformation to solve the following equations.

a. $w^3 - 3w^2 + 6w - 4 = 0$

b. $w^3 - 6w^2 + 21w - 26 = 0$

Note: It is actually very rare that the transformation produces a value of $q = 0$ for the "depressed" cubic $x^3 + px + q = 0$, and general solutions must be found using what has become known as *Cardano's formula.* For a complete treatment of cubic equations and their solutions, visit our website at www.mhhe.com/coburn. Here we'll focus on the primary root of selected cubics.

115. For each of the following complex polynomials, one of its zeroes is given. Use this zero to help write the polynomial in completely factored form. (*Hint:* Synthetic division and the quadratic formula can be applied to *all polynomials,* even those with complex coefficients.)

a. $C(z) = z^3 + (1 - 4i)z^2 + (-6 - 4i)z + 24i;$
$z = 4i$

b. $C(z) = z^3 + (5 - 9i)z^2 + (4 - 45i)z - 36i;$
$z = 9i$

c. $C(z) = z^3 + (-2 - 3i)z^2 + (5 + 6i)z - 15i;$
$z = 3i$

d. $C(z) = z^3 + (-4 - i)z^2 + (29 + 4i)z - 29i;$
$z = i$

e. $C(z) = z^3 + (-2 - 6i)z^2 + (4 + 12i)z - 24i;$
$z = 6i$

f. $C(z) = z^3 + (-6 + 4i)z^2 + (11 - 24i)z + 44i;$
$z = -4i$

g. $C(z) = z^3 + (-2 - i)z^2 + (5 + 4i)z + (-6 + 3i);$
$z = 2 - i$

h. $C(z) = z^3 - 2z^2 + (19 + 6i)z + (-20 + 30i);$
$z = 2 - 3i$

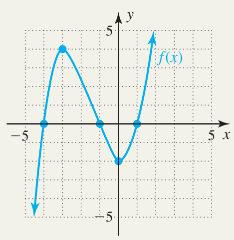
▶ MAINTAINING YOUR SKILLS

116. **(2.6)** Graph the piecewise-defined function and find the value of $f(-3)$, $f(2)$, and $f(5)$.

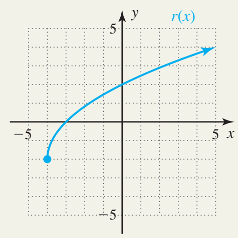
$$f(x) = \begin{cases} 2 & x \leq -1 \\ |x - 1| & -1 < x < 5 \\ 4 & x \geq 5 \end{cases}$$

117. **(3.1)** For a county fair, officials need to fence off a large rectangular area, then subdivide it into three equal (rectangular) areas. If the county provides 1200 ft of fencing, (a) what dimensions will maximize the area of the larger (outer) rectangle? (b) What is the area of each smaller rectangle?

118. **(2.7)** Use the graph given to (a) state intervals where $f(x) \geq 0$, (b) locate local maximum and minimum values, and (c) state intervals where $f(x)\uparrow$ and $f(x)\downarrow$.



119. **(2.5)** Write the equation of the function shown.



3.4 Graphing Polynomial Functions

Learning Objectives

In Section 3.4 you will learn how to:

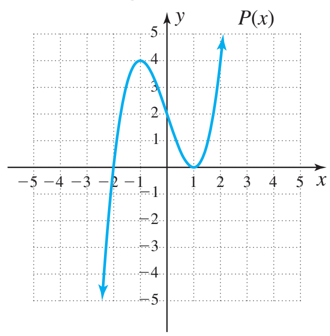
- ❑ **A.** Identify the graph of a polynomial function and determine its degree
- ❑ **B.** Describe the end behavior of a polynomial graph
- ❑ **C.** Discuss the attributes of a polynomial graph with zeroes of multiplicity
- ❑ **D.** Graph polynomial functions in standard form
- ❑ **E.** Solve applications of polynomials

As with linear and quadratic functions, understanding graphs of *polynomial* functions will help us apply them more effectively as mathematical models. Since all real polynomials can be written in terms of their linear and quadratic factors (Section 3.3), these functions provide the basis for our continuing study.

A. Identifying the Graph of a Polynomial Function

Consider the graphs of $f(x) = x + 2$ and $g(x) = (x - 1)^2$, which we know are smooth, continuous curves. The graph of f is a straight line with positive slope, that crosses the x-axis at -2. The graph of g is a parabola, opening upward, shifted 1 unit to the right, and touching the x-axis at $x = 1$. When f and g are "combined" into the single function $P(x) = (x + 2)(x - 1)^2$, the behavior of the graph at these zeroes is still evident. In Figure 3.11, the graph of P crosses the x-axis at $x = -2$, "bounces" off the x-axis at $x = 1$, and is still a smooth, continuous curve. This observation could be

Figure 3.11



> **WORTHY OF NOTE**
>
> While defined more precisely in a future course, we will take "smooth" to mean the graph has no sharp turns or jagged edges, and "continuous" to mean the entire graph can be drawn without lifting your pencil.

extended to include additional linear or quadratic factors, and helps affirm that the graph of a polynomial function is a *smooth, continuous curve*.

Further, after the graph of P crosses the axis at $x = -2$, it must "turn around" at some point to reach the zero at $x = 1$, then turn again as it touches the x-axis without crossing. By combining this observation with our work in Section 3.3, we can state the following:

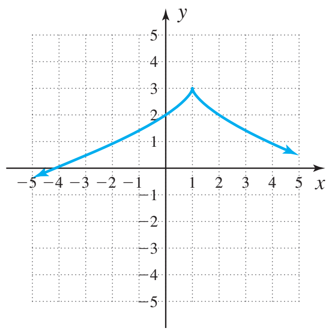
Polynomial Graphs and Turning Points

1. If $P(x)$ is a polynomial function of degree n, then the graph of P has at most $n - 1$ turning points.
2. If the graph of a function P has $n - 1$ turning points, then the degree of $P(x)$ is at least n.

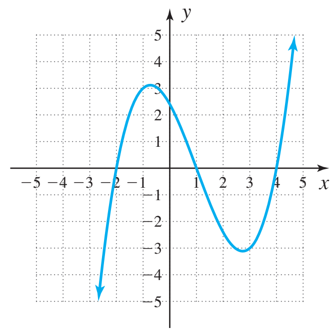
EXAMPLE 1 ▶ Identifying Polynomial Graphs

Determine whether each graph could be the graph of a polynomial. If not, discuss why. If so, use the number of turning points and zeroes to identify the least possible degree of the function.

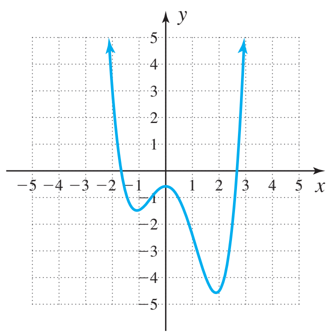
a.



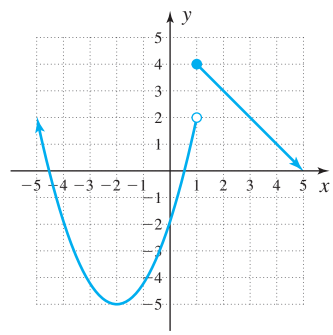
b.



c.



d.



Solution ▶

a. This is not a polynomial graph, as it has a sharp turn (called a **cusp**) at (1, 3). A polynomial graph is always smooth.

b. This graph is smooth and continuous, and could be that of a polynomial. With two turning points and three zeroes, the function is at least degree 3.

c. This graph is smooth and continuous, and could be that of a polynomial. With three turning points and two zeroes, the function is at least degree 4.

d. This is not a polynomial graph, as it has a break (discontinuity) at $x = 1$. A polynomial graph is always continuous.

A. You've just learned how to identify the graph of a polynomial function and determine its degree

Now try Exercises 7 through 12 ▶

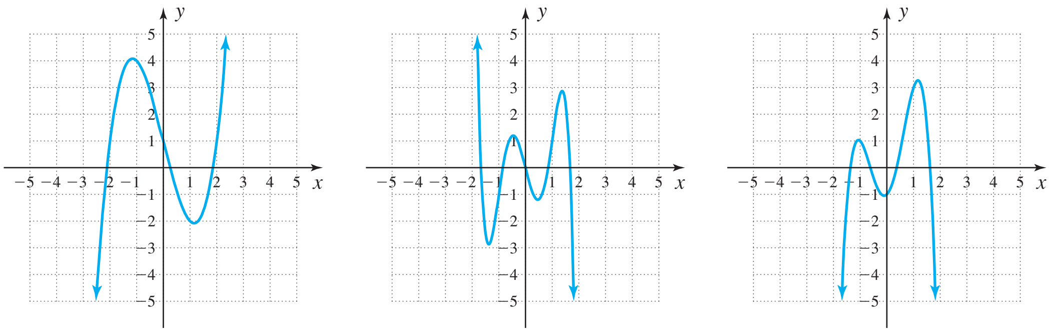
B. The End Behavior of a Polynomial Graph

Once the graph of a function has "made its last turn" and crossed or touched its last real zero, it will continue to increase or decrease without bound as $|x|$ becomes large. As before, we refer to this as the **end behavior** of the graph. In previous sections we noted that quadratic functions (degree 2) with a positive leading coefficient ($a > 0$), had the end behavior "up on the left" and "up on the right (up/up)." If the leading coefficient was negative ($a < 0$), end behavior was "down on the left" and "down on the right (down/down)." These descriptions were also applied to the graph of a linear function $y = mx + b$ (degree 1). A positive leading coefficient ($m > 0$) indicates the graph will be down on the left, up on the right (down/up), and so on. All polynomial graphs exhibit some form of end behavior, which can be likewise described.

EXAMPLE 2 ▶ Identifying the End Behavior of a Graph

State the end behavior of each graph shown:

a. $f(x) = x^3 - 4x + 1$ **b.** $g(x) = -2x^5 + 7x^3 - 4x$ **c.** $h(x) = -2x^4 + 5x^2 + x - 1$



Solution ▶

a. down on the left, up on the right

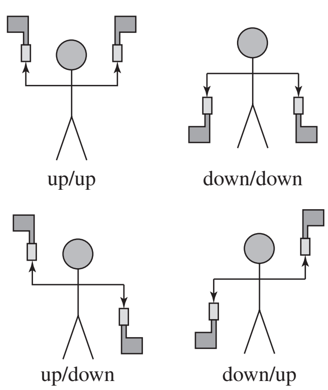
b. up on the left, down on the right

c. down on the left, down on the right

Now try Exercises 13 through 16 ▶

WORTHY OF NOTE

As a visual aid to end behavior, it might help to picture a signalman using semaphore code as illustrated here. As you view the end behavior of a polynomial graph, there is a striking resemblance.

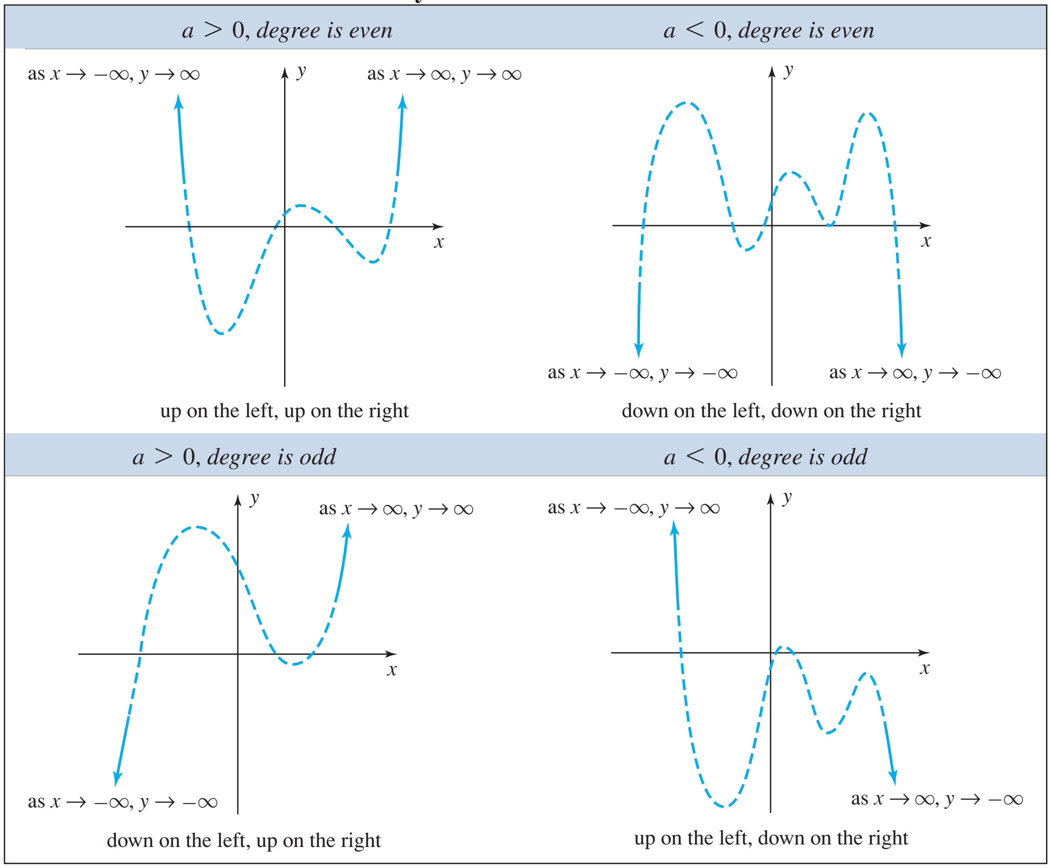


The leading term ax^n of a polynomial function is said to be the **dominant term,** because for large values of $|x|$, the value of ax^n is much larger than all other terms combined. This means that like linear and quadratic graphs, polynomial end behavior can be predicted in advance by analyzing this term alone.

1. For ax^n when n is even, any nonzero number raised to an even power is positive, so the ends of the graph must point in the same direction. If $a > 0$, both point upward. If $a < 0$, both point downward.
2. For ax^n when n is odd, any number raised to an odd power has the same sign as the input value, so the ends of the graph must point in opposite directions. If $a > 0$, end behavior is down on the left, up on the right. If $a < 0$, end behavior is up on the left, down on the right.

From this we find that end behavior depends on two things: *the degree of the function* (even or odd) and the *sign of the leading coefficient* (positive or negative). In more formal terms, this is described in terms of how the graph "behaves" for large values of x. For end behavior that is "up on the right," we mean that as x becomes a large positive number, y becomes a large positive number. This is indicated using the notation: as $x \to \infty$, $y \to \infty$. Similar notation is used for the other possibilities. These facts are summarized in Table 3.1. The interior portion of each graph is dashed since the actual number of turning points may vary, although a polynomial of odd degree will have an even number of turning points, and a polynomial of even degree will have an odd number of turning points.

Table 3.1
Polynomial End Behavior



Note the end behavior of $y = mx$ can be used as a representative of all odd degree functions, and the end behavior of $y = ax^2$ as a representative of all even degree functions.

The End Behavior of a Polynomial Graph

Given a polynomial $P(x)$ with leading term ax^n and $n \geq 1$.
If n is **even,** ends will point in the **same direction,**

1. for $a > 0$: up on the left, up on the right (*as with* $y = x^2$);
   as $x \to -\infty, y \to \infty$; as $x \to \infty, y \to \infty$
2. for $a < 0$: down on the left, down on the right (*as with* $y = -x^2$);
   as $x \to -\infty, y \to -\infty$; as $x \to \infty, y \to -\infty$

If n is **odd,** the ends will point in **opposite directions,**

1. for $a > 0$: down on the left, up on the right (*as with* $y = x$);
   as $x \to -\infty, y \to -\infty$; as $x \to \infty, y \to \infty$
2. for $a < 0$: up on the left, down on the right (*as with* $y = -x$);
   as $x \to -\infty, y \to \infty$; as $x \to \infty, y \to -\infty$

EXAMPLE 3 ▶ Identifying the End Behavior of a Function

State the end behavior of each function, without actually graphing.

a. $f(x) = 0.5x^4 + 3x^3 - 5x + 6$ **b.** $g(x) = -2x^5 - 5x^3 - 3$

Solution ▶ **a.** The function has degree 4 (even), and the ends will point in the same direction. The leading coefficient is positive, so end behavior is up/up.

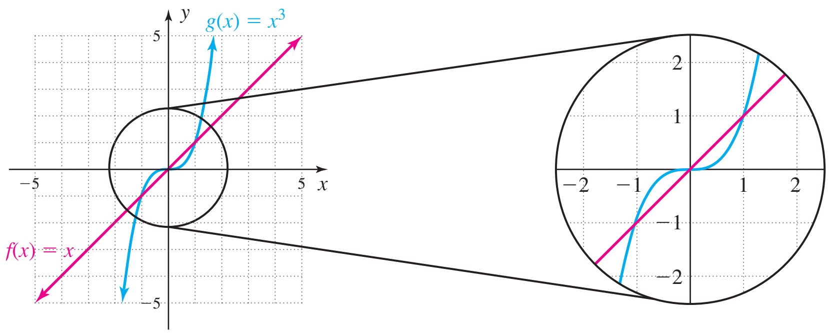
b. The function has degree 5 (odd), and the ends will point in opposite directions. The leading coefficient is negative, so the end behavior is up/down.

Now try Exercises 17 through 22 ▶

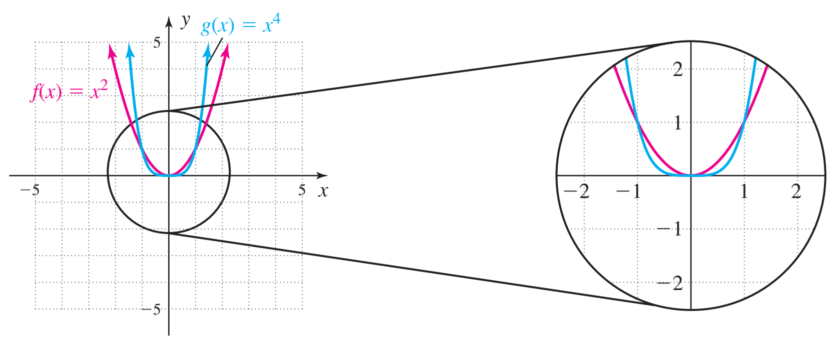
B. You've just learned how to describe the end behavior of a polynomial graph

C. Attributes of Polynomial Graphs with Zeroes of Multiplicity

Another important aspect of polynomial functions is the behavior of a graph near its zeroes. In the simplest case, consider the functions $f(x) = x$ and $g(x) = x^3$. Both have odd degree, like end behavior (down/up), and a zero at $x = 0$. But the zero of f has multiplicity 1, while the zero from g has multiplicity 3. Notice the graph of g is vertically compressed near $x = 0$ and seems to approach this zero "more gradually."



This behavior can be explained by noting that for $x = -1$ and 1, $f(x) = g(x)$. But for $|x| < 1$, the graph of g *will be closer to the x-axis* since the cube of a fractional number is smaller than the fraction itself. We further note that for $|x| > 1$, g increases much faster than f, and $|g(x)| > |f(x)|$. Similar observations can be made regarding $f(x) = x^2$ and $g(x) = x^4$. Both functions have even degree, a zero at $x = 0$, and $f(x) = g(x)$ for $x = -1$ and 1. But for $|x| < 1$, the function with higher degree is once again closer to the x-axis.



These observations can be generalized and applied to all real zeroes of a function.

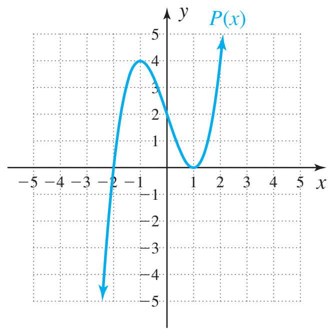
Polynomial Graphs and Zeroes of Multiplicity

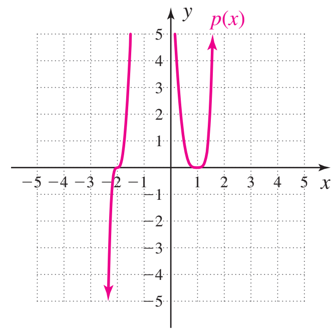
Given $P(x)$ is a polynomial with factors of the form $(x - c)^m$, with c a real number,

- If m is odd, the graph will cross through the x-axis.
- If m is even, the graph will bounce off the *x-axis* (touching at just one point).

In each case, the graph will be more compressed (flatter) near c for larger values of m.

To illustrate, compare the graph of $P(x) = (x + 2)(x - 1)^2$ from page 320, with the graph of $p(x) = (x + 2)^3(x - 1)^4$ shown, noting the increased multiplicity of each zero.



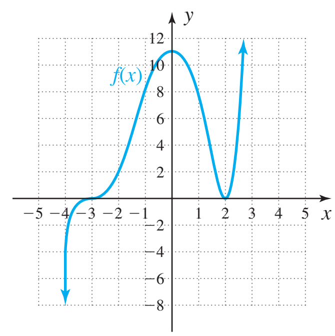


Both graphs show the expected zeroes at $x = -2$ and $x = 1$, but the graph of $p(x)$ is flatter near $x = -2$ and $x = 1$, due to the increased multiplicity of each zero. We lose sight of the graph of $p(x)$ between $x = -2$ and $x = 0$, since the increased multiplicities produce larger values than the original grid could display.

EXAMPLE 4 ▶ Naming Attributes of a Function from Its Graph

The graph of a polynomial $f(x)$ is shown.

a. State whether the degree of f is even or odd.
b. Use the graph to name the zeroes of f, then state whether their multiplicity is even or odd.
c. State the minimum possible degree of f.
d. State the domain and range of f.



Solution ▶

a. Since the ends of the graph point in opposite directions, the degree of the function must be odd.
b. The graph crosses the x-axis at $x = -3$ and is compressed near -3, meaning it must have odd multiplicity with $m > 1$. The graph bounces off the x-axis at $x = 2$ and 2 must be a zero of even multiplicity.
c. The minimum possible degree of f is 5, as in $f(x) = a(x - 2)^2(x + 3)^3$.
d. $x \in \mathbb{R}$, $y \in \mathbb{R}$.

Now try Exercises 23 through 28 ▶

To find the degree of a polynomial from its factored form, add the exponents on all linear factors, then add 2 for each irreducible quadratic factor (the degree of any quadratic factor is 2). The sum gives the degree of the polynomial, from which end behavior can be determined. To find the y-intercept, substitute 0 for x as before, noting this is equivalent to applying the exponent to the constant from each factor.

EXAMPLE 5 ▶ Naming Attributes of a Function from Its Factored Form

State the degree of each function, then describe the end behavior and name the y-intercept of each graph.

a. $f(x) = (x + 2)^3(x - 3)$ **b.** $g(x) = -(x + 1)^2(x^2 + 3)(x - 6)$

Solution ▶

a. The degree of f is $3 + 1 = 4$. With even degree and positive leading coefficient, end behavior is up/up. For $f(0) = (2)^3(-3) = -24$, the y-intercept is $(0, -24)$.
b. The degree of g is $2 + 2 + 1 = 5$. With odd degree and negative leading coefficient, end behavior is up/down. For $g(0) = -1(1)^2(3)(-6) = 18$, the y-intercept is $(0, 18)$.

Now try Exercises 29 through 36 ▶

EXAMPLE 6 ▶ Matching Graphs to Functions Using Zeroes of Multiplicity

The following functions all have zeroes at $x = -2, -1$, and 1. Match each function to the corresponding graph *using its degree and the multiplicity of each zero.*

a. $y = (x + 2)(x + 1)^2(x - 1)^3$ **b.** $y = (x + 2)(x + 1)(x - 1)^3$

c. $y = (x + 2)^2(x + 1)^2(x - 1)^3$ **d.** $y = (x + 2)^2(x + 1)(x - 1)^3$

Solution ▶ The functions in Figures 3.12 and 3.14 must have even degree due to end behavior, so each corresponds to (a) or (d). At $x = -1$ the graph in Figure 3.12 "crosses," while the graph in Figure 3.14 "bounces." This indicates Figure 3.12 matches equation (d), while Figure 3.14 matches equation (a).

The graphs in Figures 3.13 and 3.15 must have odd degree due to end behavior, so each corresponds to (b) or (c). Here, one graph "bounces" at $x = -2$, while the other "crosses." The graph in Figure 3.13 matches equation (c), the graph in Figure 3.15 matches equation (b).

Figure 3.12

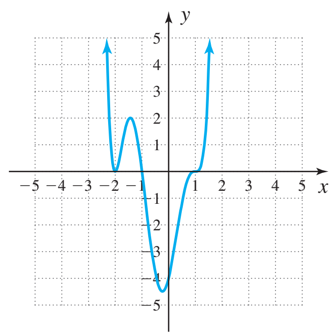


Figure 3.13

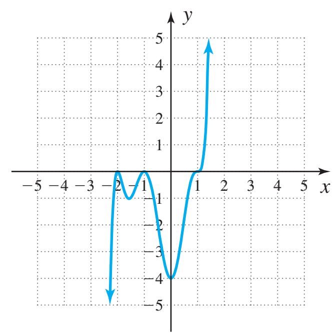


Figure 3.14

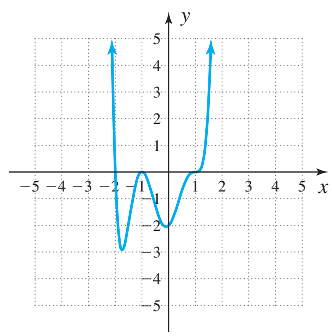
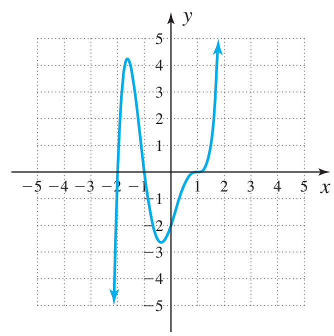


Figure 3.15



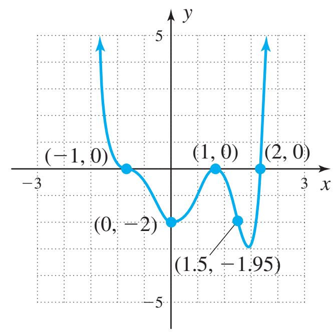
Now try Exercises 37 through 42 ▶

Using the ideas from Examples 5 and 6, we're able to draw a fairly accurate graph given the factored form of a polynomial. Convenient values between two zeroes, called **mid-interval points,** can be used to help complete the graph.

EXAMPLE 7 ▶ Graphing a Function Given the Factored Form

Sketch the graph of $f(x) = (x - 2)(x - 1)^2(x + 1)^3$ using end behavior; the x- and y-intercepts, and zeroes of multiplicity.

Solution ▶ Adding the exponents of each factor, we find that f is a function of degree 6 with a positive lead coefficient, so end behavior will be up/up. Since $f(0) = -2$, the y-intercept is $(0, -2)$. The graph will bounce off the x-axis at $x = 1$ (even multiplicity), and cross the axis at $x = -1$ and 2 (odd multiplicities). The graph will "flatten out" near $x = -1$ because of its higher multiplicity. To help "round-out" the graph we evaluate f at $x = 1.5$, giving $(-0.5)^2(0.5)^3(2.5) \approx -1.95$ (note scaling of the x- and y-axes).



Now try Exercises 43 through 56 ▶

C. You've just learned how to discuss the attributes of a polynomial graph with zeroes of multiplicity

D. The Graph of a Polynomial Function

Using the cumulative observations from this and previous sections, a general strategy emerges for the graphing of polynomial functions.

WORTHY OF NOTE

Although of somewhat limited value, symmetry (item f in the guidelines) can sometimes aid in the graphing of polynomial functions. If all terms of the function have even degree, the graph will be symmetric to the y-axis (even). If all terms have odd degree, the graph will be symmetric about the origin. Recall that a constant term has degree zero, an even number.

Guidelines for Graphing Polynomial Functions

1. Determine the end behavior of the graph.
2. Find the y-intercept $(0, a_0)$
3. Find the zeroes using any combination of the rational zeroes theorem, the factor and remainder theorems, tests for 1 and -1 (p. 310), factoring, and the quadratic formula.
4. Use the y-intercept, end behavior, the multiplicity of each zero, and midinterval points as needed to sketch a smooth, continuous curve.

Additional tools include (a) polynomial zeroes theorem, (b) complex conjugates theorem, (c) number of turning points, (d) Descartes' rule of signs, (e) upper and lower bounds, and (f) symmetry.

EXAMPLE 8 ▶ **Graphing a Polynomial Function**

Sketch the graph of $g(x) = -x^4 + 9x^2 - 4x - 12$.

Solution ▶

1. End behavior: The function has degree 4 (even) with a negative leading coefficient, so end behavior is *down on the left, down on the right.*
2. Since $g(0) = -12$, the y-intercept is $(0, -12)$.
3. Zeroes: Using the test for $x = 1$ gives $-1 + 9 - 4 - 12 = -8$, showing $x = 1$ is not a zero but $(1, -8)$ is a point on the graph. Using the test for $x = -1$ gives $-1 + 9 + 4 - 12 = 0$, so -1 is a zero and $(x + 1)$ is a factor. Using $x = -1$ with the factor theorem yields

-1	-1	0	9	-4	-12
		1	-1	-8	12
	-1	1	8	-12	0

The quotient polynomial is not easily factorable so we continue with synthetic division. Using the rational zeroes theorem, the possible rational zeroes are $\{\pm1, \pm12, \pm2, \pm6, \pm3, \pm4\}$, so we try $x = 2$.

use 2 as a "divisor" on the quotient polynomial

2	-1	1	8	-12
		-2	-2	12
	-1	-1	6	0

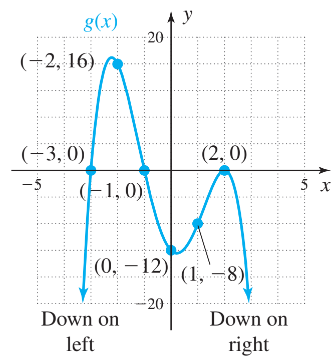
This shows $x = 2$ is a zero, $x - 2$ is a factor, and the function can now be written as

$$g(x) = (x + 1)(x - 2)(-x^2 - x + 6).$$

Factoring -1 from the trinomial gives

$$\begin{aligned} g(x) &= -1(x + 1)(x - 2)(x^2 + x - 6) \\ &= -1(x + 1)(x - 2)(x + 3)(x - 2) \\ &= -1(x + 1)(x - 2)^2(x + 3) \end{aligned}$$

The zeroes of g are $x = -1$ and -3, both with multiplicity 1, and $x = 2$ with multiplicity 2.



4. To help "round-out" the graph we evaluate the midinterval point $x = -2$ using the remainder theorem, which shows that $(-2, 16)$ is also a point on the graph.

use -2 as a "divisor"

-2	-1	0	9	-4	-12
		2	-4	-10	28
	-1	2	5	-1	16

The final result is the graph shown.

Now try Exercises 57 through 72 ►

CAUTION ► Sometimes using a midinterval point to help draw a graph will give the illusion that a maximum or minimum value has been located. This is rarely the case, as demonstrated in the figure in Example 8, where the maximum value in Quadrant II is actually closer to $(-2.22, 16.95)$.

EXAMPLE 9 ► Using the Guidelines to Sketch a Polynomial Graph

Sketch the graph of $h(x) = x^7 - 4x^6 + 7x^5 - 12x^4 + 12x^3$.

Solution ►

1. End behavior: The function has degree 7 (odd) and the ends will point in opposite directions. The leading coefficient is positive and the end behavior will be *down on the left* and *up on the right*.
2. y-intercept: Since $h(0) = 0$, the y-intercept is $(0, 0)$.
3. Zeroes: Testing 1 and -1 shows neither are zeroes but $(1, 4)$ and $(-1, -36)$ are points on the graph. Factoring out x^3 produces $h(x) = x^3(x^4 - 4x^3 + 7x^2 - 12x + 12)$, and we see that $x = 0$ is a zero of multiplicity 3. We next use synthetic division with $x = 2$ on the fourth-degree polynomial:

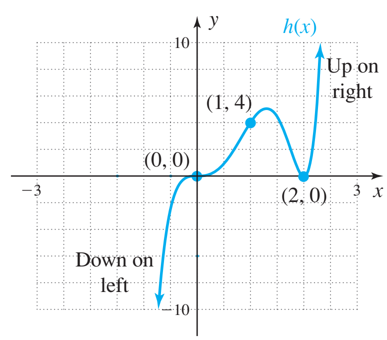
use 2 as a "divisor"

2	1	-4	7	-12	12
		2	-4	6	-12
	1	-2	3	-6	0

This shows $x = 2$ is a zero and $x - 2$ is a factor. At this stage, it appears the quotient can be factored by grouping. From $h(x) = x^3(x - 2)(x^3 - 2x^2 + 3x - 6)$, we obtain $h(x) = x^3(x - 2)(x^2 + 3)(x - 2)$ after factoring and

$$h(x) = x^3(x - 2)^2(x^2 + 3)$$

as the completely factored form. We find that $x = 2$ is a zero of multiplicity 2, and the remaining two zeroes are complex.



4. Using this information produces the graph shown in the figure.

Now try Exercises 73 through 76 ►

D. You've just learned how to graph polynomial functions in standard form

For practice with these ideas using a graphing calculator, **see Exercises 77 through 80.** Similar to our work in previous sections, **Exercises 81 and 82** ask you to reconstruct the complete equation of a polynomial from its given graph.

E. Applications of Polynomials

EXAMPLE 10 ▶ **Modeling the Value of an Investment**

In the year 2000, Marc and his wife Maria decided to invest some money in precious metals. As expected, the value of the investment fluctuated over the years, sometimes being worth more than they paid, other times less. Through 2008, the value of the investment was modeled by $v(t) = t^4 - 11t^3 + 38t^2 - 40t$, where $v(t)$ represents the gain or loss (in hundreds of dollars) in year t ($t = 0 \rightarrow 2000$).

a. Use the rational zeroes theorem to find the years when their gain/loss was zero.

b. Sketch the graph of the function.

c. In what years was the investment worth less than they paid?

d. What was their gain or loss in 2008?

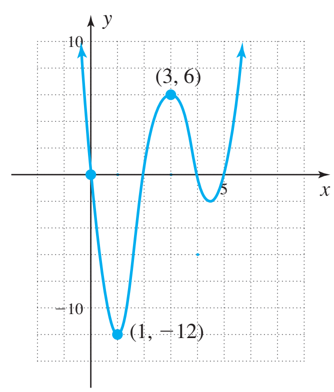
Solution ▶ **a.** Writing the function as $v(t) = t(t^3 - 11t^2 + 38t - 40)$, we note $t = 0$ shows no gain or loss on purchase, and attempt to find the remaining zeroes. Testing for 1 and -1 shows neither is a zero, but $(1, -12)$ and $(-1, 90)$ are points on the graph of v. Next we try $t = 2$ with the factor theorem and the cubic polynomial.

2⌋	1	−11	38	−40
		2	−18	40
	1	−9	2	⌊0

We find that 2 is a zero and write $v(t) = t(t - 2)(t^2 - 9t + 20)$, then factor to obtain $v(t) = t(t - 2)(t - 4)(t - 5)$. Since $v(t) = 0$ for $t = 0$, 2, 4, and 5, they "broke even" in years 2000, 2002, 2004, and 2005.

b. With even degree and a positive leading coefficient, the end behavior is up/up. All zeroes have multiplicity 1. As an additional midinterval point we find $v(3) = 6$:

3⌋	1	−11	38	−40	0
		3	−24	42	6
	1	−8	14	2	⌊6



The complete graph is shown.

c. The investment was worth less than what they paid (outputs are negative) from 2000 to 2002 and 2004 to 2005.

d. In 2008, they were "sitting pretty," as their investment had gained $576.

8⌋	1	−11	38	−40	0
		8	−24	112	576
	1	−3	14	72	⌊576

WORTHY OF NOTE

Due to the context, the domain of $v(t)$ in Example 10 actually begins at $x = 0$, which we could designate with a point at (0, 0). In addition, note there are three sign changes in the terms of $v(t)$, indicating there will be 3 or 1 positive roots (we found 3).

E. You've just learned how to solve an application of polynomials

Now try Exercises 85 through 88 ▶

3.4 EXERCISES

▶ CONCEPTS AND VOCABULARY

Fill in each blank with the appropriate word or phrase. Carefully reread the section if needed.

1. For a polynomial with factors of the form $(x - c)^m$, c is called a ________ of multiplicity ________.

2. A polynomial function of degree n has ________ zeroes and at most ________ "turning points."

3. The graphs of $Y_1 = (x - 2)^2$ and $Y_2 = (x - 2)^4$ both ________ at $x = 2$, but the graph of Y_2 is ________ than the graph of Y_1 at this point.

4. Since $x^4 > 0$ for all x, the ends of its graph will always point in the ________ direction. Since $x^3 > 0$ when $x > 0$ and $x^3 < 0$ when $x < 0$, the ends of its graph will always point in the ________ direction.

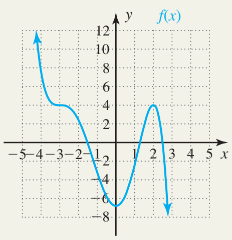
5. In your own words, explain/discuss how to find the degree and y-intercept of a function that is given in factored form. Use $f(x) = (x + 1)^3(x - 2)(x + 4)^2$ to illustrate.

6. Name all of the "tools" at your disposal that play a role in the graphing of polynomial functions. Which tools are indispensable and always used? Which tools are used only as the situation merits?

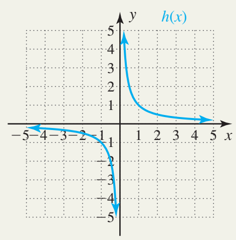
▶ DEVELOPING YOUR SKILLS

Determine whether each graph is the graph of a polynomial function. If yes, state the least possible degree of the function. If no, state why.

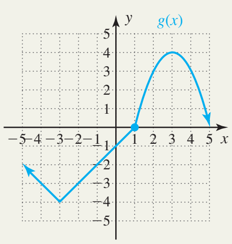
7.



8.



9.



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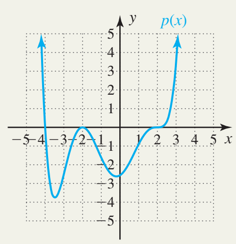

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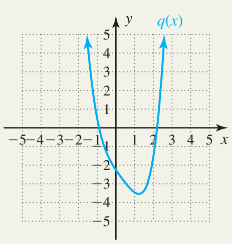

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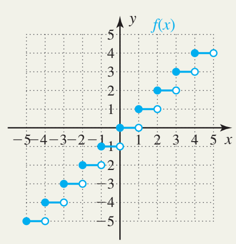
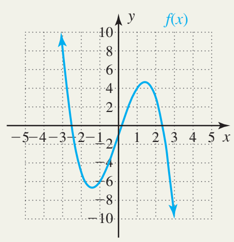
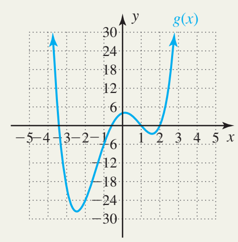


State the end behavior of the functions given.

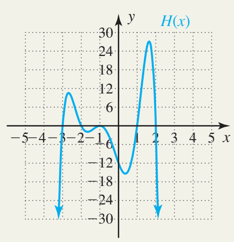
13. $f(x)$



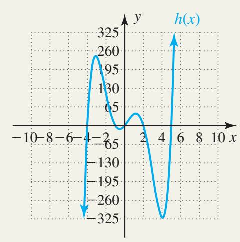
14. $g(x)$



15. $H(x)$



16. $h(x)$



State the end behavior and y-intercept of the functions given. Do not graph.

17. $f(x) = x^3 + 6x^2 - 5x - 2$

18. $g(x) = x^4 - 4x^3 - 2x^2 + 16x - 12$

19. $p(x) = -2x^4 + x^3 + 7x^2 - x - 6$

20. $q(x) = -2x^3 - 18x^2 + 7x + 3$

21. $Y_1 = -3x^5 + x^3 + 7x^2 - 6$

22. $Y_2 = -x^6 - 4x^5 + 4x^3 + 16x - 12$

For each polynomial graph, (a) state whether the degree of the function is even or odd; (b) use the graph to name the zeroes of *f*, then state whether their multiplicity is even or odd; (c) state the minimum possible degree of *f* and write it in factored form; and (d) estimate the domain and range. Assume all zeroes are real.

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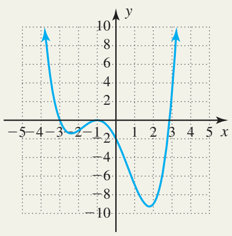

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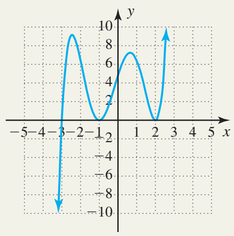

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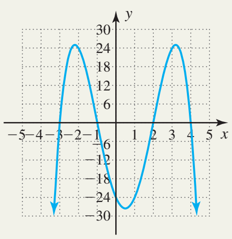

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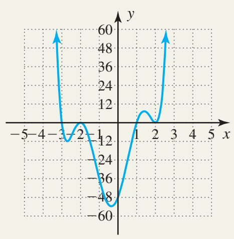

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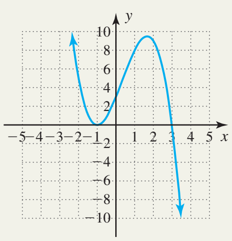

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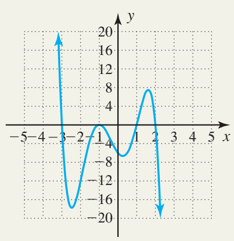


State the degree of each function, the end behavior, and *y*-intercept of its graph.

29. $f(x) = (x - 3)(x + 1)^3(x - 2)^2$

30. $g(x) = (x + 2)^2(x - 4)(x + 1)$

31. $Y_1 = -(x + 1)^2(x - 2)(2x - 3)(x + 4)$

32. $Y_2 = -(x + 1)(x - 2)^3(5x - 3)$

33. $r(x) = (x^2 + 3)(x + 4)^3(x - 1)$

34. $s(x) = (x + 2)^2(x - 1)^2(x^2 + 5)$

35. $h(x) = (x^2 + 2)(x - 1)^2(1 - x)$

36. $H(x) = (x + 2)^2(2 - x)(x^2 + 4)$

Every function in Exercises 37 through 42 has the zeroes $x = -1$, $x = -3$, and $x = 2$. Match each to its corresponding graph using degree, end behavior, and the multiplicity of each zero.

37. $f(x) = (x + 1)^2(x + 3)(x - 2)$

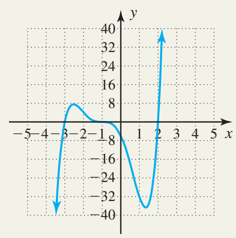
38. $F(x) = (x + 1)(x + 3)^2(x - 2)$

39. $g(x) = (x + 1)(x + 3)(x - 2)^3$

40. $G(x) = (x + 1)^3(x + 3)(x - 2)$

41. $Y_1 = (x + 1)^2(x + 3)(x - 2)^2$

42. $Y_2 = (x + 1)^3(x + 3)(x - 2)^2$

a.


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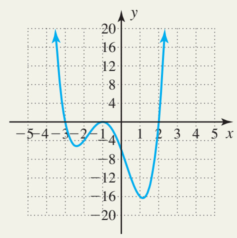

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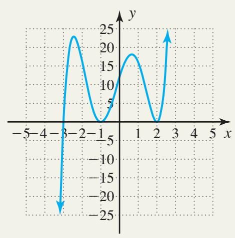
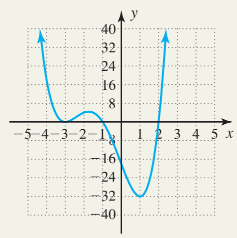
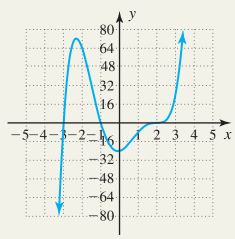
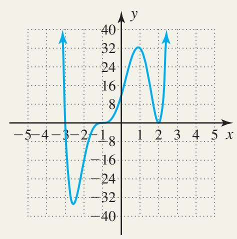


d.


e.


f.


Sketch the graph of each function using the degree, end behavior, *x*- and *y*-intercepts, zeroes of multiplicity, and a few midinterval points to round-out the graph. Connect all points with a smooth, continuous curve.

43. $f(x) = (x + 3)(x + 1)(x - 2)$

44. $g(x) = (x + 2)(x - 4)(x - 1)$

45. $p(x) = -(x + 1)^2(x - 3)$

46. $q(x) = -(x + 2)(x - 2)^2$

47. $Y_1 = (x + 1)^2(3x - 2)(x + 3)$

48. $Y_2 = (x + 2)(x - 1)^2(5x - 2)$

49. $r(x) = -(x + 1)^2(x - 2)^2(x - 1)$

50. $s(x) = -(x - 3)(x - 1)^2(x + 1)^2$

51. $f(x) = (2x + 3)(x - 1)^3$

52. $g(x) = (3x - 4)(x + 1)^3$

53. $h(x) = (x + 1)^3(x - 3)(x - 2)$

54. $H(x) = (x + 3)(x + 1)^2(x - 2)^2$

55. $Y_3 = (x + 1)^3(x - 1)^2(x - 2)$

56. $Y_4 = (x - 3)(x - 1)^3(x + 1)^2$

Use the *Guidelines for Graphing Polynomial Functions* to graph the polynomials.

57. $y = x^3 + 3x^2 - 4$

58. $y = x^3 - 13x + 12$

59. $f(x) = x^3 - 3x^2 - 6x + 8$

60. $g(x) = x^3 + 2x^2 - 5x - 6$

61. $h(x) = -x^3 - x^2 + 5x - 3$

62. $H(x) = -x^3 - x^2 + 8x + 12$

63. $p(x) = -x^4 + 10x^2 - 9$

64. $q(x) = -x^4 + 13x^2 - 36$

65. $r(x) = x^4 - 9x^2 - 4x + 12$

66. $s(x) = x^4 - 5x^3 + 20x - 16$

67. $Y_1 = x^4 - 6x^3 + 8x^2 + 6x - 9$

68. $Y_2 = x^4 - 4x^3 - 3x^2 + 10x + 8$

69. $Y_3 = 3x^4 + 2x^3 - 36x^2 + 24x + 32$

70. $Y_4 = 2x^4 - 3x^3 - 15x^2 + 32x - 12$

71. $F(x) = 2x^4 + 3x^3 - 9x^2$

72. $G(x) = 3x^4 + 2x^3 - 8x^2$

73. $f(x) = x^5 + 4x^4 - 16x^2 - 16x$

74. $g(x) = x^5 - 3x^4 + x^3 - 3x^2$

75. $h(x) = x^6 - 2x^5 - 4x^4 + 8x^3$

76. $H(x) = x^6 + 3x^5 - 4x^4$

In preparation for future course work, it becomes helpful to recognize the most common square roots in mathematics: $\sqrt{2} \approx 1.414$, $\sqrt{3} \approx 1.732$, and $\sqrt{6} \approx 2.449$. Graph the following polynomials *on a graphing calculator,* and use the calculator to locate the maximum/minimum values and all zeroes. Use the zeroes to write the polynomial in factored form, then verify the *y*-intercept from the factored form and polynomial form.

77. $h(x) = x^5 + 4x^4 - 9x - 36$

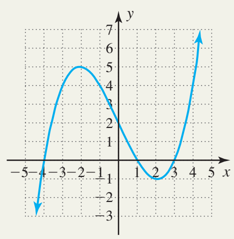
78. $H(x) = x^5 + 5x^4 - 4x - 20$

79. $f(x) = 2x^5 + 5x^4 - 10x^3 - 25x^2 + 12x + 30$

80. $g(x) = 3x^5 + 2x^4 - 24x^3 - 16x^2 + 36x + 24$

Use the graph of each function to construct its equation in factored form and in polynomial form. Be sure to check the *y*-intercept and adjust the lead coefficient if necessary.

81.



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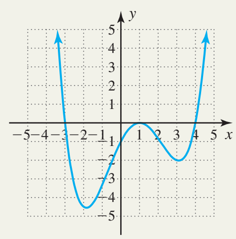


▶ WORKING WITH FORMULAS

83. Root tests for quartic polynomials: $ax^4 + bx^3 + cx^2 + dx + e = 0$
If u, v, w, and z represent the roots of a quartic polynomial, then the following relationships are true: (a) $u + v + w + z = -b$, (b) $u(v + z) + v(w + z) + w(u + z) = c$, (c) $u(vw + wz) + v(uz + wz) = -d$, and (d) $u \cdot v \cdot w \cdot z = e$. Use these tests to verify that $x = -3, -1, 2, 4$ are the solutions to $x^4 - 2x^3 - 13x^2 + 14x + 24 = 0$, then use these zeroes and the factored form to write the equation in polynomial form to confirm results.

84. It is worth noting that the root tests in Exercise 83 still apply when the roots are irrational and/or complex. Use these tests to verify that $x = -\sqrt{3}$, $\sqrt{3}$, $1 + 2i$, and $1 - 2i$ are the solutions to $x^4 - 2x^3 + 2x^2 + 6x - 15 = 0$, then use these zeroes and the factored form to write the equation in polynomial form to confirm results.

▶ APPLICATIONS

85. Traffic volume: Between the hours of 6:00 A.M. and 6.00 P.M., the volume of traffic at a busy intersection can be modeled by the polynomial $v(t) = -t^4 + 25t^3 - 192t^2 + 432t$, where $v(t)$ represents the number of vehicles above/below average, and t is number of hours past 6:00 A.M. (6:00 A.M. $\rightarrow 0$). (a) Use the remainder theorem to find the volume of traffic during rush hour (8:00 A.M.), lunch time (12 noon), and the trip home (5:00 P.M.). (b) Use the rational zeroes theorem to find the times when the volume of traffic is at its average $[v(t) = 0]$. (c) Use this information to graph $v(t)$, then use the graph to estimate the maximum and minimum flow of traffic and the time at which each occurs.

86. Insect population: The population of a certain insect varies dramatically with the weather, with spring-like temperatures causing a population boom and extreme weather (summer heat and winter cold) adversely affecting the population. This phenomena can be modeled by the polynomial $p(m) = -m^4 + 26m^3 - 217m^2 + 588m$, where $p(m)$

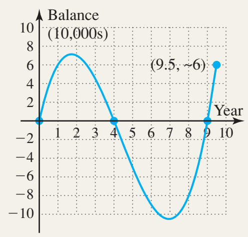
represents the number of live insects (in hundreds of thousands) in month m ($m = 1 \rightarrow$ Jan). (a) Use the remainder theorem to find the population of insects during the cool of spring (March) and the fair weather of fall (October). (b) Use the rational zeroes theorem to find the times when the population of insects becomes dormant $[p(m) = 0]$. (c) Use this information to graph $p(m)$, then use the graph to estimate the maximum and minimum population of insects, and the month at which each occurs.

87. Balance of payments: The graph shown represents the balance of payments (surplus versus deficit) for a large county over a 9-yr period. Use it to answer the following:

a. What is the minimum possible degree polynomial that can model this graph?

b. How many years did this county run a deficit?

c. Construct an equation model in factored form and in polynomial form, adjusting the lead coefficient as needed. How large was the deficit in year 8?

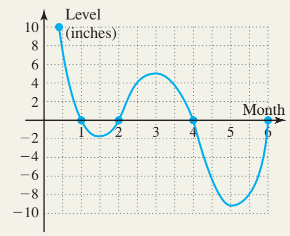


88. Water supply: The graph shown represents the water level in a reservoir (above and below normal) that supplies water to a metropolitan area, over a 6-month period. Use it to answer the following:

a. What is the minimum possible degree polynomial that can model this graph?

b. How many months was the water level below normal in this 6-month period?

c. At the beginning of this period ($m = 0$), the water level was 36 in. above normal, due to a long period of rain. Use this fact to help construct an equation model in factored form and in polynomial form, adjusting the lead coefficient as needed. Use the equation to determine the water level in months three and five.



▶ EXTENDING THE CONCEPT

89. As discussed in this section, the study of end behavior looks at what happens to the graph of a function as $|x| \rightarrow \infty$. Notice that as $|x| \rightarrow \infty$, both $\frac{1}{x}$ and $\frac{1}{x^2}$ approach zero. This fact can be used to study the end behavior of polynomial graphs.

a. For $f(x) = x^3 + x^2 - 3x + 6$, factoring out x^3 gives the expression $f(x) = x^3\left(1 + \frac{1}{x} - \frac{3}{x^2} + \frac{6}{x^3}\right)$. What happens to the value of the expression as $x \rightarrow \infty$? As $x \rightarrow -\infty$?

b. Factor out x^4 from $g(x) = x^4 + 3x^3 - 4x^2 + 5x - 1$. What happens to the value of the expression as $x \rightarrow \infty$? As $x \rightarrow -\infty$? How does this affirm the end behavior must be up/up?

90. For what value of c will three of the four real roots of $x^4 + 5x^3 + x^2 - 21x + c = 0$ be shared by the polynomial $x^3 + 2x^2 - 5x - 6 = 0$?

Show that the following equations have no rational roots.

91. $x^5 - x^4 - x^3 + x^2 - 2x + 3 = 0$

92. $x^5 - 2x^4 - x^3 + 2x^2 - 3x + 4 = 0$

▶ MAINTAINING YOUR SKILLS

93. **(2.8)** Given $f(x) = x^2 - 2x$ and $g(x) = \frac{1}{x}$, find the compositions $h(x) = (f \circ g)(x)$ and $H(x) = (g \circ f)(x)$, then state the domain of each.

94. **(1.5)** By direct substitution, verify that $x = 1 - 2i$ is a solution to $x^2 - 2x + 5 = 0$ and name the second solution.

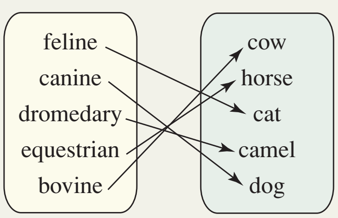
95. **(1.1/1.6)** Solve each of the following equations.

a. $-(2x + 5) - (6 - x) + 3 = x - 3(x + 2)$

b. $\sqrt{x + 1} + 3 = \sqrt{2x} + 2$

c. $\frac{2}{x - 3} + 5 = \frac{21}{x^2 - 9} + 4$

96. **(2.2)** Determine if the relation shown is a function. If not, explain how the definition of a function is violated.



MID-CHAPTER CHECK

1. Compute $(x^3 + 8x^2 + 7x - 14) \div (x + 2)$ using long division and write the result in two ways: (a) dividend = (quotient)(divisor) + remainder and (b) $\frac{\text{dividend}}{\text{divisor}} = (\text{quotient}) + \frac{\text{remainder}}{\text{divisor}}$.

2. Given that $x - 2$ is a factor of $f(x) = 2x^4 - x^3 - 8x^2 + x + 6$, use the rational zeroes theorem to write $f(x)$ in completely factored form.

3. Use the remainder theorem to evaluate $f(-2)$, given $f(x) = -3x^4 + 7x^2 - 8x + 11$.

4. Use the factor theorem to find a third-degree polynomial having $x = -2$ and $x = 1 + i$ as roots.

5. Use the intermediate value theorem to show that $g(x) = x^3 - 6x - 4$ has a root in the interval (2, 3).

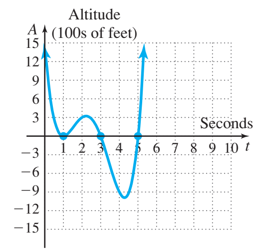
6. Use the rational zeroes theorem, tests for -1 and 1, synthetic division, and the remainder theorem to write $f(x) = x^4 + 5x^3 - 20x - 16$ in completely factored form.

7. Find all the zeroes of h, real and complex: $h(x) = x^4 + 3x^3 + 10x^2 + 6x - 20$.

8. Sketch the graph of p using its degree, end behavior, y-intercept, zeroes of multiplicity, and any midinterval points needed, given $p(x) = (x + 1)^2(x - 1)(x - 3)$.

9. Use the *Guidelines for Graphing* to draw the graph of $q(x) = x^3 + 5x^2 + 2x - 8$.

10. When fighter pilots train for dogfighting, a "hard-deck" is usually established below which no competitive activity can take place. The polynomial graph given shows Maverick's altitude above and below this hard-deck during a 5-sec interval.



a. What is the minimum possible degree polynomial that could form this graph? Why?

b. How many seconds (total) was Maverick below the hard-deck for these 5 sec of the exercise?

c. At the beginning of this time interval ($t = 0$), Maverick's altitude was 1500 ft above the hard-deck. Use this fact and the graph given to help construct an equation model in factored form and in polynomial form, adjusting the lead coefficient if needed. Use the equation to determine Maverick's altitude in relation to the hard-deck at $t = 2$ and $t = 4$.

REINFORCING BASIC CONCEPTS

Approximating Real Zeroes

Consider the equation $x^4 + x^3 + x - 6 = 0$. Using the rational zeroes theorem, the possible rational zeroes are $\{\pm 1, \pm 6, \pm 2, \pm 3\}$. The tests for 1 and -1 indicate that neither is a zero: $f(1) = -3$ and $f(-1) = -7$. Descartes' rule of signs reveals there must be one positive real zero since the coefficients of $f(x)$ change sign one time: $f(x) = x^4 + x^3 + x - 6$, and one negative real zero since $f(-x)$ also changes sign one time: $f(-x) = x^4 - x^3 - x - 6$. The remaining two zeroes must be complex. Using $x = 2$ with synthetic division shows 2 is not a zero, but the coefficients in the quotient row are all positive, so 2 is an upper bound:

2⌋	1	1	0	1	−6	coefficients of $f(x)$
		2	6	12	26	
	1	3	6	13	20	$q(x)$

Using $x = -2$ shows that -2 is a zero *and a lower bound* for all other zeroes (quotient row alternates in sign):

−2⌋	1	1	0	1	−6	coefficients of $f(x)$
		−2	2	−4	6	
	1	−1	2	−3	0	$q_1(x)$

This means the remaining real zero must be a positive irrational number less than 2 (all other possible rational zeroes were eliminated). The quotient polynomial $q_1(x) = x^3 - x^2 + 2x - 3$ is not factorable, yet we're left with the challenge of finding this final zero. While there are many advanced techniques available for approximating irrational zeroes, at this level either technology or a technique called **bisection** is commonly used. The bisection method combines the intermediate value theorem with successively smaller intervals of the input variable, to narrow down the location of the irrational zero. Although "bisection" implies halving the interval each time, any number within the interval will do. The bisection method may be most efficient using a succession of short input/output tables as shown, with the number of tables increased if greater accuracy is desired. Since $f(1) = -3$ and $f(2) = 20$, the intermediate value theorem tells us the zero must be in the interval [1, 2]. We begin our search here, rounding noninteger outputs to the nearest 100th. As a visual aid, positive outputs are in blue, negative outputs in red.

x	$f(x)$	Conclusion
1	−3	← Zero is here, use $x = 1.25$ next
1.5	3.94	
2	20	

x	$f(x)$	Conclusion
1	−3	Zero is here, ← use $x = 1.30$ next
1.25	−0.36	
1.5	3.94	

x	$f(x)$	Conclusion
1.25	−0.36	← Zero is here, use $x = 1.275$ next
1.30	0.35	
1.5	3.94	

A reasonable estimate for the zero appears to be $x = 1.275$. Evaluating the function at this point gives $f(1.275) \approx 0.0098$, which is very close to zero.

Naturally, a closer approximation is obtained using the capabilities of a graphing calculator. To seven decimal places the zero is $x \approx 1.2756822$.

Exercise 1: Use the intermediate value theorem to show that $f(x) = x^3 - 3x + 1$ has a zero in the interval [1, 2], then use bisection to locate the zero to three decimal place accuracy.

Exercise 2: The function $f(x) = x^4 + 3x - 15$ has two real zeroes in the interval $[-5, 5]$. Use the intermediate value theorem to locate the zeroes, then use bisection to find the zeroes accurate to three decimal places.

3.5 Graphing Rational Functions

Learning Objectives

In Section 3.5 you will learn how to:

- ❑ **A.** Identify horizontal and vertical asymptotes
- ❑ **B.** Find the domain of a rational function
- ❑ **C.** Apply the concept of "multiplicity" to rational graphs
- ❑ **D.** Find the horizontal asymptotes of a rational function
- ❑ **E.** Graph general rational functions
- ❑ **F.** Solve applications of rational functions

In this section we introduce an entirely new kind of relation, called a **rational function.** While we've already studied a variety of functions, we still lack the ability to model a large number of important situations. For example, functions that model the amount of medication remaining in the bloodstream over time, the relationship between altitude and weightlessness, and the relationship between predator and prey populations are all rational functions.

A. Rational Functions and Asymptotes

Just as a rational number is the ratio of two integers, a **rational function** is the ratio of two polynomials. In general,

Rational Functions

A rational function $V(x)$ is one of the form

$$V(x) = \frac{p(x)}{d(x)},$$

where p and d are polynomials and $d(x) \neq 0$.
The domain of $V(x)$ is all real numbers, *except the zeroes of d.*

The simplest rational functions are the reciprocal function $y = \frac{1}{x}$ and the reciprocal square function $y = \frac{1}{x^2}$, as both have a constant numerator and a single term in the denominator, with the domain of both excluding $x = 0$.

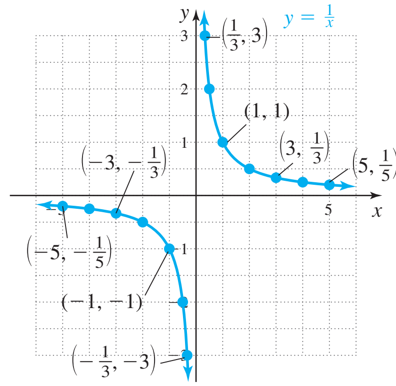
The Reciprocal Function: $y = \frac{1}{x}$

The reciprocal function takes any input (other than zero) and gives its reciprocal as the output. This means large inputs produce small outputs and vice versa. A table of values (Table 3.2) and the resulting graph (Figure 3.16) are shown.

Table 3.2

x	y	x	y
−1000	−1/1000	1/1000	1000
−5	−1/5	1/3	3
−4	−1/4	1/2	2
−3	−1/3	1	1
−2	−1/2	2	1/2
−1	−1	3	1/3
−1/2	−2	4	1/4
−1/3	−3	5	1/5
−1/1000	−1000	1000	1/1000
0	undefined		

Figure 3.16



WORTHY OF NOTE

The notation used for graphical behavior always begins by describing what is happening to the x-values, and the resulting effect on the y-values. Using Figure 3.17, visualize that for a point (x, y) on the graph of $y = \frac{1}{x}$, as x gets larger, y must become smaller, particularly since their product must always be 1 ($y = \frac{1}{x} \Rightarrow xy = 1$).

Figure 3.17

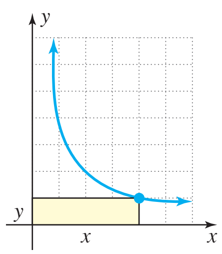


Table 3.2 and Figure 3.16 reveal some interesting features. First, the graph passes the vertical line test, verifying $y = \frac{1}{x}$ is indeed a function. Second, since division by zero is undefined, there can be no corresponding point on the graph, *creating a break at* $x = 0$. In line with our definition of rational functions, the domain is $x \in (-\infty, 0) \cup (0, \infty)$. Third, this is an odd function, with a "branch" of the graph in the first quadrant and one in the third quadrant, as the reciprocal of any input maintains its sign. Finally, we note in QI that as x becomes an infinitely large positive number, y gets closer and closer to zero. It seems convenient to symbolize this end behavior using the notation adopted in Section 3.4, and we write as $x \to \infty$, $y \to 0$. Graphically, the curve becomes very close to, or *approaches the x-axis*.

We also note that as x approaches zero from the right, y becomes an infinitely large positive number: as $x \to 0^+$, $y \to \infty$. Note a superscript + or − sign is used to indicate the *direction of the approach,* meaning *from the positive side* (right) or *from the negative side* (left).

EXAMPLE 1 ▶ Describing the End Behavior of Rational Functions

For $y = \frac{1}{x}$ in QIII,

a. Describe the end behavior of the graph.

b. Describe what happens as x approaches zero.

Solution ▶ Similar to the graph's behavior in QI, we have

a. In words: As x becomes an infinitely large negative number, y approaches zero. In notation: As $x \to -\infty$, $y \to 0$.

b. In words: As x approaches zero from the left, y becomes an infinitely large negative number. In notation: As $x \to 0^-$, $y \to -\infty$.

Now try Exercises 7 and 8 ▶

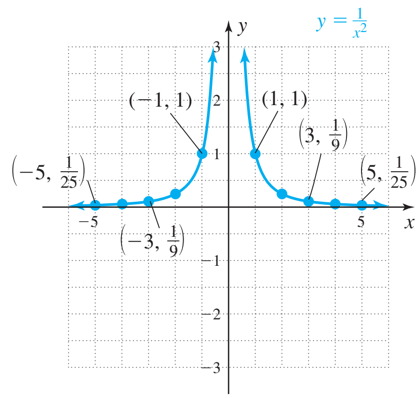
The Reciprocal Square Function: $y = \frac{1}{x^2}$

From our previous work, we anticipate this graph will also have a break at $x = 0$. But since the square of any negative number is positive, the branches of the **reciprocal square function** are both *above the x-axis*. Note the result is the graph of an even function. See Table 3.3 and Figure 3.18.

Table 3.3

x	y	x	y
-1000	1/1,000,000	1/1000	1,000,000
-5	1/25	1/3	9
-4	1/16	1/2	4
-3	1/9	1	1
-2	1/4	2	1/4
-1	1	3	1/9
$-1/2$	4	4	1/16
$-1/3$	9	5	1/25
$-1/1000$	1,000,000	1000	1/1,000,000
0	undefined		

Figure 3.18



Similar to $y = \frac{1}{x}$, large positive inputs generate small, positive outputs: as $x \to \infty$, $y \to 0$. This is one indication of **asymptotic behavior** in the horizontal direction, and we say the line $y = 0$ is a **horizontal asymptote** for the reciprocal and reciprocal square functions. In general,

Horizontal Asymptotes

Given a constant k, the line $y = k$ is a horizontal asymptote for a function V if as x increases without bound, $V(x)$ approaches k:

$$\text{as } x \to -\infty,\ V(x) \to k \qquad \text{or} \qquad \text{as } x \to \infty,\ V(x) \to k$$

Figure 3.19 shows a horizontal asymptote at $y = 1$, which suggests the graph of $f(x)$ is the graph of $y = \frac{1}{x}$ shifted up 1 unit. Figure 3.20 shows a horizontal asymptote at $y = -2$, which suggests the graph of $g(x)$ is the graph of $y = \frac{1}{x^2}$ shifted down 2 units.

Figure 3.19

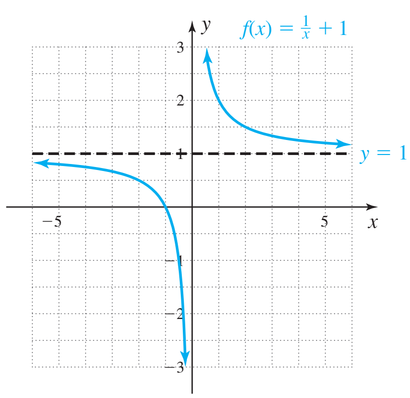
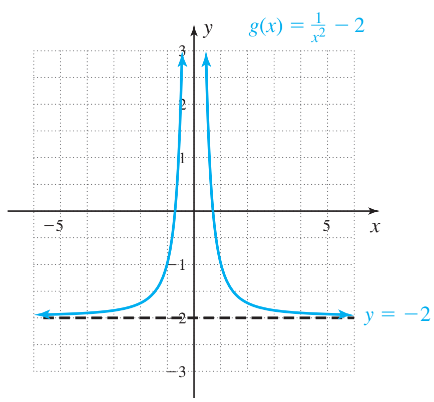


Figure 3.20



WORTHY OF NOTE

As seen in Figures 3.19 and 3.20, asymptotes appear graphically as dashed lines that seem to "guide" the branches of the graph.

EXAMPLE 2 ▶ **Describing the End Behavior of Rational Functions**

For the graph in Figure 3.20, use mathematical notation to

a. Describe the end behavior of the graph.

b. Describe what happens as x approaches zero.

Solution ▶ **a.** as $x \to -\infty$, $g(x) \to -2$
as $x \to \infty$, $g(x) \to -2$

b. as $x \to 0^-$, $g(x) \to \infty$
as $x \to 0^+$, $g(x) \to \infty$

Now try Exercises 9 and 10 ▶

From Example 2b, we note that as x becomes *smaller and close to 0*, g becomes very large and *increases without bound*. This is an indication of asymptotic behavior in the vertical direction, and we say the line $x = 0$ is a **vertical asymptote** for g ($x = 0$ is also a vertical asymptote for f). In general,

Vertical Asymptotes

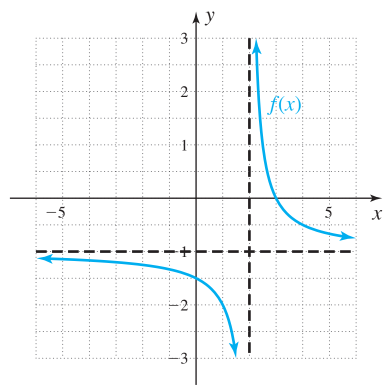
Given a constant h, the line $x = h$ is a vertical asymptote for a function V if as x approaches h, $V(x)$ increases or decreases without bound:

$$\text{as } x \to h^+,\ V(x) \to \pm\infty \qquad \text{or} \qquad \text{as } x \to h^-,\ V(x) \to \pm\infty$$

Identifying these asymptotes is useful because the graphs of $y = \frac{1}{x}$ and $y = \frac{1}{x^2}$ can be transformed *in exactly the same way as the toolbox functions.* When their graphs shift—the vertical and horizontal asymptotes shift with them and can be used as guides to redraw the graph. In shifted form, $f(x) = \dfrac{a}{(x \pm h)} \pm k$ for the reciprocal function, and $g(x) = \dfrac{a}{(x \pm h)^2} \pm k$ for the reciprocal square function.

EXAMPLE 3 ▶ Writing the Equation of a Basic Rational Function, Given Its Graph

Identify the function family for the graph given, then use the graph to write the equation of the function in "shifted form." Assume $|a| = 1$.



Solution ▶ The graph appears to be from the reciprocal function family, and has been shifted 2 units right (the vertical asymptote is at $x = 2$), and 1 unit down (the horizontal asymptote is at $y = -1$). From $y = \frac{1}{x}$, we obtain $f(x) = \frac{1}{x - 2} - 1$ as the shifted form.

Now try Exercises 11 through 22 ▶

A. You've just learned how to identify and name horizontal and vertical asymptotes

WORTHY OF NOTE

In Section 2.7, we studied special cases of $\frac{p(x)}{d(x)}$, where p and d shared a common factor, creating a "hole" in the graph. In this section, we'll assume the functions are given in simplest form (the numerator and denominator have no common factors).

B. Vertical Asymptotes and the Domain

Much of what we know about these basic functions can be generalized and applied to general rational functions. The graphs in Figures 3.21 through 3.24 show that rational graphs come in many shapes, often in "pieces," and exhibit asymptotic behavior.

Figure 3.21

$$f(x) = \frac{1}{x + 2}$$

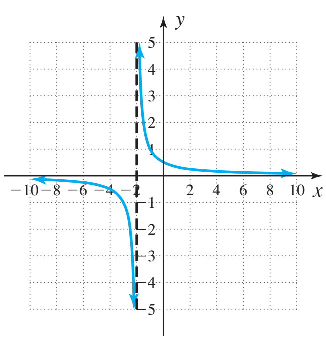


Figure 3.22

$$g(x) = \frac{2x}{x^2 - 1}$$

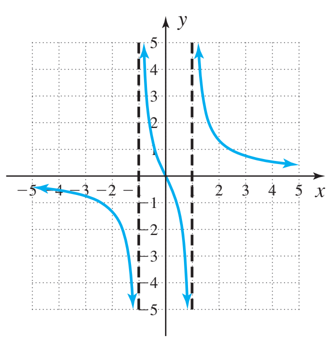


Figure 3.23

$$w(x) = \frac{3}{x^2 + 1}$$

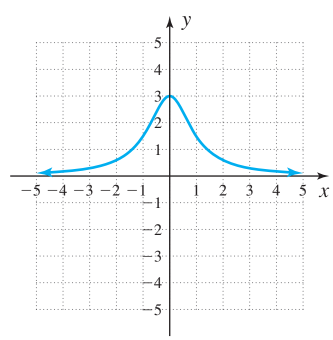
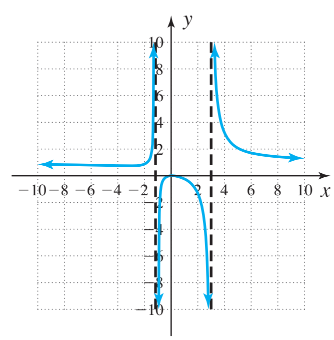


Figure 3.24

$$H(x) = \frac{x^2}{x^2 - 2x - 3}$$



WORTHY OF NOTE

Breaks created by vertical asymptotes are said to be **nonremovable,** because there is no way to repair the break, even if a piecewise-defined function were used. See Example 5, Section 2.7.

For $y = \frac{1}{x}$ and $y = \frac{1}{x^2}$, a vertical asymptote occurred at the zero of the denominator. This actually applies to all rational functions *in simplified form*. For $V(x) = \frac{p(x)}{d(x)}$, if c is a zero of $d(x)$, the function can be evaluated at every point near c, but not *at* c. This creates a **break** or **discontinuity** in the graph, resulting in the asymptotic behavior.

Vertical Asymptotes of a Rational Function

Given $V(x) = \frac{p(x)}{d(x)}$ is a rational function in simplest form, vertical asymptotes will occur at the real zeroes of d.

EXAMPLE 4 ▶ **Finding Vertical Asymptotes**

Locate the vertical asymptote(s) of each function given, then state its domain.

a. $f(x) = \dfrac{2x}{x^2 - 1}$ **b.** $g(x) = \dfrac{3}{x^2 + 1}$ **c.** $v(x) = \dfrac{x^2}{x^2 - 2x - 3}$

Solution ▶ **a.** Setting the denominator equal to zero gives $x^2 - 1 = 0$, so vertical asymptotes will occur at $x = -1$ and $x = 1$. The domain of f is $x \in (-\infty, -1) \cup (-1, 1) \cup (1, \infty)$.

b. Since the equation $x^2 + 1 = 0$ has no real zeroes, there are no vertical asymptotes and the domain of g is unrestricted: $x \in \mathbb{R}$.

c. Solving $x^2 - 2x - 3 = 0$ gives $(x + 1)(x - 3) = 0$, with solutions $x = -1$ and $x = 3$. There are vertical asymptotes at $x = -1$ and $x = 3$, and the domain of v is $x \in (-\infty, -1) \cup (-1, 3) \cup (3, \infty)$.

B. You've just learned how to find the domain of a rational function

Now try Exercises 23 through 30 ▶

C. Vertical Asymptotes and Multiplicities

The "cross" and "bounce" concept used for polynomial graphs can also be applied to rational graphs, particularly when viewed in terms of sign changes in the dependent variable. As you can see in Figures 3.25 to 3.27, the function $f(x) = \dfrac{1}{x + 2}$ changes sign at the asymptote $x = -2$ (negative on one side, positive on the other), and the denominator has multiplicity 1 (odd). The function $g(x) = \dfrac{1}{(x - 1)^2}$ does not change sign at the asymptote $x = 1$ (positive on both sides), and its denominator has multiplicity 2 (even). As with our earlier study of multiplicities, when these two are combined into the single function $v(x) = \dfrac{1}{(x + 2)(x - 1)^2}$, the function still changes sign at $x = -2$, and does not change sign at $x = 1$.

Figure 3.25

$f(x) = \dfrac{1}{x + 2}$

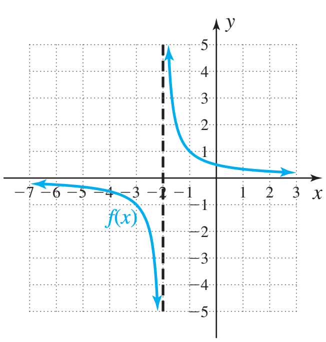


Figure 3.26

$g(x) = \dfrac{1}{(x - 1)^2}$

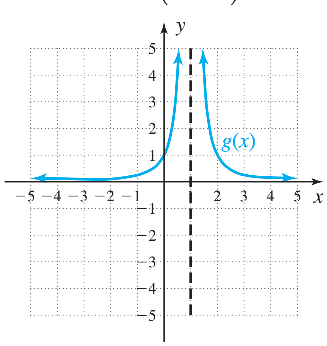
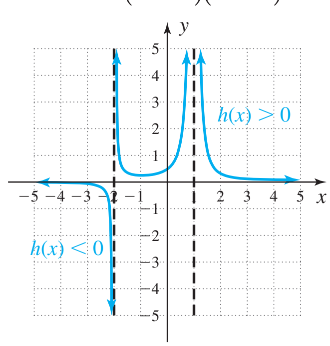


Figure 3.27

$h(x) = \dfrac{1}{(x + 2)(x - 1)^2}$



EXAMPLE 5 ▶ Finding Sign Changes at Vertical Asymptotes

Locate the vertical asymptotes of each function and state whether the function will change sign from one side of the asymptote(s) to the other.

a. $f(x) = \dfrac{x^2 - 4x + 4}{x^2 - 2x - 3}$ **b.** $g(x) = \dfrac{x^2 + 2}{x^2 + 2x + 1}$

Solution ▶

a. Factoring the denominator of f and setting it equal to zero gives $(x + 1)(x - 3) = 0$, and vertical asymptotes will occur at $x = -1$ and $x = 3$ (both multiplicity 1). The function will change sign at each asymptote (see Figure 3.28).

b. Factoring the denominator of g and setting it equal to zero gives $(x + 1)^2 = 0$. There will be a vertical asymptote at $x = -1$, but the function will not change sign since it's a zero of even multiplicity (see Figure 3.29).

Figure 3.28

$f(x) = \dfrac{x^2 - 4x + 4}{x^2 - 2x - 3}$

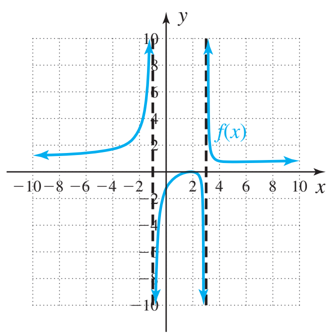
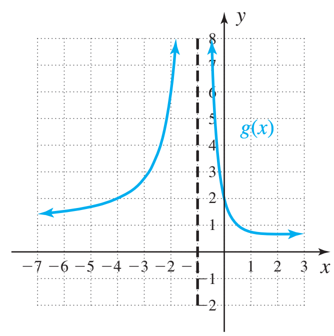


Figure 3.29

$g(x) = \dfrac{x^2 + 2}{x^2 + 2x + 1}$



C. You've just learned how to apply the concept of "multiplicity" to rational graphs

Now try Exercises 31 through 36 ▶

D. Finding Horizontal Asymptotes

A study of horizontal asymptotes is closely related to our study of "dominant terms" in Section 3.4. Recall the highest degree term in a polynomial tends to dominate all other terms as $|x| \to \infty$. For $v(x) = \dfrac{2x^2 + 4x + 3}{x^2 + 2x + 1}$, both polynomials *have the same degree,* so $\dfrac{2x^2 + 4x + 3}{x^2 + 2x + 1} \approx \dfrac{2x^2}{x^2} = 2$ for large values of x: as $|x| \to \infty$, $y \to 2$ and $y = 2$ is a horizontal asymptote for v. When the degree of the numerator is *smaller* than the degree of the denominator, our earlier work with $y = \frac{1}{x}$ and $y = \frac{1}{x^2}$ showed there was a horizontal asymptote at $y = 0$ (the x-axis), since as $|x| \to \infty$, $y \to 0$. In general,

LOOKING AHEAD

In Section 3.6 we will explore two additional kinds of asymptotic behavior, (1) oblique (slant) asymptotes and (2) asymptotes that are nonlinear.

Horizontal Asymptotes

Given $V(x) = \frac{p(x)}{d(x)}$ is a rational function in lowest terms, where the leading term of p is ax^n and the leading term of d is bx^m (polynomial p has degree n, polynomial d has degree m).

I. If $n < m$, there is a horizontal asymptote at $y = 0$ (the x-axis).
II. If $n = m$, there is a horizontal asymptote at $y = \frac{a}{b}$.
III. If $n > m$, the graph has no horizontal asymptote.

Finally, while the graph of a rational function can never "cross" the vertical asymptote $x = h$ (since the function simply cannot be evaluated at h), it is possible for a graph to cross the horizontal asymptote $y = k$ (some do, others do not). To find out which is the case, we set the function equal to k and solve.

EXAMPLE 6 ▶ Locating Horizontal Asymptotes

Locate the horizontal asymptote for each function, if one exists. Then determine if the graph will cross the asymptote.

a. $f(x) = \dfrac{3x}{x^2 + 2}$ **b.** $g(x) = \dfrac{x^2 - 4}{x^2 - 1}$ **c.** $v(x) = \dfrac{3x^2 - x - 6}{x^2 + x - 6}$

Solution ▶

a. For $f(x)$, the degree of the numerator $<$ degree of the denominator, indicating a horizontal asymptote at $y = 0$. Solving $f(x) = 0$, we find $x = 0$ is the only solution and the graph will cross the horizontal asymptote at (0, 0) (see Figure 3.30).

b. For $g(x)$, the degree of the numerator and the denominator are equal. This means $g(x) \approx \dfrac{x^2}{x^2} = 1$ for large values of x, and there is a horizontal asymptote at $y = 1$. Solving $g(x) = 1$ gives

$$\frac{x^2 - 4}{x^2 - 1} = 1 \qquad y = 1 \rightarrow \text{horizontal asymptote}$$
$$x^2 - 4 = x^2 - 1 \qquad \text{multiply by } x^2 - 1$$
$$-4 = -1 \qquad \text{no solution}$$

The graph will not cross the asymptote (see Figure 3.31).

c. For $v(x)$, the degree of the numerator and denominator are once again equal, so $v(x) \approx \dfrac{3x^2}{x^2} = 3$ and there is a horizontal asymptote at $y = 3$.
Solving $v(x) = 3$ gives

$$\frac{3x^2 - x - 6}{x^2 + x - 6} = 3 \qquad y = 3 \rightarrow \text{horizontal asymptote}$$
$$3x^2 - x - 6 = 3(x^2 + x - 6) \qquad \text{multiply by } x^2 + x - 6$$
$$3x^2 - x - 6 = 3x^2 + 3x - 18 \qquad \text{distribute}$$
$$-4x + 12 = 0 \qquad \text{simplify}$$
$$x = 3 \qquad \text{result}$$

The graph will cross its asymptote at $x = 3$ (see Figure 3.32).

☑ **D.** You've just learned how to find the horizontal asymptotes of a rational function

Figure 3.30

$f(x) = \dfrac{3x}{x^2 + 2}$

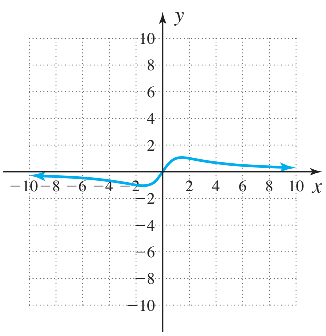


Figure 3.31

$g(x) = \dfrac{x^2 - 4}{x^2 - 1}$

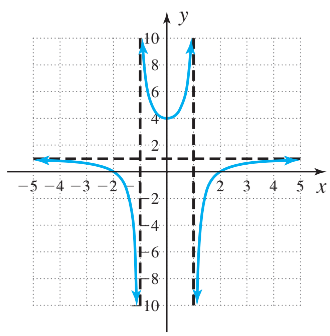
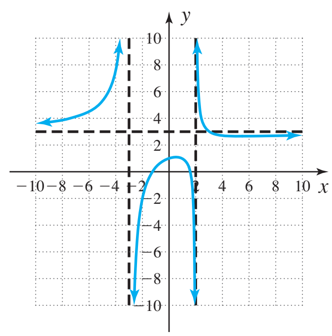


Figure 3.32

$v(x) = \dfrac{3x^2 - x - 6}{x^2 + x - 6}$



Now try Exercises 37 through 42 ▶

WORTHY OF NOTE

It's helpful to note that all nonvertical asymptotes and whether they cross the graph can actually be found using long division. The quotient $q(x)$ gives the equation of the asymptote, and the zeroes of the remainder $r(x)$ will indicate if or where the two will cross. From Example 6c, long division gives $q(x) = 3$ and $r(x) = -4x + 12$ (verify this), showing there is a horizontal asymptote at $y = 3$, which the graph crosses at $x = 3$ [the zero of $r(x)$].

E. The Graph of a Rational Function

Our observations to this point lead us to this general strategy for graphing rational functions. Not all graphs require every step, but together they provide an effective approach.

Guidelines for Graphing Rational Functions

Given $V(x) = \frac{p(x)}{d(x)}$, $d(x) \neq 0$, is a rational function in lowest terms,

1. Find the y-intercept at $V(0)$.
2. Find vertical asymptotes (if any) at $d(x) = 0$.
3. Find x-intercepts at $p(x) = 0$.
4. Locate the horizontal asymptote (if any).
5. Determine if the graph will cross the horizontal asymptote.
6. If needed, compute "midinterval" points to help complete the graph.
7. Draw the asymptotes, plot the intercepts and additional points, and use intervals where $V(x)$ changes sign to complete the graph.

EXAMPLE 7 ▶ Graphing Rational Functions

Graph each function given.

a. $f(x) = \dfrac{x^2 - x - 6}{x^2 + x - 6}$ **b.** $g(x) = \dfrac{2x^2 - 4x + 2}{x^2 - 7}$

Solution ▶ **a.** Begin by writing f in factored form: $f(x) = \dfrac{(x + 2)(x - 3)}{(x + 3)(x - 2)}$.

1. y-intercept: $f(0) = \dfrac{(2)(-3)}{(3)(-2)} = 1$, so the y-intercept is $(0, 1)$.

2. Vertical asymptote(s): Setting the denominator equal to zero gives $(x + 3)(x - 2) = 0$, showing there will be vertical asymptotes at $x = -3$, $x = 2$.

3. x-intercepts: Setting the numerator equal to zero gives $(x + 2)(x - 3) = 0$, showing the x-intercepts will be $(-2, 0)$ and $(3, 0)$.

4. Horizontal asymptote: Since the degree of the numerator and the degree of the denominator are equal, $y = \dfrac{x^2}{x^2} = 1$ is a horizontal asymptote.

5. Solving

$$\frac{x^2 - x - 6}{x^2 + x - 6} = 1 \qquad f(x) = 1 \rightarrow \text{horizontal asymptote}$$

$$x^2 - x - 6 = x^2 + x - 6 \qquad \text{multiply by } x^2 + x - 6$$

$$-2x = 0 \qquad \text{simplify}$$

$$x = 0 \qquad \text{solve}$$

The graph will cross the horizontal asymptote at $(0, 1)$.

The information from steps 1 through 5 is shown in Figure 3.33, and indicates we have no information about the graph in the interval $(-\infty, -3)$. Since rational functions are defined for all real numbers except the zeroes of d, we know there must be a "piece" of the graph in this interval.

6. Selecting $x = -4$ to compute one additional point, we find $f(-4) = \frac{(-2)(-7)}{(-1)(-6)} = \frac{14}{6} = \frac{7}{3}$. The point is $(-4, \frac{7}{3})$.

7. All factors of f are linear, so function values will alternate sign in the intervals created by x-intercepts and vertical asymptotes. The y-intercept $(0, 1)$ shows $f(x)$ is positive in the interval containing 0. To meet all necessary conditions, we complete the graph, as shown in Figure 3.34.

Figure 3.33

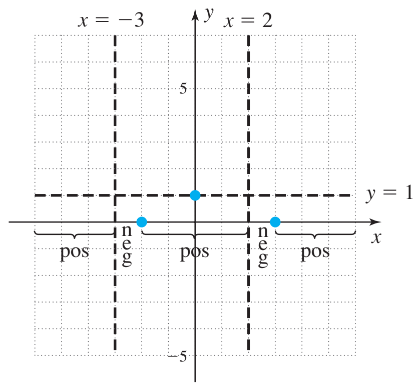
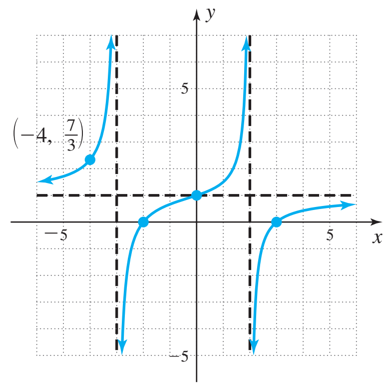


Figure 3.34



b. Writing g in factored form gives $g(x) = \dfrac{2(x^2 - 2x + 1)}{x^2 - 7} = \dfrac{2(x-1)^2}{(x+\sqrt{7})(x-\sqrt{7})}$.

1. y-intercept: $g(0) = \dfrac{2(-1)^2}{(\sqrt{7})(-\sqrt{7})} = -\dfrac{2}{7}$. The y-intercept is $(0, -\frac{2}{7})$.

2. Vertical asymptote(s): Setting the denominator equal to zero gives $(x + \sqrt{7})(x - \sqrt{7}) = 0$, showing there will be asymptotes at $x = -\sqrt{7}, x = \sqrt{7}$.

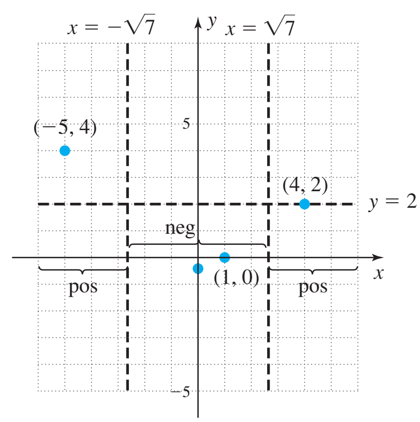
3. x-intercept(s): Setting the numerator equal to zero gives $2(x - 1)^2 = 0$, with $x = 1$ a zero of multiplicity 2. The x-intercept is $(1, 0)$.

4. Horizontal asymptote: The degree of the numerator is equal to the degree of denominator, so $y = \dfrac{2x^2}{x^2} = 2$ is a horizontal asymptote.

5. Solve
$$\begin{aligned} \frac{2x^2 - 4x + 2}{x^2 - 7} &= 2 && g(x) = 2 \rightarrow \text{horizontal asymptote} \\ 2x^2 - 4x + 2 &= 2x^2 - 14 && \text{multiply by } x^2 - 7 \\ -4x &= -16 && \text{simplify} \\ x &= 4 && \text{solve} \end{aligned}$$

The graph will cross its horizontal asymptote at $(4, 2)$. The information from steps 1 to 5 is shown in Figure 3.35, and indicates we have no information about the graph in the interval $(-\infty, -\sqrt{7})$.

Figure 3.35



6. Selecting $x = -5$,
$$\begin{aligned} g(-5) &= \frac{2(-5-1)^2}{(-5)^2 - 7} \\ &= \frac{2(-6)^2}{25 - 7} \\ &= \frac{2(36)}{18} \\ &= 4 \end{aligned}$$

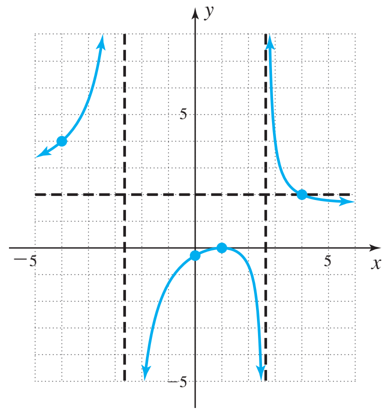
The point $(-5, 4)$ is on the graph.

WORTHY OF NOTE

It's useful to note that the number of "pieces" forming a rational graph will always be one more than the number of vertical asymptotes. The graph of $f(x) = \dfrac{3x}{x^2 + 2}$ (Figure 3.30) has no vertical asymptotes and one piece, $y = \frac{1}{x}$ has one vertical asymptote and two pieces, $g(x) = \dfrac{x^2 - 4}{x^2 - 1}$ (Figure 3.31) has two vertical asymptotes and three pieces, and so on.

7. Since factors of the denominator have odd multiplicity, function values will alternate sign on either side of the asymptotes. The factor in the numerator has even multiplicity, so the graph will "bounce off" the x-axis at $x = 1$ (no change in sign). The y-intercept $(0, -\frac{2}{7})$ shows the function is negative in the interval containing 0. This information and the completed graph are shown in Figure 3.36.

Figure 3.36



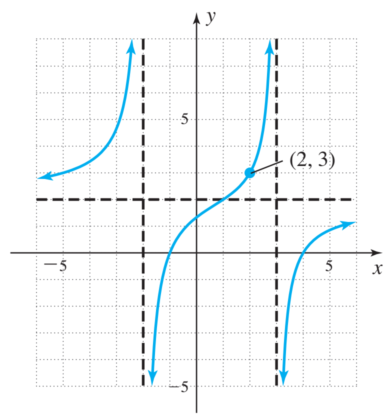
Now try Exercises 43 through 66 ►

Examples 6 and 7 demonstrate that graphs of rational functions come in a large variety. Once the components of the graph have been found, completing the graph presents an intriguing and puzzle-like challenge as we attempt to sketch a graph that meets all conditions. As we've done with other functions, can you reverse this process? That is, given the graph of a rational function, can you construct its equation?

EXAMPLE 8 ► Finding the Equation of a Rational Function from Its Graph

Use the graph of $f(x)$ shown to construct its equation.

Solution ► The x-intercepts are $(-1, 0)$ and $(4, 0)$, so the numerator must contain the factors $(x + 1)$ and $(x - 4)$. The vertical asymptotes are $x = -2$ and $x = 3$, so the denominator must have the factors $(x + 2)$ and $(x - 3)$. So far we have:



$$f(x) = \frac{a(x + 1)(x - 4)}{(x + 2)(x - 3)}$$

Since (2, 3) is on the graph, we substitute 2 for x and 3 for $f(x)$ to solve for a:

$$3 = \frac{a(2 + 1)(2 - 4)}{(2 + 2)(2 - 3)} \quad \text{substitute 3 for } f(x) \text{ and 2 for } x$$

$$3 = \frac{3a}{2} \quad \text{simplify}$$

$$2 = a \quad \text{solve}$$

The result is $f(x) = \frac{2(x + 1)(x - 4)}{(x + 2)(x - 3)} = \frac{2x^2 - 6x - 8}{x^2 - x - 6}$, with a horizontal asymptote at $y = 2$ and a y-intercept of $(0, \frac{4}{3})$, which fits the graph very well.

☑ **E.** You've just learned how to graph general rational functions

Now try Exercises 67 through 70 ►

F. Applications of Rational Functions

In many applications of rational functions, the coefficients can be rather large and the graph should be scaled appropriately.

EXAMPLE 9 ► **Modeling the Cost to Remove Chemical Waste**

For a large urban-centered county, the cost to remove chemical waste from a local river is modeled by $C(p) = \frac{180p}{100 - p}$, where $C(p)$ represents the cost (in thousands of dollars) to remove p percent of the pollutants.

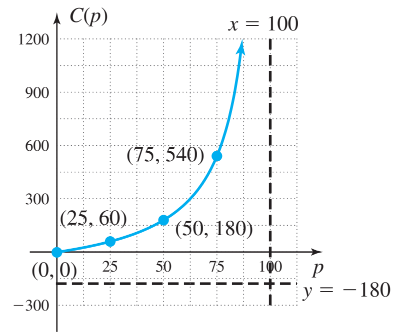
a. Find the cost to remove 25%, 50%, and 75% of the pollutants and comment.

b. Graph the function using an appropriate scale.

c. In mathematical notation, state what happens if the county attempts to remove 100% of the pollutants.

Solution ► **a.** Evaluating the function for the values indicated, we find $C(25) = 60$, $C(50) = 180$, and $C(75) = 540$. The cost is escalating rapidly. The change from 25% to 50% brought a \$120,000 increase, but the change from 50% to 75% brought *a \$360,000 increase!*

b. From $C(p) = \frac{180p}{100 - p}$, we see that C has a y-intercept at (0, 0) and a vertical asymptote at $p = 100$. Since the degree of the numerator and denominator are equal, there is a horizontal asymptote at $y = \frac{180p}{-p} = -180$. From the context we need only graph the portion from $0 \le p < 100$, producing the following graph:



c. As the percentage of the pollutants removed approaches 100%, the cost of the cleanup skyrockets. Notationally, as $p \to 100^-$, $C \to \infty$.

F. You've just learned how to solve an application of rational functions

Now try Exercises 73 through 84 ►

TECHNOLOGY HIGHLIGHT

Rational Functions and Appropriate Domains

In Example 9, portions of the graph were ignored due to the context of the application. To see the full graph, we use the fact that a second branch of C occurs on the opposite side of the vertical and horizontal asymptotes, and set a window size like the one shown in Figure 3.37. After entering $C(p)$ as Y_1 on the Y= screen and pressing GRAPH, the full graph shown in Figure 3.38 appears (the horizontal asymptote was drawn using $Y_2 = -180$).

Figure 3.37

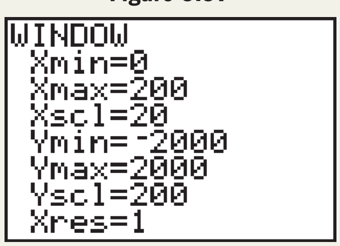
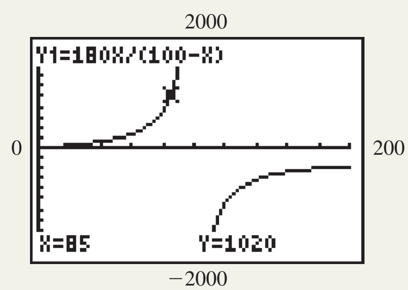


Figure 3.38



Exercise 1: Use the TRACE feature to verify that as $p \to 100^-$, $C \to \infty$. Approximately how much money must be spent to remove 95% of the pollutants? What happens when you TRACE to 100%? Past 100%?

Exercise 2: Calculate the rate of change $\frac{\Delta C}{\Delta p}$ for the intervals [60, 65], [85, 90], and [90, 95] (use the *Technology Extension* from Chapter 3 at www.mhhe.com/coburn if desired). Comment on what you notice.

Exercise 3: Reset the window size changing only Xmax to 100 and Ymin to 0 for a more relevant graph. How closely does it resemble the graph from Example 9?

3.5 EXERCISES

▶ CONCEPTS AND VOCABULARY

Fill in each blank with the appropriate word or phrase. Carefully reread the section if needed.

1. Write the following in direction/approach notation. *As x becomes an infinitely large negative number, y approaches 2.* ________

2. For any constant k, the notation as $|x| \to +\infty$, $y \to k$ is an indication of a ________ asymptote, while $x \to k$, $|y| \to +\infty$ indicates a ________ asymptote.

3. Vertical asymptotes are found by setting the ________ equal to zero. The x-intercepts are found by setting the ________ equal to zero.

4. If the degree of the numerator is equal to the degree of the denominator, a horizontal asymptote occurs at $y = \frac{a}{b}$, where $\frac{a}{b}$ represents the ratio of the ________ ________.

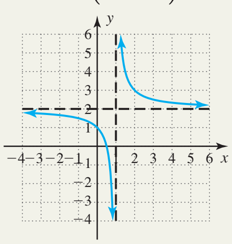
5. Use the function $g(x) = \dfrac{3x^2 - 2x}{2x^2 - 3}$ and a table of values to discuss the concept of horizontal asymptotes. At what positive value of x is the graph of g within 0.01 of its horizontal asymptote?

6. Name all of the "tools" at your disposal that play a role in the graphing of rational functions. Which tools are indispensable and always used? Which are used only as the situation merits?

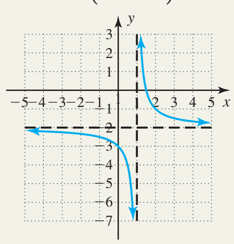
▶ DEVELOPING YOUR SKILLS

For each graph given, (a) use mathematical notation to describe the end behavior of each graph and (b) describe what happens as x approaches 1.

7. $V(x) = \dfrac{1}{(x - 1)} + 2$

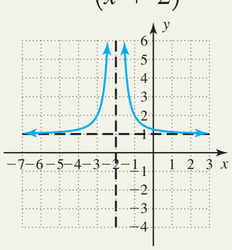


8. $v(x) = \dfrac{1}{(x - 1)} - 2$

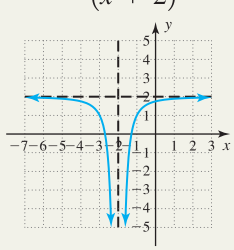


For each graph given, (a) use mathematical notation to describe the end behavior of each graph, and (b) describe what happens as x approaches -2.

9. $Q(x) = \dfrac{1}{(x + 2)^2} + 1$

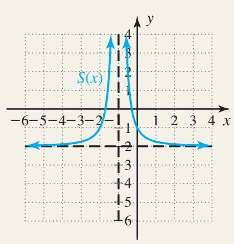


10. $q(x) = \dfrac{-1}{(x + 2)^2} + 2$



Identify the parent function for each graph given, then use the graph to construct the equation of the function in shifted form. Assume $|a| = 1$.

11.



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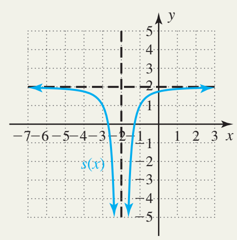

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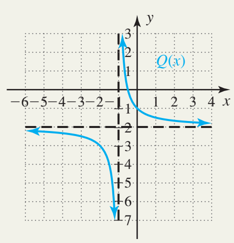

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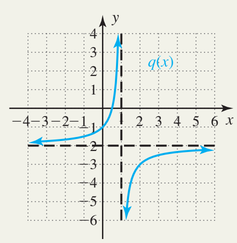

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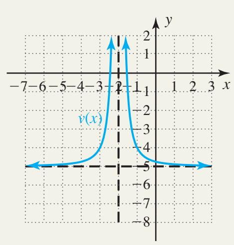

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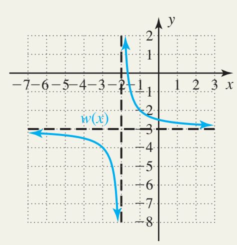
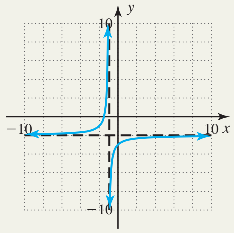


Use the graph shown to complete each statement using the direction/approach notation.

Exercises 17 through 22



17. As $x \to -\infty$, y ______.

18. As $x \to \infty$, y ______.

19. As $x \to -1^+$, y ______.

20. As $x \to -1^-$, y ______.

21. The line $x = -1$ is a vertical asymptote, since: as $x \to$ ____, $y \to$ ____.

22. The line $y = -2$ is a horizontal asymptote, since: as $x \to$ ____, $y \to$ ____.

▶ DEVELOPING YOUR SKILLS

Give the location of the vertical asymptote(s) if they exist, and state the function's domain.

23. $f(x) = \dfrac{x+2}{x-3}$

24. $F(x) = \dfrac{4x}{2x-3}$

25. $g(x) = \dfrac{3x^2}{x^2-9}$

26. $G(x) = \dfrac{x+1}{9x^2-4}$

27. $h(x) = \dfrac{x^2-1}{2x^2+3x-5}$

28. $H(x) = \dfrac{x-5}{2x^2-x-3}$

29. $p(x) = \dfrac{2x+3}{x^2+x+1}$

30. $q(x) = \dfrac{2x^3}{x^2+4}$

Give the location of the vertical asymptote(s) if they exist, and state whether function values will change sign (positive to negative or negative to positive) from one side of the asymptote to the other.

31. $Y_1 = \dfrac{x+1}{x^2-x-6}$

32. $Y_2 = \dfrac{2x+3}{x^2-x-20}$

33. $r(x) = \dfrac{x^2+3x-10}{x^2-6x+9}$

34. $R(x) = \dfrac{x^2-2x-15}{x^2-4x+4}$

35. $Y_1 = \dfrac{x}{x^3+2x^2-4x-8}$

36. $Y_2 = \dfrac{-2x}{x^3+x^2-x-1}$

For the functions given, (a) determine if a horizontal asymptote exists and (b) determine if the graph will cross the asymptote, and if so, where it crosses.

37. $Y_1 = \dfrac{2x-3}{x^2+1}$

38. $Y_2 = \dfrac{4x+3}{2x^2+5}$

39. $r(x) = \dfrac{4x^2-9}{x^2-3x-18}$

40. $R(x) = \dfrac{2x^2-x-10}{x^2+5}$

41. $p(x) = \dfrac{3x^2-5}{x^2-1}$

42. $P(x) = \dfrac{3x^2-5x-2}{x^2-4}$

Give the location of the *x*- and *y*-intercepts (if they exist), and discuss the behavior of the function (bounce or cross) at each *x*-intercept.

43. $f(x) = \dfrac{x^2-3x}{x^2-5}$

44. $F(x) = \dfrac{2x-x^2}{x^2+2x-3}$

45. $g(x) = \dfrac{x^2+3x-4}{x^2-1}$

46. $G(x) = \dfrac{x^2+7x+6}{x^2-2}$

47. $h(x) = \dfrac{x^3-6x^2+9x}{4-x^2}$

48. $H(x) = \dfrac{4x+4x^2+x^3}{x^2-1}$

Use the *Guidelines for Graphing Rational Functions* to graph the functions given.

49. $f(x) = \dfrac{x+3}{x-1}$

50. $g(x) = \dfrac{x-4}{x+2}$

51. $F(x) = \dfrac{8x}{x^2+4}$

52. $G(x) = \dfrac{-12x}{x^2+3}$

53. $p(x) = \dfrac{-2x^2}{x^2-4}$

54. $P(x) = \dfrac{3x^2}{x^2-9}$

55. $q(x) = \dfrac{2x-x^2}{x^2+4x-5}$

56. $Q(x) = \dfrac{x^2+3x}{x^2-2x-3}$

57. $h(x) = \dfrac{-3x}{x^2-6x+9}$

58. $H(x) = \dfrac{2x}{x^2-2x+1}$

59. $Y_1 = \dfrac{x-1}{x^2-3x-4}$

60. $Y_2 = \dfrac{1-x}{x^2-2x}$

61. $s(x) = \dfrac{4x^2}{2x^2+4}$

62. $S(x) = \dfrac{-2x^2}{x^2+1}$

63. $Y_1 = \dfrac{x^2-4}{x^2-1}$

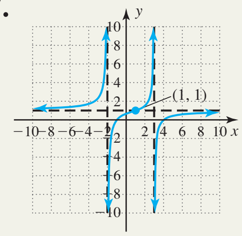
64. $Y_2 = \dfrac{x^2-x-6}{x^2+x-6}$

65. $v(x) = \dfrac{-2x}{x^3+2x^2-4x-8}$

66. $V(x) = \dfrac{3x}{x^3+x^2-x-1}$

Use the vertical asymptotes, x-intercepts, and their multiplicities to construct an equation that corresponds to each graph. Be sure the y-intercept estimated from the graph matches the value given by your equation for $x = 0$. Check work on a graphing calculator.

67.



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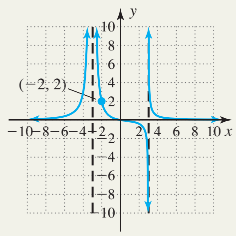

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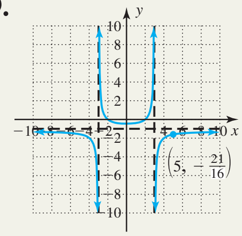

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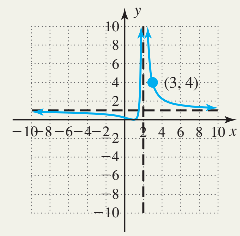


▶ WORKING WITH FORMULAS

71. Population density: $D(x) = \dfrac{ax}{x^2 + b}$

The population density of urban areas (in people per square mile) can be modeled by the formula shown, where a and b are constants related to the overall population and sprawl of the area under study, and $D(x)$ is the population density (in hundreds), x mi from the center of downtown.

Graph the function for $a = 63$ and $b = 20$ over the interval $x \in [0, 25]$, and then use the graph to answer the following questions.

a. What is the significance of the *horizontal asymptote* (what does it mean in this context)?

b. How far from downtown does the population density fall below 525 people per square mile? How far until the density falls below 300 people per square mile?

c. Use the graph and a table to determine how far from downtown the population density reaches a maximum? What is this maximum?

72. Cost of removing pollutants: $C(x) = \dfrac{kx}{100 - x}$

Some industries resist cleaner air standards because the cost of removing pollutants rises dramatically as higher standards are set. This phenomenon can be modeled by the formula given, where $C(x)$ is the cost (in thousands of dollars) of removing x% of the pollutant and k is a constant that depends on the type of pollutant and other factors.

Graph the function for $k = 250$ over the interval $x \in [0, 100]$, and then use the graph to answer the following questions.

a. What is the significance of the *vertical asymptote* (what does it mean in this context)?

b. If new laws are passed that require 80% of a pollutant to be removed, while the existing law requires only 75%, how much will the new legislation cost the company? Compare the cost of the 5% increase from 75% to 80% with the cost of the 1% increase from 90% to 91%.

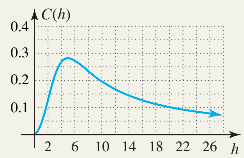
c. What percent of the pollutants can be removed if the company budgets 2250 thousand dollars?

▶ APPLICATIONS

73. For a certain coal-burning power plant, the cost to remove pollutants from plant emissions can be modeled by $C(p) = \dfrac{80p}{100 - p}$, where $C(p)$ represents the cost (in thousands of dollars) to remove p percent of the pollutants. (a) Find the cost to remove 20%, 50%, and 80% of the pollutants, then comment on the results; (b) graph the function using an appropriate scale; and (c) use the direction/approach notation to state what happens if the power company attempts to remove 100% of the pollutants.

74. A large city has initiated a new recycling effort, and wants to distribute recycling bins for use in separating various recyclable materials. City planners anticipate the cost of the program can be modeled by the function $C(p) = \dfrac{220p}{100 - p}$, where $C(p)$ represents the cost (in \$10,000) to distribute the bins to p percent of the population. (a) Find the cost to distribute bins to 25%, 50%, and 75% of the population, then comment on the results; (b) graph the function using an appropriate scale; and (c) use the direction/approach notation to state what happens if the city attempts to give recycling bins to 100% of the population.

75. The concentration C of a certain medicine in the bloodstream h hours after being injected into the shoulder is given by the function: $C(h) = \dfrac{2h^2 + h}{h^3 + 70}$. Use the given graph of the function to answer the following questions.

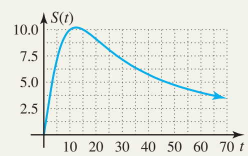


a. Approximately how many hours after injection did the maximum concentration occur? What was the maximum concentration?

b. Use $C(h)$ to *compute* the rate of change for the intervals $h = 8$ to $h = 10$ and $h = 20$ to $h = 22$. What do you notice?

c. Use the direction/approach notation to state what happens to the concentration C as the number of hours becomes infinitely large. What role does the h-axis play for this function?

76. In response to certain market demands, manufacturers will quickly get a product out on the market to take advantage of consumer interest. Once the product is released, it is not uncommon for sales to initially skyrocket, taper off and then gradually decrease as consumer interest wanes. For a certain product, sales can be modeled by the function $S(t) = \dfrac{250t}{t^2 + 150}$, where $S(t)$ represents the daily sales (in \$10,000) t days after the product has debuted. Use the given graph of the function to answer the following questions.



a. Approximately how many days after the product came out did sales reach a maximum? What was the maximum sales?

b. Use $S(t)$ to compute the rate of change for the intervals $t = 7$ to $t = 8$ and $t = 60$ to $t = 62$. What do you notice?

c. Use the direction/approach notation to state what happens to the daily sales S as the number of days becomes infinitely large. What role does the t-axis play for this function?

Memory retention: Due to their asymptotic behavior, rational functions are often used to model the mind's ability to retain information over a long period of time—the "use it or lose it" phenomenon.

77. A large group of students is asked to memorize a list of 50 Italian words, a language that is unfamiliar to them. The group is then tested regularly to see how many of the words are retained over a period of time. The average number of words retained is modeled by the function $W(t) = \dfrac{6t + 40}{t}$, where $W(t)$ represents the number of words remembered after t days.

a. Graph the function over the interval $t \in [0, 40]$. How many days until only half the words are remembered? How many days until only one-fifth of the words are remembered?

b. After 10 days, what is the average number of words retained? How many days until only 8 words can be recalled?

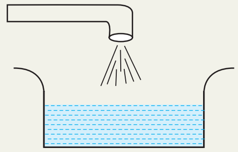
c. What is the significance of the horizontal asymptote (what does it mean in this context)?

78. A similar study asked students to memorize 50 Hawaiian words, a language that is both unfamiliar and phonetically foreign to them (see Exercise 77). The average number of words retained is modeled by the function $W(t) = \dfrac{4t + 20}{t}$, where $W(t)$ represents the number of words after t days.

a. Graph the function over the interval $t \in [0, 40]$. How many days until only half the words are remembered? How does this compare to Exercise 77? How many days until only one-fifth of the words are remembered?

b. After 7 days, what is the average number of words retained? How many days until only 5 words can be recalled?

c. What is the significance of the horizontal asymptote (what does it mean in this context)?



Concentration and dilution: When antifreeze is mixed with water, it becomes diluted—less than 100% antifreeze. The more water added, the less concentrated the antifreeze becomes, with this process continuing until a desired concentration is met. This application and many similar to it can be modeled by rational functions.

79. A 400-gal tank currently holds 40 gal of a 25% antifreeze solution. To raise the concentration of the antifreeze in the tank, x gal of a 75% antifreeze solution is pumped in.

a. Show the formula for the resulting concentration is $C(x) = \dfrac{40 + 3x}{160 + 4x}$ after simplifying, and graph the function over the interval $x \in [0, 360]$.

b. What is the concentration of the antifreeze in the tank after 10 gal of the new solution are added? After 120 gal have been added? How much liquid is now in the tank?

c. If the concentration level is now at 65%, how many gallons of the 75% solution have been added? How many gallons of liquid are in the tank now?

d. What is the maximum antifreeze concentration that can be attained in a tank of this size? What is the maximum concentration that can be attained in a tank of "unlimited" size?

80. A sodium chloride solution has a concentration of 0.2 oz (weight) per gallon. The solution is pumped into an 800-gal tank currently holding 40 gal of pure water, at a rate of 10 gal/min.

a. Find a function $A(t)$ modeling the amount of liquid in the tank after t min, and a function $S(t)$ for the amount of sodium chloride in the tank after t min.

b. The concentration $C(t)$ in ounces per gallon is measured by the ratio $\dfrac{S(t)}{A(t)}$, a rational function. Graph the function on the interval $t \in [0, 100]$. What is the concentration level (in ounces per gallon) after 6 min? After 28 min? How many gallons of liquid are in the tank at this time?

c. If the concentration level is now 0.184 oz/gal, how long have the pumps been running? How many gallons of liquid are in the tank now?

d. What is the maximum concentration that can be attained in a tank of this size? What is the maximum concentration that can be attained in a tank of "unlimited" size?

Average cost of manufacturing an item: The cost "C" to manufacture an item depends on the relatively fixed costs "K" for remaining in business (utilities, maintenance, transportation, etc.) and the actual cost "c" of manufacturing the item (labor and materials). For x items the cost is $C(x) = K + cx$. The average cost "A" of manufacturing an item is then $A(x) = \dfrac{C(x)}{x}$.

81. A company that manufactures water heaters finds their fixed costs are normally $50,000 per month, while the cost to manufacture each heater is $125. Due to factory size and the current equipment, the company can produce a maximum of 5000 water heaters per month during a good month.

a. Use the average cost function to find the average cost if 500 water heaters are manufactured each month. What is the average cost if 1000 heaters are made?

b. What level of production will bring the average cost down to $150 per water heater?

c. If the average cost is currently $137.50, how many water heaters are being produced that month?

d. What's the significance of the horizontal asymptote for the average cost function (what does it mean in this context)? Will the company ever break the $130 average cost level? Why or why not?

82. An enterprising company has finally developed a disposable diaper that is biodegradable. The brand becomes wildly popular and production is soaring. The fixed cost of production is $20,000 per month, while the cost of manufacturing is $6.00 per case (48 diapers). Even while working three shifts around-the-clock, the maximum production level is 16,000 cases per month. The company figures it will be profitable if it can bring costs down to an average of $7 per case.

a. Use the average cost function to find the average cost if 2000 cases are produced each month. What is the average cost if 4000 cases are made?

b. What level of production will bring the average cost down to $8 per case?

c. If the average cost is currently $10 per case, how many cases are being produced?

d. What's the significance of the horizontal asymptote for the average cost function (what does it mean in this context)? Will the company ever reach its goal of $7/case at its maximum production? What level of production would help them meet their goal?

Test averages and grade point averages: To calculate a test average we sum all test points P and divide by the number of tests N: $\frac{P}{N}$. To compute the score or scores needed on future tests to raise the average grade to a desired grade G, we add the number of additional tests n to the denominator, and the number of additional tests times the projected grade g on each test to the numerator: $G(n) = \frac{P + ng}{N + n}$. The result is a rational function with some "eye-opening" results.

83. After four tests, Bobby Lou's test average was an 84. [*Hint:* $P = 4(84) = 336$.]

a. Assume that she gets a 95 on all remaining tests ($g = 95$). Graph the resulting function on a calculator using the window $n \in [0, 20]$ and $G(n) \in [80 \text{ to } 100]$. Use the calculator to determine how many tests are required to lift her grade to a 90 under these conditions.

b. At some colleges, the range for an "A" grade is 93–100. How many tests would Bobby Lou have to score a 95 on, to raise her average to higher than 93? Were you surprised?

c. Describe the significance of the horizontal asymptote of the average grade function. Is a test average of 95 possible for her under these conditions?

d. Assume now that Bobby Lou scores 100 on all remaining tests ($g = 100$). Approximately how many more tests are required to lift her grade average to higher than 93?

84. At most colleges, $A \rightarrow 4$ grade points, $B \rightarrow 3$, $C \rightarrow 2$, and $D \rightarrow 1$. After taking 56 credit hours, Aurelio's GPA is 2.5. [*Hint:* In the formula given, $P = 2.5(56) = 140$.]

a. Assume Aurelio is determined to get A's (4 grade points or $g = 4$), for all remaining credit hours. Graph the resulting function on a calculator using the window $n \in [0, 60]$ and $G(n) \in [2, 4]$. Use the calculator to determine the number of credit hours required to lift his GPA to over 2.75 under these conditions.

b. At some colleges, scholarship money is available only to students with a 3.0 average or higher. How many (perfect 4.0) credit hours would Aurelio have to earn, to raise his GPA to 3.0 or higher? Were you surprised?

c. Describe the significance of the horizontal asymptote of the GPA function. Is a GPA of 4.0 possible for him under these conditions?

▶ EXTENDING THE CONCEPT

85. In addition to determining *if* a function has a vertical asymptote, we are often interested in *how fast* the graph approaches the asymptote. As in previous investigations, this involves the function's rate of change over a small interval. Exercise 72 describes the rising cost of removing pollutants from the air. As noted there, the rate of increase in the cost changes as higher requirements are set. To quantify this change, we'll compute the rate of change

$$\frac{\Delta C}{\Delta x} = \frac{C(x_2) - C(x_1)}{x_2 - x_1} \text{ for } C(x) = \frac{250x}{100 - x}.$$

a. Find the rate of change of the function in the following intervals:

$x \in [60, 61]$ $\quad x \in [70, 71]$

$x \in [80, 81]$ $\quad x \in [90, 91]$

b. What do you notice? How much did the rate increase from the first interval to the second? From the second to the third? From the third to the fourth?

c. Recompute parts (a) and (b) using the function $C(x) = \frac{350x}{100 - x}$. Comment on what you notice.

86. Consider the function $f(x) = \frac{ax^2 + k}{bx^2 + h}$, where a, b, k, and h are constants and $a, b > 0$.

a. What can you say about asymptotes and intercepts of this function if $h, k > 0$?

b. Now assume $k < 0$ and $h > 0$. How does this affect the asymptotes? The intercepts?

c. If $b = 1$ and $a > 1$, how does this affect the results from part (b)?

d. How is the graph affected if $k > 0$ and $h < 0$?

e. Find values of a, b, h, and k that create a function with a horizontal asymptote at $y = \frac{3}{2}$, x-intercepts at $(-2, 0)$ and $(2, 0)$, a y-intercept of $(0, -4)$, and no vertical asymptotes.

87. The horizontal asymptotes of a rational function, and whether or not a graph crosses this asymptote, can be found using long division. The quotient polynomial $q(x)$ gives the equation of the asymptote, and the zeroes of the remainder $r(x)$ will indicate if and where the graph crosses it. Use this idea to help graph these functions.

a. $V(x) = \dfrac{3x^2 - 16x - 20}{x^2 - 3x - 10}$

b. $v(x) = \dfrac{-2x^2 + 4x + 13}{x^2 - 2x - 3}$

▶ MAINTAINING YOUR SKILLS

88. **(R.1/1.4)** Describe/Define each set of numbers: complex C, rational Q, and integers Z.

89. **(2.3)** Find the equation of a line that is perpendicular to $3x - 4y = 12$ and contains the point $(2, -3)$.

90. **(1.5)** Solve the following equation using the quadratic formula, then write the equation in factored form: $12x^2 + 55x - 48 = 0$.

91. **(3.2)** Use synthetic division and the remainder theorem to find the value of $f(4)$, $f(\frac{3}{2})$, and $f(2)$: $f(x) = 2x^3 - 7x^2 + 5x + 3$.

3.6 Additional Insights into Rational Functions

Learning Objectives

In Section 3.6 you will learn how to:

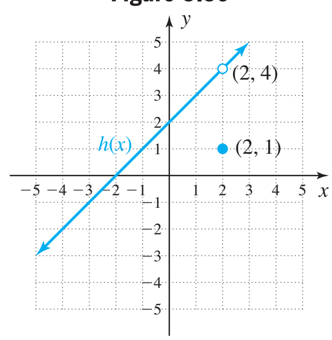
- ☐ **A.** Graph rational functions with removable discontinuities
- ☐ **B.** Graph rational functions with oblique or nonlinear asymptotes
- ☐ **C.** Solve applications involving rational functions

In Section 3.5, we saw that rational graphs can have both a horizontal and vertical asymptote. In this section, we'll study functions with asymptotes that are *neither* horizontal nor vertical. In addition, we'll further explore the "break" we saw in graphs of certain piecewise-defined functions, that of a simple "hole" created when the numerator and denominator share a common variable factor.

A. Rational Functions and Removable Discontinuities

In Example 5 of Section 2.7, we graphed the piecewise-defined function $h(x) = \begin{cases} \dfrac{x^2 - 4}{x - 2} & x \neq 2 \\ 1 & x = 2 \end{cases}$.

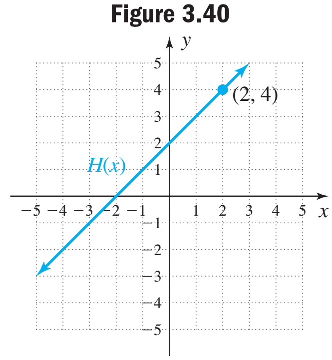
Figure 3.39



The second piece is simply the point (2, 1). The first piece is a rational function, but instead of a vertical asymptote at $x = 2$ (the zero of the denominator), its graph was actually the line $y = x + 2$ with a "hole" at (2, 4), called a **removable discontinuity** (Figure 3.39). As the name implies, we can *remove* or fix this break by redefining the second piece as $h(x) = 4$, when $x = 2$. This would create a new and continuous function,

$H(x) = \begin{cases} \dfrac{x^2 - 4}{x - 2} & x \neq 2 \\ 4 & x = 2 \end{cases}$ (Figure 3.40). It's possible for a rational graph to have more than one removable discontinuity, or to be nonlinear with a removable discontinuity. For cases where we elect to repair the break, we will adopt the convention of using the corresponding upper case letter to name the new function, as we did here.

Figure 3.40



WORTHY OF NOTE

The graph of $f(x) = \dfrac{1}{x - 2}$ also has a break at $x = 2$, but this time the result is a *vertical asymptote*. The difference is the numerator and denominator of $h(x) = \dfrac{x^2 - 4}{x - 2}$ share a common factor, and canceling these factors leaves $y = x + 2$, which is a continuous function. However, the *original function* is not defined at $x = 2$, so we must remove the single point (2, 4) from the domain of $y = x + 2$ (Figure 3.39).

EXAMPLE 1 ▶ **Graphing Rational Functions with Removable Discontinuities**

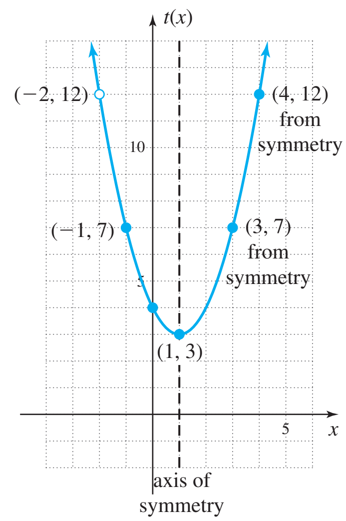
Graph the function $t(x) = \frac{x^3 + 8}{x + 2}$. If there is a removable discontinuity, repair the break using an appropriate piecewise-defined function.

Solution ▶ Note the domain of t does not include $x = -2$. We begin by factoring as before to identify zeroes and asymptotes, but find the numerator and denominator share a common factor, which we remove.

$$\begin{aligned} t(x) &= \frac{x^3 + 8}{x + 2} \\ &= \frac{\cancel{(x + 2)}(x^2 - 2x + 4)}{\cancel{x + 2}} \\ &= x^2 - 2x + 4;\ \text{where } x \neq -2 \end{aligned}$$

The graph of t will be the same as $y = x^2 - 2x + 4$ *for all values except* $x = -2$. Here we have a parabola, opening upward, with y-intercept (0, 4). From the vertex formula, the x-coordinate of the vertex will be $\frac{-b}{2a} = \frac{-(-2)}{2(1)} = 1$, giving $y = 3$ after substitution. The vertex is (1, 3). Evaluating $t(-1)$ we find $(-1, 7)$ is on the graph, giving the point (3, 7) using the axis of symmetry. We draw a parabola through these points, noting the original function is not defined at -2, and there will be a "hole" in the graph at $(-2, y)$. The value of y is found by substituting -2 for x in the simplified form: $(-2)^2 - 2(-2) + 4 = 12$. This information produces the graph shown. We can repair the break using the function

$$T(x) = \begin{cases} \frac{x^3 + 8}{x + 2} & x \neq -2 \\ 12 & x = -2 \end{cases}$$



WORTHY OF NOTE

For more on removable discontinuities, see the *Technology Highlight* feature on page 370.

A. You've just learned how to graph rational functions with removable discontinuities

Now try Exercises 7 through 18 ▶

B. Rational Functions with Oblique and Nonlinear Asymptotes

In Section 3.5, we found that for $V(x) = \frac{p(x)}{d(x)}$, the location of nonvertical asymptotes was determined by comparing the degree of p with the degree of d. As review, for $p(x)$ with leading term ax^n and $d(x)$ with leading term degree bx^m,

- If $n < m$, the line $y = 0$ is a horizontal asymptote.
- If $n = m$, the line $y = \frac{a}{b}$ is a horizontal asymptote.

But what happens if the degree of the numerator is *greater than* the degree of the denominator? To investigate, consider the functions f, g, and h in Figures 3.41 to 3.43, whose only difference is the degree of the numerator.

Figure 3.41

$f(x) = \dfrac{2x}{x^2 + 1}$

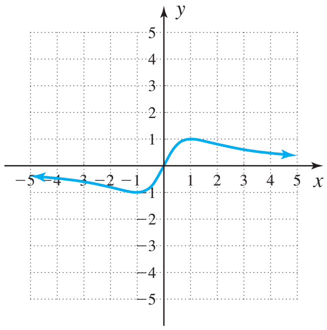


Figure 3.42

$g(x) = \dfrac{2x^2}{x^2 + 1}$

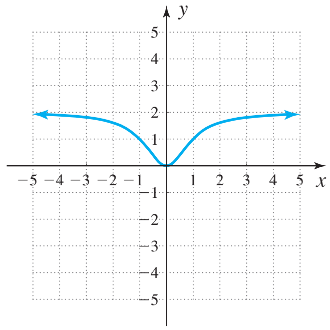
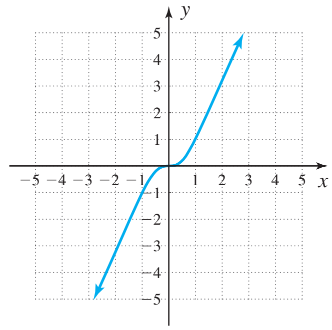


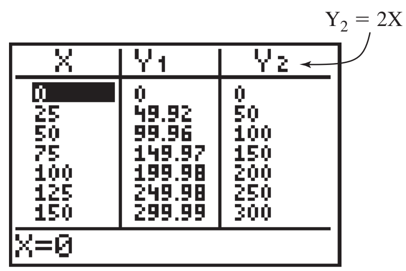
Figure 3.43

$h(x) = \dfrac{2x^3}{x^2 + 1}$



The graph of f has a horizontal asymptote at $y = 0$ since the denominator is of larger degree (as $|x| \to \infty$, $y \to 0$). As we might have anticipated, the horizontal asymptote for g is $y = 2$, the ratio of leading coefficients (as $|x| \to \infty$, $y \to 2$). The graph of h has no horizontal asymptote, yet appears to be asymptotic to some slanted line. The table in Figure 3.44 suggests that as $|x| \to \infty$, $y \to 2x = Y_2$. To see why, note the function $h(x) = \dfrac{2x^3}{x^2 + 1}$ can be considered an "improper fraction," similar to how we apply this designation to the fraction $\frac{3}{2}$. To write h in "proper" form, we use long division, writing the dividend as $2x^3 + 0x^2 + 0x + 0$, and the divisor as $x^2 + 0x + 1$.

Figure 3.44



X	Y1	Y2
0	0	0
25	49.92	50
50	99.96	100
75	149.97	150
100	199.98	200
125	249.98	250
150	299.99	300

X=0

The ratio $\dfrac{2x^3}{x^2}$ (from dividend / from divisor) shows $\mathbf{2x}$ will be our first multiplier.

$$\begin{array}{r}
2x \\
\text{divisor} \to x^2 + 0x + 1\overline{)\,2x^3 + 0x^2 + 0x + 0} \\
-(2x^3 + 0x^2 + 2x) \\
\hline
-\,2x
\end{array}$$

multiply $2x(x^2 + 0x + 1)$

subtract, next term is 0

The result shows $h(x) = 2x + \dfrac{-2x}{x^2 + 1}$. Note as $|x| \to \infty$, the term $\dfrac{-2x}{x^2 + 1}$ becomes very small and closer to zero, so $h(x) \approx 2x$ for large x. This is an example of an **oblique asymptote.** In general,

Oblique and Nonlinear Asymptotes

Given $V(x) = \frac{p(x)}{d(x)}$ is a rational function in simplest form, where the degree of p is greater than the degree of d, the graph will have an oblique or nonlinear asymptote as determined by $q(x)$, where $q(x)$ is the quotient polynomial after division.

We conclude that an oblique or slant asymptote occurs when the degree of the numerator is one more than the degree of the denominator, and a nonlinear asymptote occurs when its degree is larger by two or more.

EXAMPLE 2 ▶ Graphing a Rational Function with an Oblique Asymptote

Graph the function $f(x) = \dfrac{x^2 - 1}{x}$.

Solution ▶ Using the *Guidelines,* we find $f(x) = \dfrac{(x + 1)(x - 1)}{x}$ and proceed:

1. y-intercept: The graph has no y-intercept.
2. Vertical asymptote(s): $x = 0$ with multiplicity 1. The function will change sign at $x = 0$.
3. x-intercepts: From $(x + 1)(x - 1) = 0$, the x-intercepts are $(-1, 0)$ and $(1, 0)$. Since both have multiplicity 1, the graph will cross the x-axis and the function will change sign at these points
4. Horizontal/oblique asymptote: Since the degree of numerator > the degree of denominator, we rewrite f using division. Using term-by-term division (the denominator is a monomial) produces $f(x) = \dfrac{x^2 - 1}{x} = \dfrac{x^2}{x} - \dfrac{1}{x} = x - \dfrac{1}{x}$. The quotient polynomial is $q(x) = x$ and the graph has the oblique asymptote $y = x$.
5. To determine if the function will cross the asymptote, we solve

$$\frac{x^2 - 1}{x} = x \quad q(x) = x \text{ is the slant asymptote}$$
$$x^2 - 1 = x^2 \quad \text{multiply by } x$$
$$-1 = 0 \quad \text{no solutions possible}$$

The graph will not cross the oblique asymptote.

The information from steps 1 through 5 is displayed in Figure 3.45. While this is sufficient to complete the graph, we select $x = -4$ and 4 to compute additional points and find $f(-4) = -\frac{15}{4}$ and $f(4) = \frac{15}{4}$. To meet all necessary conditions, we complete the graph as shown in Figure 3.46.

WORTHY OF NOTE

If the denominator is a monomial, term-by-term division is the most efficient means of computing the quotient. If the denominator is not a monomial, either synthetic division or long division must be used.

Figure 3.45

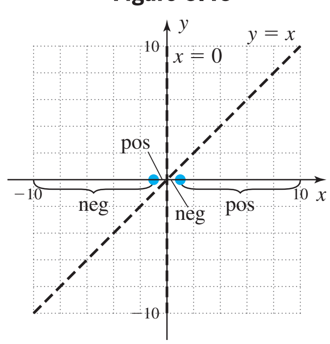
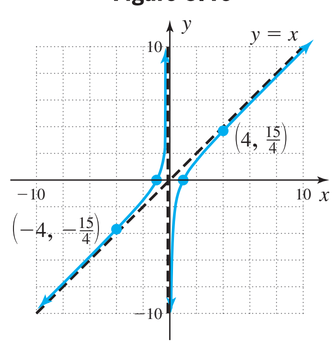


Figure 3.46



Now try Exercises 19 through 24 ▶

EXAMPLE 3 ▶ Graphing a Rational Function with an Oblique Asymptote

Graph the function: $h(x) = \dfrac{x^2}{x - 1}$

Solution ▶ The function is already in "factored form."

1. y-intercept: Since $h(0) = 0$, the y-intercept is $(0, 0)$.
2. Vertical asymptote: Solving $x - 1 = 0$ gives $x = 1$ with multiplicity one. There is a vertical asymptote at $x = 1$ and the function will change sign here.

3. x-intercept: (0, 0); From, $x^2 = 0$, we have $x = 0$ with multiplicity two. The x-intercept is (0, 0) and the function will not change sign here.

4. Horizontal/oblique asymptote: Since the degree of numerator > the degree of denominator, we rewrite h using division. The denominator is linear so we use synthetic division:

$$\begin{array}{r|rrr} 1 & 1 & 0 & 0 \\ & \downarrow & 1 & 1 \\ \hline & 1 & 1 & 1 \end{array}$$

use 1 as a "divisor"; coefficients of dividend; quotient and remainder

Since $q(x) = x + 1$ the graph has an oblique asymptote at $y = x + 1$.

5. To determine if the function crosses the asymptote, we solve

$$\frac{x^2}{x-1} = x + 1 \quad q(x) = x + 1 \text{ is the slant asymptote}$$

$$x^2 = x^2 - 1 \quad \text{cross multiply}$$

$$0 = -1 \quad \text{no solutions possible}$$

The graph will not cross the slant asymptote.

The information gathered in steps 1 through 5 is shown Figure 3.47, and is actually sufficient to complete the graph. If you feel a little unsure about how to "puzzle" out the graph, find additional points in the first and third quadrants: $h(2) = 4$ and $h(-2) = -\frac{4}{3}$. Since the graph will "bounce" at $x = 0$ and output values must change sign at $x = 1$, all conditions are met with the graph shown in Figure 3.48.

Figure 3.47

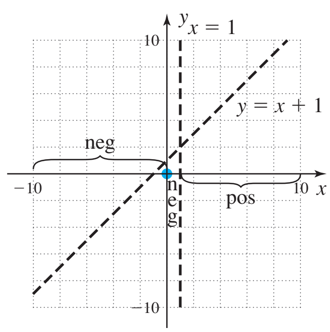
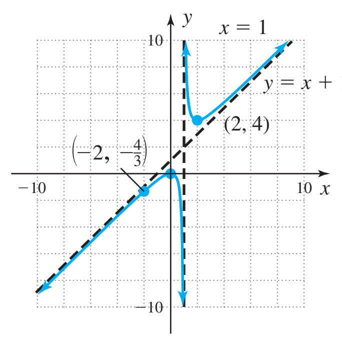
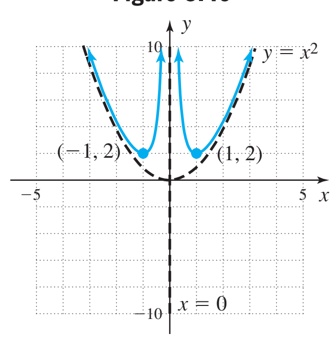


Figure 3.48



Now try Exercises 25 through 46 ▶

Figure 3.49



Finally, it would be a mistake to think that all asymptotes are linear. In fact, when the degree of the numerator is two more than the degree of the denominator, a parabolic asymptote results. Functions of this type often occur in applications of rational functions, and are used to minimize cost, materials, distances, or other considerations of great importance to business and industry. For $f(x) = \frac{x^4 + 1}{x^2}$, term-by-term division gives $x^2 + \frac{1}{x^2}$ and the quotient $q(x) = x^2$ is a nonlinear, parabolic asymptote (see Figure 3.49). For more on nonlinear asymptotes, **see Exercises 47 through 50.**

☑ **B.** You've just learned how to graph rational functions with oblique or nonlinear asymptotes

C. Applications of Rational Functions

Rational functions have applications in a wide variety of fields, including environmental studies, manufacturing, and various branches of medicine. In most practical applications, only the values from Quadrant I have meaning since inputs and outputs must often be positive **(see Exercises 51 and 52).** Here we investigate an application involving manufacturing and average cost.

EXAMPLE 4 ▶ **Solving an Application of Rational Functions**

Suppose the cost (in thousands of dollars) of manufacturing x thousand of a given item is modeled by the function $C(x) = x^2 + 4x + 3$. The *average cost* of each item would then be expressed by

$$A(x) = \frac{x^2 + 4x + 3}{x} = \frac{\text{total cost}}{\text{number of items}}$$

a. Graph the function $A(x)$.

b. Find how many thousand items are manufactured when the average cost is \$8.

c. Determine how many thousand items should be manufactured to minimize the average cost (use the graph to estimate this minimum average cost).

Solution ▶ **a.** The function is already in simplest form.

1. y-intercept: none $[A(0)$ is undefined$]$
2. Vertical asymptote: $x = 0$, multiplicity one; the function will change sign at $x = 0$.
3. x-intercept(s): After factoring we obtain $(x + 3)(x + 1) = 0$, and the zeroes of the numerator are $x = -1$ and $x = -3$, both with multiplicity one. The graph will cross the x-axis at each intercept.
4. Horizontal/oblique asymptote: The degree of numerator $>$ the degree of denominator, so we divide using term-by-term division:

$$\frac{x^2 + 4x + 3}{x} = \frac{x^2}{x} + \frac{4x}{x} + \frac{3}{x}$$
$$= x + 4 + \frac{3}{x}$$

The line $q(x) = x + 4$ is an oblique asymptote.

5. Solve

$$\frac{x^2 + 4x + 3}{x} = x + 4 \qquad q(x) = x + 4 \text{ is a slant asymptote}$$
$$x^2 + 4x + 3 = x^2 + 4x \qquad \text{cross multiply}$$
$$3 = 0 \qquad \text{no solutions possible}$$

The graph will not cross the slant asymptote.

The function changes sign at both x-intercepts and at the asymptote $x = 0$. The information from steps 1 through 5 is shown in Figure 3.50 and perhaps an additional point in Quadrant I would help to complete the graph: $A(1) = 8$. The point (1, 8) is on the graph, showing A is positive in the interval containing 1. Since output values will alternate in sign as stipulated above, all conditions are met with the graph shown in Figure 3.51.

Figure 3.50

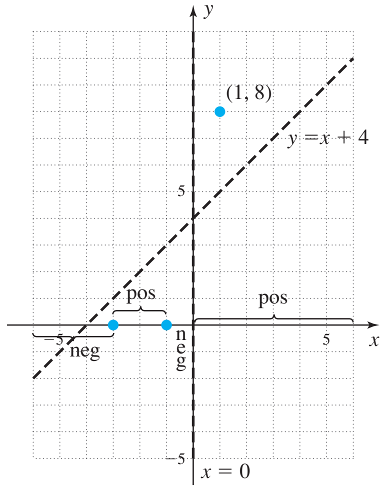
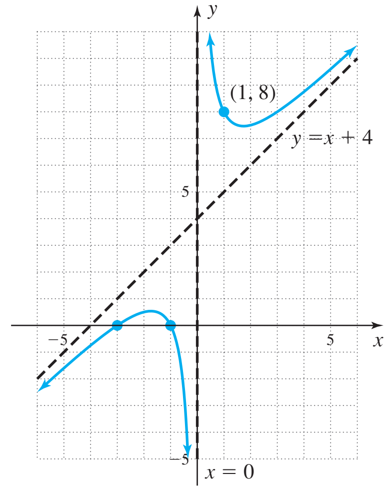


Figure 3.51



b. To find the number of items manufactured when average cost is \$8, we replace $A(x)$ with 8 and solve: $\frac{x^2 + 4x + 3}{x} = 8$:

$$x^2 + 4x + 3 = 8x$$
$$x^2 - 4x + 3 = 0$$
$$(x - 1)(x - 3) = 0$$
$$x = 1 \quad \text{or} \quad x = 3$$

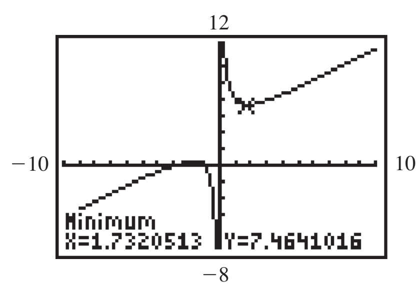
The average cost is \$8 when 1000 items or 3000 items are manufactured.

c. From the graph, it appears that the minimum average cost is close to \$7.50, when approximately 1500 to 1800 items are manufactured.

Now try Exercises 55 and 56 ►

GRAPHICAL SUPPORT

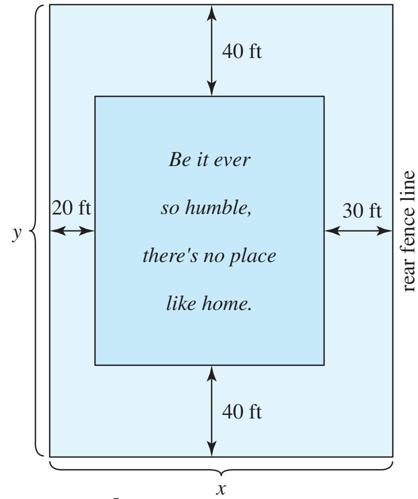
In the *Technology Highlight* from Section 2.5, we saw how a graphing calculator can be used to locate the extreme values of a function. Applying this technology to the graph from Example 4 we find that the minimum average cost is approximately \$7.46, when about 1732 items are manufactured.



In some applications, the functions we use are initially defined in *two variables* rather than just one, as in $H(x, y) = (x - 50)(y - 80)$. However, in the solution process a substitution is used to rewrite the relationship as a rational function in one variable and we can proceed as before.

EXAMPLE 5 ▶ **Using a Rational Function to Solve a Layout Application**

The building codes in a new subdivision require that a rectangular home be built at least 20 ft from the street, 40 ft from the neighboring lots, and 30 ft from the house to the rear fence line.



a. Find a function $A(x, y)$ for the area of the lot, and a function $H(x, y)$ for the area of the home (the inner rectangle).

b. If a new home is to have a floor area of 2000 ft^2, $H(x, y) = 2000$. Substitute 2000 for $H(x, y)$ and solve for y, then substitute the result in $A(x, y)$ to write the area A as a function of x alone (simplify the result).

c. Graph $A(x)$ on a calculator, using the window $X \in [-50, 150]$; $Y \in [-30{,}000, 30{,}000]$. Then graph $y = 80x + 2000$ on the same screen. How are these two graphs related?

d. Use the graph of $A(x)$ in Quadrant I to determine the minimum dimensions of a lot that satisfies the subdivision's requirements (to the nearest tenth of a foot). Also state the dimensions of the house.

Solution ▶ **a.** The area of the lot is simply width times length, so $A(x, y) = xy$. For the house, these dimensions are decreased by 50 ft and 80 ft, respectively, so $H(x, y) = (x - 50)(y - 80)$.

b. Given $H(x, y) = 2000$ produces the equation $2000 = (x - 50)(y - 80)$, and solving for y gives

$$2000 = (x - 50)(y - 80) \quad \text{given equation}$$

$$\frac{2000}{x - 50} = y - 80 \quad \text{divide by } x - 50$$

$$\frac{2000}{x - 50} + 80 = y \quad \text{add 80}$$

$$\frac{2000}{x - 50} + \frac{80(x - 50)}{x - 50} = y \quad \text{find LCD}$$

$$\frac{80x - 2000}{x - 50} = y \quad \text{combine terms}$$

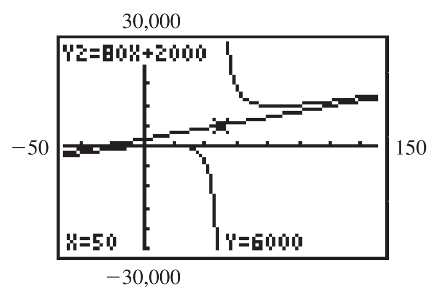
Substituting this expression for y in $A(x, y) = xy$ produces

$$A(x) = x\left(\frac{80x - 2000}{x - 50}\right) \quad \text{substitute } \frac{80x - 2000}{x - 50} \text{ for } y$$

$$= \frac{80x^2 - 2000x}{x - 50} \quad \text{multiply}$$

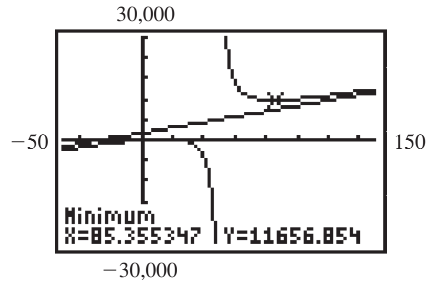
c. The graph of $Y1 = A(x)$ appears in Figure 3.52 using the prescribed window. $Y_2 = 80x + 2000$ appears to be an oblique asymptote for A, which can be verified using synthetic division.

Figure 3.52



d. Using the 2nd TRACE (CALC) **3:minimum** feature of a calculator, the minimum width is $x \approx 85.4$ ft. Substituting 85.4 for x in $y = \frac{80x - 2000}{x - 50}$, gives the length $y \approx 136.5$ ft. The dimensions of the house must be $85.4 - 50 = 35.4$ ft, by $136.5 - 80 = 56.5$ ft (see Figure 3.53).

Figure 3.53



As expected, the area of the house will be $(35.4)(56.5) \approx 2000$ ft^2.

C. You've just learned how to solve applications involving rational functions

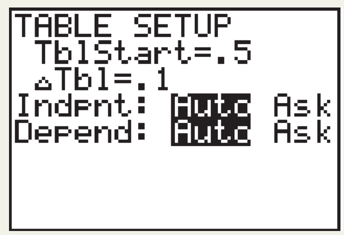
Now try Exercises 57 through 60 ▶

TECHNOLOGY HIGHLIGHT

Removable Discontinuities

Graphing calculators offer both numerical and visual representations of removable discontinuities. For instance, enter the function $r(x) = \frac{x^2 - 4x + 3}{x - 1}$ on the Y= screen, then use the TBLSET feature to set up the table as shown in Figure 3.54. Pressing 2nd GRAPH displays the expected table, which shows the function cannot be evaluated at $x = 1$ (see Figure 3.55). Now change the TBLSET screen so that **ΔTbl = 0.01.** Note again that the function is defined for all values except $x = 1$. Reset the table to **ΔTbl = 0.001** and investigate further.

Figure 3.54



We can actually see the gap or hole in the graph using a "friendly window." Since the screen of the TI-84 Plus is 95 pixels wide and 63 pixels high, multiples of 4.7 for Xmin and Xmax, and multiples of 3.1 for Ymin and Ymax, display what happens at integer (and other) values (see Figure 3.56). Pressing GRAPH gives Figure 3.57, which shows a noticeable gap at $(1, -2)$. With the TRACE feature, move the cursor over to the gap and notice what happens.

Use these ideas to view the discontinuities in the following rational functions. State the ordered pair location of each discontinuity.

Figure 3.55

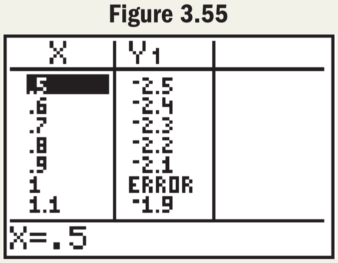


Figure 3.56

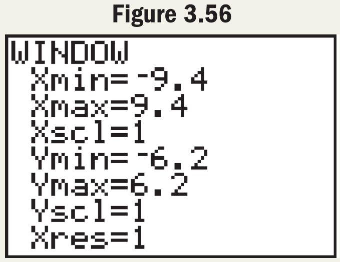
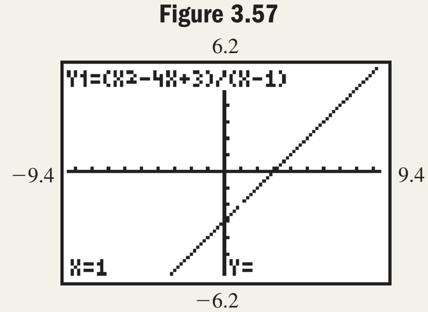


Figure 3.57



Exercise 1: $r(x) = \frac{x^2 - 4}{x + 2}$

Exercise 2: $f(x) = \frac{x^2 - 2x - 3}{x + 1}$

Exercise 3: $r(x) = \frac{x^3 + 1}{x + 1}$

Exercise 4: $f(x) = \frac{x^3 - 7x + 6}{x^2 + x - 6}$

3.6 EXERCISES

CONCEPTS AND VOCABULARY

Fill in each blank with the appropriate word or phrase. Carefully reread the section if needed.

1. The discontinuity in the graph of $y = \dfrac{1}{(x + 3)^2}$ is called a ________ discontinuity, since it cannot be "repaired."
2. If the degree of the numerator is greater than the degree of the denominator, the graph will have an ________ or ________ asymptote.
3. If the degree of the numerator is ________ more than the degree of the denominator, the graph will have a parabolic asymptote.
4. If the denominator is a ________, use term by term division to find the quotient. Otherwise ________ or long division must be used.
5. Discuss/Explain how you would create a function with a parabolic asymptote and two vertical asymptotes.
6. Complete Exercise 7 in expository form. That is, work this exercise out completely, discussing each step of the process as you go.

DEVELOPING YOUR SKILLS

Graph each function. If there is a removable discontinuity, repair the break using an appropriate piecewise-defined function.

7. $f(x) = \dfrac{x^2 - 4}{x + 2}$
8. $f(x) = \dfrac{x^2 - 9}{x + 3}$
9. $g(x) = \dfrac{x^2 - 2x - 3}{x + 1}$
10. $g(x) = \dfrac{x^2 - 3x - 10}{x - 5}$
11. $h(x) = \dfrac{3x - 2x^2}{2x - 3}$
12. $h(x) = \dfrac{4x - 5x^2}{5x - 4}$
13. $p(x) = \dfrac{x^3 - 8}{x - 2}$
14. $p(x) = \dfrac{8x^3 - 1}{2x - 1}$
15. $q(x) = \dfrac{x^3 - 7x - 6}{x + 1}$
16. $q(x) = \dfrac{x^3 - 3x + 2}{x + 2}$
17. $r(x) = \dfrac{x^3 + 3x^2 - x - 3}{x^2 + 2x - 3}$
18. $r(x) = \dfrac{x^3 - 2x^2 - 4x + 8}{x^2 - 4}$

Graph each function using the *Guidelines for Graphing Rational Functions*, which is simply modified to include nonlinear asymptotes. Clearly label all intercepts and asymptotes and any additional points used to sketch the graph.

19. $Y_1 = \dfrac{x^2 - 4}{x}$
20. $Y_2 = \dfrac{x^2 - x - 6}{x}$
21. $v(x) = \dfrac{3 - x^2}{x}$
22. $V(x) = \dfrac{7 - x^2}{x}$
23. $w(x) = \dfrac{x^2 + 1}{x}$
24. $W(x) = \dfrac{x^2 + 4}{2x}$
25. $h(x) = \dfrac{x^3 - 2x^2 + 3}{x^2}$
26. $H(x) = \dfrac{x^3 + x^2 - 2}{x^2}$
27. $Y_1 = \dfrac{x^3 + 3x^2 - 4}{x^2}$
28. $Y_2 = \dfrac{x^3 - 3x^2 + 4}{x^2}$
29. $f(x) = \dfrac{x^3 - 3x + 2}{x^2}$
30. $F(x) = \dfrac{x^3 - 12x - 16}{x^2}$
31. $Y_3 = \dfrac{x^3 - 5x^2 + 4}{x^2}$
32. $Y_4 = \dfrac{x^3 + 5x^2 - 6}{x^2}$
33. $r(x) = \dfrac{x^3 - x^2 - 4x + 4}{x^2}$
34. $R(x) = \dfrac{x^3 - 2x^2 - 9x + 18}{x^2}$

35. $g(x) = \dfrac{x^2 + 4x + 4}{x + 3}$ **36.** $G(x) = \dfrac{x^2 - 2x + 1}{x - 2}$

37. $f(x) = \dfrac{x^2 + 1}{x + 1}$ **38.** $F(x) = \dfrac{x^2 + x + 1}{x - 1}$

39. $Y_3 = \dfrac{x^2 - 4}{x + 1}$ **40.** $Y_4 = \dfrac{x^2 - x - 6}{x - 1}$

41. $v(x) = \dfrac{x^3 - 4x}{x^2 - 1}$ **42.** $V(x) = \dfrac{9x - x^3}{x^2 - 4}$

43. $w(x) = \dfrac{16x - x^3}{x^2 + 4}$ **44.** $W(x) = \dfrac{x^3 - 7x + 6}{2 + x^2}$

45. $Y_1 = \dfrac{x^3 - 3x + 2}{x^2 - 9}$ **46.** $Y_2 = \dfrac{x^3 - x^2 - 12x}{x^2 - 7}$

47. $p(x) = \dfrac{x^4 + 4}{x^2 + 1}$ **48.** $P(x) = \dfrac{x^4 - 5x^2 + 4}{x^2 + 2}$

49. $q(x) = \dfrac{10 + 9x^2 - x^4}{x^2 + 5}$ **50.** $Q(x) = \dfrac{x^4 - 2x^2 + 3}{x^2}$

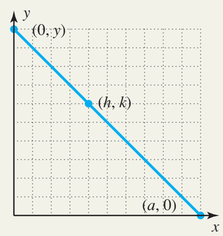
Graph each function and its nonlinear asymptote on the same screen, using the window specified. Then locate the minimum value of *f* in the first quadrant.

51. $f(x) = \dfrac{x^3 + 500}{x}$;
$x \in [-24, 24], y \in [-500, 500]$

52. $f(x) = \dfrac{2\pi x^3 + 750}{x}$;
$x \in [-12, 12], y \in [-750, 750]$

▶ WORKING WITH FORMULAS

53. Area of a first quadrant triangle: $A(a) = \dfrac{1}{2}\left(\dfrac{ka^2}{a - h}\right)$



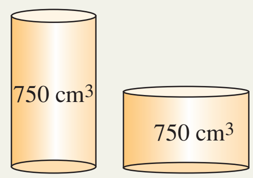
The area of a right triangle in the first quadrant, formed by a line with negative slope through the point (h, k) and legs that lie along the positive axes is given by the formula shown, where a represents the x-intercept of the resulting line ($h < a$). The area of the triangle varies with the slope of the line. Assume the line contains the point (5, 6).

a. Find the equation of the vertical and slant asymptotes.

b. Find the area of the triangle if it has an x-intercept of (11, 0).

c. Use a graphing calculator to graph the function on an appropriate window. Does the shape of the graph look familiar? Use the calculator to find the value of a that minimizes $A(a)$. That is, find the x-intercept that results in a triangle with the smallest possible area.

54. Surface area of a cylinder with fixed volume: $S = \dfrac{2\pi r^3 + 2V}{r}$



It's possible to construct many different cylinders that will hold a specified volume, by changing the radius and height. This is critically important to producers who want to minimize the cost of packing canned goods and marketers who want to present an attractive product. The surface area of the cylinder can be found using the formula shown, where the radius is r and $V = \pi r^2 h$ is known. Assume the fixed volume is 750 cm^3.

a. Find the equation of the vertical asymptote. How would you describe the nonlinear asymptote?

b. If the radius of the cylinder is 2 cm, what is its surface area?

c. Use a graphing calculator to graph the function on an appropriate window, and use it to find the value of r that minimizes $S(r)$. That is, find the radius that results in a cylinder with the smallest possible area, while still holding a volume of 750 cm^3.

▶ APPLICATIONS

Costs of manufacturing: As in Example 4, the cost $C(x)$ of manufacturing is sometimes nonlinear and can increase dramatically with each item. For the average cost function $A(x) = \dfrac{C(x)}{x}$, consider the following.

55. Assume the monthly cost of manufacturing custom-crafted storage sheds is modeled by the function $C(x) = 4x^2 + 53x + 250$.

a. Write the average cost function and state the equation of the vertical and oblique asymptotes.

b. Enter the cost function $C(x)$ as Y_1 on a graphing calculator, and the average cost function $A(x)$ as Y_2. Using the TABLE feature, find the cost and average cost of making 1, 2, and 3 sheds.

c. Scroll down the table to where it appears that average cost is a minimum. According to the table, how many sheds should be made each month to minimize costs? What is the minimum cost?

d. Graph the average cost function and its asymptotes, using a window that shows the entire function. Use the graph to confirm the result from part (c).

56. Assume the monthly cost of manufacturing playground equipment that combines a play house, slides, and swings is modeled by the function $C(x) = 5x^2 + 94x + 576$. The company has projected that they will be profitable if they can bring their average cost down to \$200 per set of playground equipment.

a. Write the average cost function and state the equation of the vertical and oblique asymptotes.

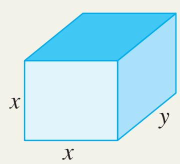
b. Enter the cost function $C(x)$ as Y_1 on a graphing calculator, and the average cost function $A(x)$ as Y_2. Using the TABLE feature, find the cost and average cost of making 1, 2, and 3 playground equipment combinations. Why would the average cost fall so dramatically early on?

c. Scroll down the table to where it appears that average cost is a minimum. According to the table, how many sets of equipment should be made each month to minimize costs? What is the minimum cost? Will the company be profitable under these conditions?

d. Graph the average cost function and its asymptotes, using a window that shows the entire function. Use the graph to confirm the result from part (c).

Minimum cost of packaging: Similar to Exercise 54, manufacturers can minimize their costs by shipping merchandise in packages that use a minimum amount of material. After all, rectangular boxes come in different sizes and there are many combinations of length, width, and height that will hold a specified volume.

57. A clothing manufacturer wishes to ship lots of 12 ft^3 of clothing in boxes with square ends and rectangular sides.



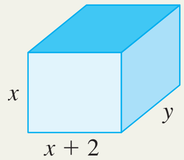
a. Find a function $S(x, y)$ for the surface area of the box, and a function $V(x, y)$ for the volume of the box.

b. Solve for y in $V(x, y) = 12$ (volume is 12 ft^3) and use the result to write the surface area as a function $S(x)$ in terms of x alone (simplify the result).

c. On a graphing calculator, graph the function $S(x)$ using the window $x \in [-8, 8]$; $y \in [-100, 100]$. Then graph $y = 2x^2$ on the same screen. How are these two graphs related?

d. Use the graph of $S(x)$ in Quadrant I to determine the dimensions that will minimize the surface area of the box, yet still hold 12 ft^3 of clothing. Clearly state the values of x and y, *in terms of feet and inches*, rounded to the nearest $\frac{1}{2}$ in.

58. A maker of packaging materials needs to ship 36 ft^3 of foam "peanuts" to his customers across the country, using boxes with the dimensions shown.



a. Find a function $S(x, y)$ for the surface area of the box, and a function $V(x, y)$ for the volume of the box.

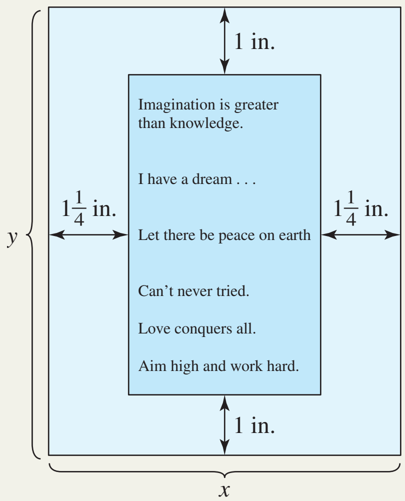
b. Solve for y in $V(x, y) = 36$ (volume is 36 ft^3), and use the result to write the surface area as a function $S(x)$ in terms of x alone (simplify the result).

c. On a graphing calculator, graph the function $S(x)$ using the window $x \in [-10, 10]$; $y \in [-200, 200]$. Then graph $y = 2x^2 + 4x$ on the same screen. How are these two graphs related?

d. Use the graph of $S(x)$ in Quadrant I to determine the dimensions that will minimize the surface area of the box, yet still hold the foam peanuts. Clearly state the values of x and y, *in terms of feet and inches*, rounded to the nearest $\frac{1}{2}$ in.

Printing and publishing: In the design of magazine pages, posters, and other published materials, an effort is made to maximize the usable area of the page while maintaining an attractive border, or minimizing the page size that will hold a certain amount of print or art work.

59. An editor has a story that requires 60 in^2 of print. Company standards require a 1-in. border at the top and bottom of a page, and 1.25-in. borders along both sides.



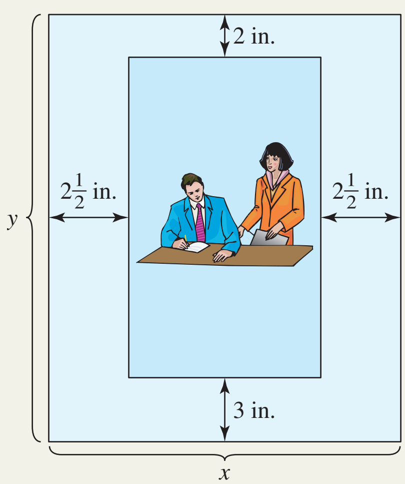
a. Find a function $A(x, y)$ for the area of the page, and a function $R(x, y)$ for the area of the inner rectangle (the printed portion).

b. Solve for y in $R(x, y) = 60$, and use the result to write the area from part (a) as a function $A(x)$ in terms of x alone (simplify the result).

c. On a graphing calculator, graph the function $A(x)$ using the window $x \in [-30, 30]$; $y \in [-100, 200]$. Then graph $y = 2x + 60$ on the same screen. How are these two graphs related?

d. Use the graph of $A(x)$ in Quadrant I to determine the page of minimum size that satisfies these border requirements and holds the necessary print. Clearly state the values of x and y, rounded to the nearest hundredth of an inch.

60. *The Poster Shoppe* creates posters, handbills, billboards, and other advertising for business customers. An order comes in for a poster with 500 in^2 of usable area, with margins of 2 in. across the top, 3 in. across the bottom, and 2.5 in. on each side.



a. Find a function $A(x, y)$ for the area of the page, and a function $R(x, y)$ for the area of the inner rectangle (the usable area).

b. Solve for y in $R(x, y) = 500$, and use the result to write the area from part (a) as a function $A(x)$ in terms of x alone (simplify the result).

c. On a graphing calculator, graph $A(x)$ using the window $x \in [-100, 100]$; $y \in [-800, 1600]$. Then graph $y = 5x + 500$ on the same screen. How are these two graphs related?

d. Use the graph of $A(x)$ in Quadrant I to determine the poster of minimum size that satisfies these border requirements and has the necessary usable area. Clearly state the values of x and y, rounded to the nearest hundredth of an inch.

61. The formula from Exercise 54 has an interesting derivation. The volume of a cylinder is $V = \pi r^2 h$, while the surface area is given by $S = 2\pi r^2 + 2\pi rh$ (the circular top and bottom + the area of the side).

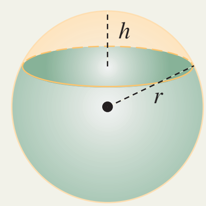
a. Solve the volume formula for the variable h.

b. Substitute the resulting expression for h into the surface area formula and simplify.

c. Combine the resulting two terms using the least common denominator, and the result is the formula from Exercise 54.

d. Assume the volume of a can must be 1200 cm^3. Use a calculator to graph the function S using an appropriate window, then use it to find the radius r and height h that will result in a cylinder with the smallest possible area, while still holding a volume of 1200 cm^3. Also see Exercise 62.

62. The surface area of a spherical cap is given by $S = 2\pi rh$, where r is the radius of the sphere and h is the perpendicular distance from the sphere's surface to the plane intersecting the sphere, forming the cap. The volume of the cap is $V = \frac{1}{3}\pi h^2(3r - h)$. Similar to Exercise 61, a formula can be found that will minimize the area of a cap that holds a specified volume.



a. Solve the volume formula for the variable r.

b. Substitute the resulting expression for r into the surface area formula and simplify. The result is a formula for surface area given solely in terms of the volume V and the height h.

c. Assume the volume of the spherical cap is 500 cm^3. Use a graphing calculator to graph the resulting function on an appropriate window, and use the graph to find the height h that will result in a spherical cap with the smallest possible area, while still holding a volume of 500 cm^3.

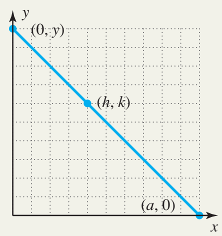
d. Use this value of h and $V = 500$ cm^3 to find the radius of the sphere.

▶ EXTENDING THE CONCEPT

63. Consider rational functions of the form $f(x) = \frac{x^2 - a}{x - b}$. Use a graphing calculator to explore cases where $a = b^2 + 1$, $a = b^2$, and $a = b^2 - 1$. What do you notice? Explain/Discuss why the graphs differ. It's helpful to note that when graphing functions of this form, the "center" of the graph will be at $(b, b^2 - a)$, and the window size can be set accordingly for an optimal view. Do some investigation on this function and determine/explain *why* the "center" of the graph is at $(b, b^2 - a)$.

64. The formula from Exercise 53 also has an interesting derivation, and the process involves this sequence:

a. Use the points $(a, 0)$ and (h, k) to find the slope of the line, and the point-slope formula to find the equation of the line in terms of y.



b. Use this equation to find the x- and y-intercepts of the line in terms of a, k, and h.

c. Complete the derivation using these intercepts and the triangle formula $A = \frac{1}{2}BH$.

d. If the lines goes through (4, 4) the area formula becomes $A = \frac{1}{2}\left(\frac{4a^2}{a - 4}\right)$. Find the minimum value of this rational function. What can you say about the triangle with minimum area through (h, k), where $h = k$? Verify using the points (5, 5), and (6, 6).

65. Referring to Exercises 54 and 61, suppose that instead of a closed cylinder, with both a top and bottom, we needed to manufacture *open cylinders,* like tennis ball cans that use a lid made from a different material. Derive the formula that will minimize the surface area of an open cylinder, and use it to find the cylinder with minimum surface area that will hold 90 in^3 of material.

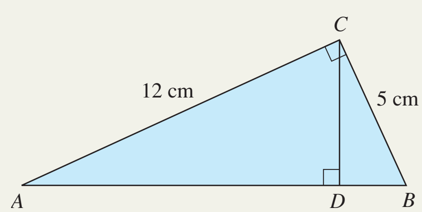
▶ MAINTAINING YOUR SKILLS

66. **(1.4)** Compute the quotient $\frac{5i}{1 + 2i}$, then check your answer using multiplication.

67. **(2.3)** Write the equation of the line in slope intercept form and state the slope and y-intercept: $-3x + 4y = -16$.

68. **(1.5)** Given $f(x) = ax^2 + bx + c$, for what real values of a, b, and c will the function have: (a) two, real/rational roots, (b) two, real/irrational roots, (c) one real and rational root, (d) one real/irrational root, (e) one complex root, and (f) two complex roots?

69. **(R.2/1.5)** For triangle ABC as shown, (a) find the perimeter; (b) find the length of $\overline{CD}$, given $(\overline{CB})^2 = \overline{AB} \cdot \overline{DB}$; (c) find the area; and (d) find the area of the two smaller triangles.



3.7 Polynomial and Rational Inequalities

Learning Objectives

In Section 3.7 you will learn how to:

- ❑ **A.** Solve quadratic inequalities
- ❑ **B.** Solve polynomial inequalities
- ❑ **C.** Solve rational inequalities
- ❑ **D.** Use interval tests to solve inequalities
- ❑ **E.** Solve applications of inequalities

The study of polynomial and rational inequalities is simply an extension of our earlier work in analyzing functions (Section 2.5). While we've developed the ability to graph a variety of new functions, solution sets will still be determined by analyzing the behavior of the function at its zeroes, and in the case of rational functions, on either side of any vertical asymptotes. The key idea is to recognize the following statements are synonymous:

1. $f(x) > 0$. **2.** Outputs are positive. **3.** The graph is *above the x-axis*.

Similar statements can be made using the other inequality symbols.

A. Quadratic Inequalities

Solving a quadratic inequality only requires that we (a) locate any real zeroes of the function and (b) determine whether the graph opens upward or downward. If there are no x-intercepts, the graph is entirely above the x-axis (output values are positive), or entirely below the x-axis (output values are negative), making the solution either all real numbers or the empty set.

EXAMPLE 1 ▶ Solving a Quadratic Inequality

For $f(x) = x^2 + x - 6$, solve $f(x) > 0$.

Solution ▶ The graph of f will open upward since $a > 0$. Factoring gives $f(x) = (x + 3)(x - 2)$, with zeroes at -3 and 2. Using a the x-axis alone (since graphing the function is not our focus), we plot $(-3, 0)$ and $(2, 0)$ and visualize a parabola opening upward through these points (Figure 3.58).

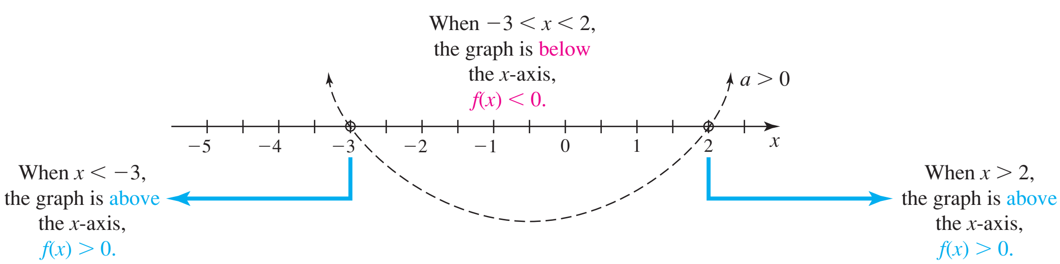
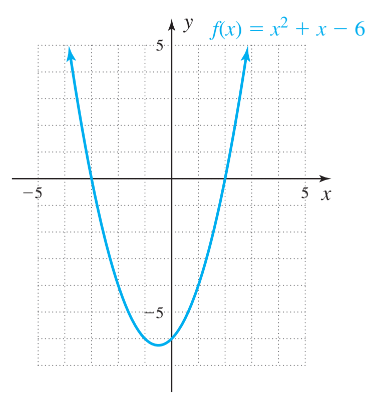


Figure 3.58

Figure 3.59



The diagram clearly shows the graph is *above* the x-axis (outputs are positive) when $x < -3$ or when $x > 2$. The solution is $x \in (-\infty, -3) \cup (2, \infty)$. For reference only, the complete graph is given in Figure 3.59.

Now try Exercises 7 through 18 ▶

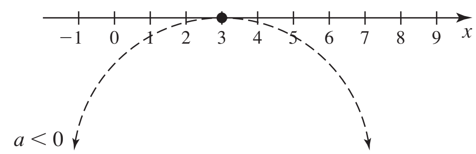
When solving general inequalities, zeroes of multiplicity continue to play a role. In Example 1, the zeroes of f were both of multiplicity 1, and the graph crossed the x-axis at these points. In other cases, the zeroes may have even multiplicity.

EXAMPLE 2 ► **Solving a Quadratic Inequality**

Solve the inequality $-x^2 + 6x \leq 9$.

Solution ► Begin by writing the inequality in standard form: $-x^2 + 6x - 9 \leq 0$. Note this is equivalent to $g(x) \leq 0$ for $g(x) = -x^2 + 6x - 9$. Since $a < 0$, the graph of g will open downward. The factored form is $g(x) = -(x - 3)^2$, showing 3 is a zero with multiplicity 2. Using the x-axis, we plot the point (3, 0) and visualize a parabola opening downward through this point.

Figure 3.60

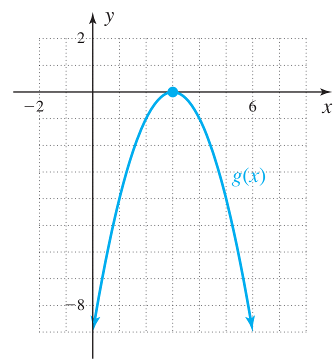


WORTHY OF NOTE

Since $x = 3$ was a zero of multiplicity 2, the graph "bounced off" the x-axis at this point, with no change of sign for g. The graph is entirely below the x-axis, except at the vertex (3, 0).

Figure 3.60 shows the graph is *below* the x-axis (outputs are negative) for *all values* of x except $x = 3$. But since this is a less than *or equal to* inequality, the solution is $x \in \mathbb{R}$. For reference only, the complete graph is given in Figure 3.61.

Figure 3.61



A. You've just learned how to solve quadratic inequalities

Now try Exercises 19 through 36 ►

B. Polynomial Inequalities

The reasoning in Examples 1 and 2 transfers seamlessly to inequalities involving higher degree polynomials. After writing the polynomial in standard form, find the zeroes, plot them on the x-axis, and determine the solution set using end behavior and the behavior at each zero (cross—sign change; or bounce—no change in sign). In this process, any irreducible quadratic factors can be ignored, as they have no effect on the solution set. In summary,

Solving Polynomial Inequalities

Given $f(x)$ is a polynomial in standard form,

1. Write f in completely factored form.
2. Plot real zeroes on the x-axis, noting their multiplicity.
   - If the multiplicity is odd the function will **change** sign.
   - If the multiplicity is even, there will be **no change** in sign.
3. Use the end behavior to determine the sign of f in the outermost intervals, then label the other intervals as $f(x) < 0$ or $f(x) > 0$ by analyzing the multiplicity of neighboring zeroes.
4. State the solution in interval notation.

EXAMPLE 3 ▶ Solving a Polynomial Inequality

Solve the inequality $x^3 - 18 < -4x^2 + 3x$.

Solution ▶ In standard form we have $x^3 + 4x^2 - 3x - 18 < 0$, which is equivalent to $f(x) < 0$ where $f(x) = x^3 + 4x^2 - 3x - 18$. The polynomial cannot be factored by grouping and testing 1 and -1 shows neither is a zero. Using $x = 2$ and synthetic division gives

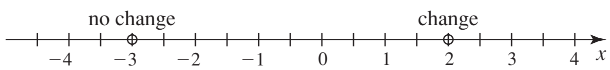
use 2 as a "divisor"

2⌋	1	4	−3	−18
	↓	2	12	18
	1	6	9	0,

with a quotient of $x^2 + 6x + 9$ and a remainder of zero.

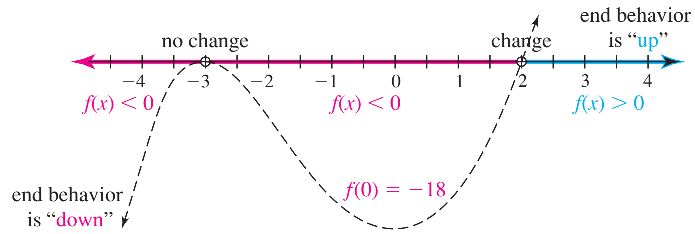
1. The factored form is $f(x) = (x - 2)(x^2 + 6x + 9) = (x - 2)(x + 3)^2$.
2. The graph will bounce off the x-axis at $x = -3$ (f will not change sign), and cross the x-axis at $x = 2$ (f will change sign). This is illustrated in Figure 3.62, which uses open dots due to the strict inequality.

Figure 3.62



3. The polynomial has odd degree with a positive lead coefficient, so end behavior is down/up, which we note in the outermost intervals. Working from the left, f will not change sign at $x = -3$, showing $f(x) < 0$ in the left and middle intervals. This is supported by the y-intercept $(0, -18)$. See Figure 3.63.

Figure 3.63

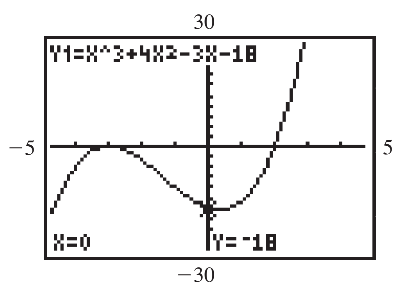


4. From the diagram, we see that $f(x) < 0$ for $x \in (-\infty, -3) \cup (-3, 2)$, which must also be the solution interval for $x^3 - 18 < -4x^2 + 3x$.

Now try Exercises 37 through 48 ▶

GRAPHICAL SUPPORT

The results from Example 3 can easily be verified using a graphing calculator. The graph shown here is displayed using a window of $X \in [-5, 5]$ and $Y \in [-30, 30]$, and definitely shows the graph is below the x-axis $[f(x) < 0]$ from $-\infty$ to 2, except at $x = -3$ where the graph touches the x-axis without crossing.



EXAMPLE 4 ▶ **Solving a Polynomial Inequality**

Solve the inequality $x^4 + 4x \leq 9x^2 - 12$.

Solution ▶ Writing the polynomial in standard form gives $x^4 - 9x^2 + 4x + 12 \leq 0$. The equivalent inequality is $f(x) \leq 0$. Testing 1 and -1 shows $x = 1$ is not a zero, but $x = -1$ is. Using synthetic division with $x = -1$ gives

use -1 as a "divisor"

-1	1	**0**	-9	4	12
	↓	-1	1	8	-12
	1	-1	-8	12	0

with a quotient of $q_1(x) = x^3 - x^2 - 8x + 12$ and a remainder of zero. As $q_1(x)$ is not easily factored, we continue with synthetic division using $x = 2$.

use 2 as a "divisor"

2	1	-1	-8	12
	↓	2	2	-12
	1	1	-6	0

The result is $q_2(x) = x^2 + x - 6$ with a remainder of zero.

1. The factored form is $f(x) = (x + 1)(x - 2)(x^2 + x - 6) = (x + 1)(x - 2)^2(x + 3)$.
2. The graph will "cross" at $x = -1$ and -3, and f will change sign. The graph will bounce at $x = 2$ and f will not change sign. This is illustrated in Figure 3.65 which uses closed dots since $f(x)$ can be equal to zero. See Figure 3.64.

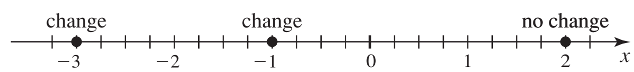


Figure 3.64

3. With even degree and positive lead coefficient, the end behavior is up/up. Working from the leftmost interval, $f(x) > 0$, the function must change sign at $x = -3$ (going below the x-axis), and again at $x = -1$ (going above the x-axis). This is supported by the y-intercept $(0, 12)$. The graph then "bounces" at $x = 2$, remaining above the x-axis (no sign change). This produces the sketch shown in Figure 3.65.

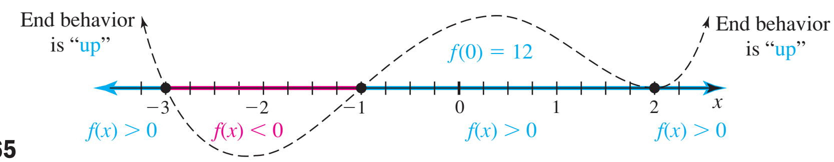


Figure 3.65

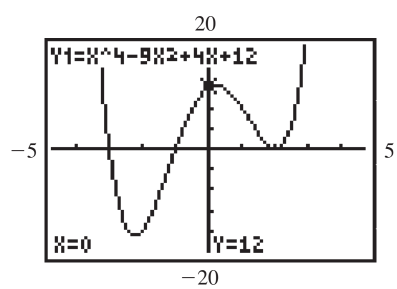
4. From the diagram, we see that $f(x) \leq 0$ for $x \in [-3, -1]$, and at the single point $x = 2$. This shows the solution for $x^4 + 4x \leq 9x^2 - 12$ is $x \in [-3, -1] \cup \{2\}$.

☑ **B.** You've just learned how to solve polynomial inequalities

Now try Exercises 49 through 54 ▶

GRAPHICAL SUPPORT

As with Example 3, the results from Example 4 can be confirmed using a graphing calculator. The graph shown here is displayed using $X \in [-5, 5]$ and $Y \in [-20, 20]$. The graph is below or touching the x-axis $[f(x) < 0]$ from -3 to -1 and at $x = 2$.



C. Rational Inequalities

In general, the solution process for polynomial and rational inequalities is virtually identical, once we recognize that vertical asymptotes also break the *x*-axis into intervals where function values may change sign. However, for rational functions it's more efficient to begin the analysis using the *y*-intercept or a test point, rather than end behavior, although either will do.

EXAMPLE 5 ▶ Solving a Rational Inequality

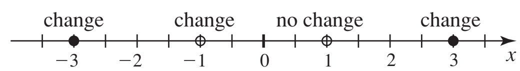
Solve $\dfrac{x^2 - 9}{x^3 - x^2 - x + 1} \le 0.$

Solution ▶ In function form, $v(x) = \dfrac{x^2 - 9}{x^3 - x^2 - x + 1}$ and we want the solution for $v(x) \le 0$.

The numerator and denominator are in standard form. The numerator factors easily, and the denominator can be factored by grouping.

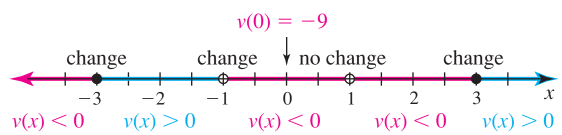
1. The factored form is $v(x) = \dfrac{(x - 3)(x + 3)}{(x - 1)^2(x + 1)}$.
2. $v(x)$ will change sign at $x = 3, -3$, and -1 as all have odd multiplicity, but will not change sign at $x = 1$ (even multiplicity). Note that zeroes of the denominator will always be indicated by open dots (Figure 3.66) as they are excluded from any solution set.

Figure 3.66



3. The *y*-intercept is $(0, -9)$, indicating that function values will be negative in the interval containing zero. Working outward from this interval using the "change/no change" approach, gives the solution indicated in Figure 3.67.

Figure 3.67



4. For $v(x) \le 0$, the solution is $x \in (-\infty, -3] \cup (-1, 1) \cup (1, 3]$.

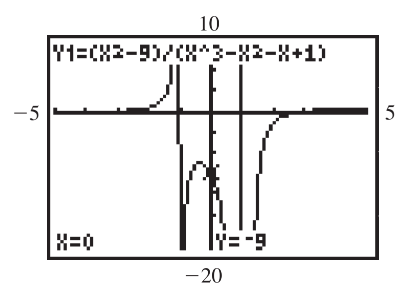
Now try Exercises 55 through 66 ▶

WORTHY OF NOTE

End behavior can also be used to analyze rational inequalities, although using the *y*-intercept may be more efficient. For the function $v(x)$ from Example 5 we have $\dfrac{x^2 - 9}{x^3 - x^2 - x + 1} \approx \dfrac{x^2}{x^3} = \dfrac{1}{x}$ for large values of x, indicating $v(x) > 0$ to the far right and $v(x) < 0$ to the far left. The analysis of each interval can then begin from either side.

GRAPHICAL SUPPORT

Sometimes finding a window that clearly displays all features of rational function can be difficult. In these cases, we can investigate each piece separately to confirm solutions. For Example 5, most of the features of $v(x)$ can be seen using a window $X \in [-5, 5]$ and $Y \in [-20, 10]$, and we note the graph displayed strongly tends to support our solution.



If the rational inequality is not given in function form or is composed of more than one term, start by writing the inequality with zero on one side, then combine terms into a single expression.

EXAMPLE 6 ▶ **Solving a Rational Inequality**

Solve $\frac{x-2}{x-3} \le \frac{1}{x+3}$.

Solution ▶ Rewrite the inequality with zero on one side: $\frac{x-2}{x-3} - \frac{1}{x+3} \le 0$. This is equivalent to $v(x) \le 0$, where $v(x) = \frac{x-2}{x-3} - \frac{1}{x+3}$. Combining the expressions on the right, we have

$$v(x) = \frac{(x-2)(x+3) - 1(x-3)}{(x+3)(x-3)} \qquad \text{LCD is } (x+3)(x-3)$$

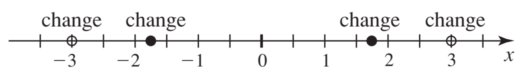
$$= \frac{x^2 + x - 6 - x + 3}{(x+3)(x-3)} \qquad \text{multiply}$$

$$= \frac{x^2 - 3}{(x+3)(x-3)} \qquad \text{simplify}$$

1. The factored form is $v(x) = \frac{(x+\sqrt{3})(x-\sqrt{3})}{(x+3)(x-3)}$. $\qquad x^2 - k = (x+\sqrt{k})(x-\sqrt{k})$

2. $v(x)$ will change sign at $x = -\sqrt{3}, \sqrt{3}, -3$, and 3, as all have odd multiplicity (Figure 3.68).

Figure 3.68



3. Since $v(0) = \frac{1}{3}$ (verify this), function values will be positive in the interval containing zero. Working outward from this interval produces the diagram shown in Figure 3.69.

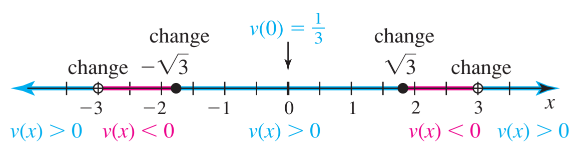


Figure 3.69

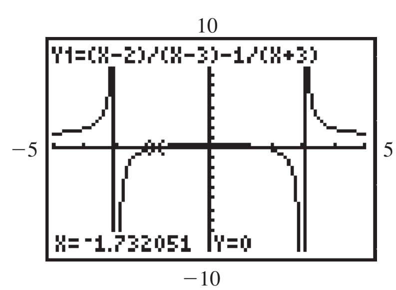
4. The solution for $\frac{x-2}{x-3} \le \frac{1}{x+3}$ is $x \in (-3, -\sqrt{3}] \cup [\sqrt{3}, 3)$.

☑ **C.** You've just learned how to solve rational inequalities

Now try Exercises 67 through 82 ▶

GRAPHICAL SUPPORT

To check the solutions to $\frac{x-2}{x-3} \le \frac{1}{x+3}$, we subtract $\frac{1}{x+3}$ and graph $Y_1 = \frac{x-2}{x-3} - \frac{1}{x+3}$ to look for intervals where the graph is below the x-axis. The graph is shown here using the window $X \in [-5, 5]$ and $y \in [-10, 10]$, and verifies our solution.



D. Solving Function Inequalities Using Interval Tests

As an alternative to the "zeroes method," an **interval test method** can be used to solve polynomial and rational inequalities. The x-intercepts and vertical asymptotes (in the case of rational functions) are noted on the x-axis, then a test number is selected from each interval. Since polynomial and rational functions are continuous over their entire domain, the sign of the function at these test values will be the sign of the function for all values of x in the chosen interval.

EXAMPLE 7 ▶ Solving a Polynomial Inequality

Solve the inequality $x^3 + 8 \le 5x^2 - 2x$.

Solution ▶ Writing the relationship in function form gives $p(x) = x^3 - 5x^2 + 2x + 8$, with solutions needed to $p(x) \le 0$. The tests for 1 and -1 show $x = -1$ is a root, and using -1 with synthetic division gives

use −1 as a "divisor"

$-1\rfloor$	1	-5	2	8
	↓	-1	6	-8
	1	-6	8	$\lfloor 0$

The quotient is $q(x) = x^2 - 6x + 8$, with a remainder of 0.

The factored form is $p(x) = (x + 1)(x^2 - 6x + 8) = (x + 1)(x - 2)(x - 4)$. The x-intercepts are $(-1, 0)$, $(2, 0)$, and $(4, 0)$. Plotting these intercepts creates four intervals on the x-axis (Figure 3.70).

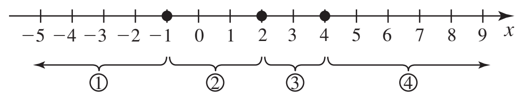


Figure 3.70

Selecting a test value from each interval gives Figure 3.71.

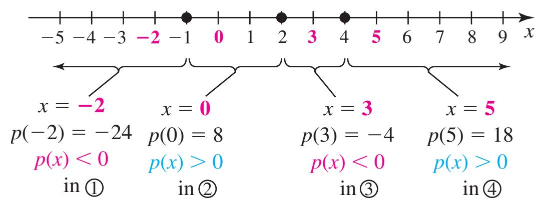


Figure 3.71

The interval tests show $x^3 + 8 \le 5x^2 - 2x$ for $x \in (-\infty, -1] \cup [2, 4]$.

Now try Exercises 83 through 90 ▶

WORTHY OF NOTE

When evaluating a function using the interval test method, it's usually easier to use the factored form instead of the polynomial form, since all you really need is whether the result will be positive or negative. For instance, you could likely tell $p(3) = (3 + 1)(3 - 2)(3 - 4)$ is going to be negative, more quickly than $p(3) = (3)^3 - 5(3)^2 + 2(3) + 8$.

☑ **D.** You've just learned how to use interval tests to solve inequalities

E. Applications of Inequalities

Applications of inequalities come in many varieties. In addition to stating the solution algebraically, these exercises often compel us to consider the *context of each application* as we state the solution set.

EXAMPLE 8 ▶ Solving Applications of Inequalities

The velocity of a particle (in feet per second) as it floats through air turbulence is given by $V(t) = t^5 - 10t^4 + 35t^3 - 50t^2 + 24t$, where t is the time in seconds and $0 < t < 4.5$. During what intervals of time is the particle moving in the positive direction $[V(t) > 0]$?

Solution ▶ Begin by writing V in factored form. Testing 1 and -1 shows $t = 1$ is a root. Factoring out t gives $V(t) = t(t^4 - 10t^3 + 35t^2 - 50t + 24)$, and using $t = 1$ with synthetic division yields

use 1 as a "divisor"

$1\rfloor$	1	-10	35	-50	24
	↓	1	-9	26	-24
	1	-9	26	-24	$\lfloor 0$

The quotient is $q_1(t) = t^3 - 9t^2 + 26t - 24$. Using $t = 2$, we continue with the division on $q_1(t)$ which gives:

use 2 as a "divisor"				
$\underline{2}\vert$	1	−9	26	−24
	↓	2	−14	24
	1	−7	12	$\vert\underline{0}$

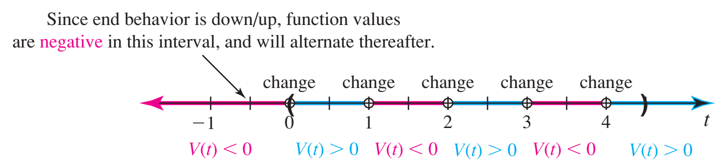
This shows $V(t) = t(t - 1)(t - 2)(t^2 - 7t + 12)$.

1. The completely factored form is $V(t) = t(t - 1)(t - 2)(t - 3)(t - 4)$.

2. All zeroes have odd multiplicity and function values will change sign.

3. With odd degree and a positive leading coefficient, end behavior is down/up.

Function values will be negative in the far left interval and alternate in sign thereafter. The solution diagram is shown in the figure.



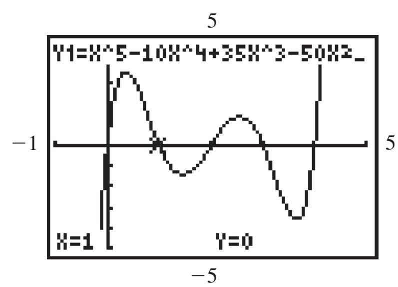
E. You've just learned how to solve applications of inequalities

4. For $V(t) > 0$, the solution is $t \in (0, 1) \cup (2, 3) \cup (4, 4.5)$. The particle is moving in the positive direction in these time intervals.

Now try Exercises 93 through 100 ►

GRAPHICAL SUPPORT

To verify our analysis of Example 8, we graph $V(t)$ using the window $X \in [-1, 5]$ and $Y \in [-5, 5]$. As the graph shows, function values are positive (graph is above the x-axis) when $t \in (0, 1) \cup (2, 3) \cup (4, 4.5)$. Also see the *Technology Highlight* for this section.

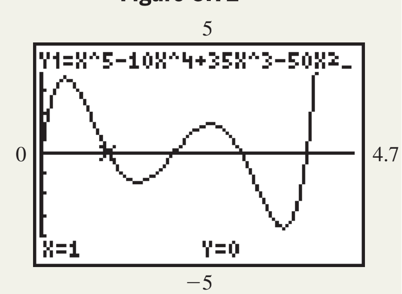


TECHNOLOGY HIGHLIGHT

Polynomial and Rational Inequalities

Consider the results from Example 8, where we solved the inequality $V(t) > 0$ for $V(t) = t^5 - 10t^4 + 35t^3 - 50t^2 + 24t$. To emphasize that we are seeking intervals where the function is above the x-axis (the horizontal line $y = 0$), we can have the calculator *shade these areas.* Begin by entering $V(t)$ as Y_1 on the Y= screen, and the line $y = 0$ as Y_2. Using $x \in [0, 4.7]$ and $y \in [-5, 5]$ (a "friendly" window) and GRAPHing the functions produces Figure 3.72. To shade all portions of the graph that are above the x-axis, go to the home screen, and press 2nd PRGM **(DRAW) 7:Shade.** This feature requires six arguments, all separated by commas. These are (in order): *lower function, upper function, left endpoint, right endpoint, pattern choice,* and *density*. The calculator will then shade the area between the lower and upper functions,

Figure 3.72



between the left and right endpoints, using the pattern and density chosen. The patterns are (1) vertical lines, (2) horizontal lines, (3) lines with negative slope, and (4) lines with positive slope. There are eight density settings, from every pixel (1), to every eight pixels (8). Figure 3.73 shows the options we've selected, with the resulting graph shown in Figure 3.74. The friendly window makes it easy to investigate the inequality further using the TRACE feature.

Use these ideas to visually study and explore the solution to the following inequality.

Figure 3.73

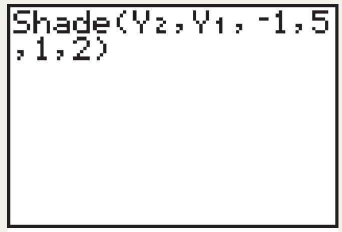
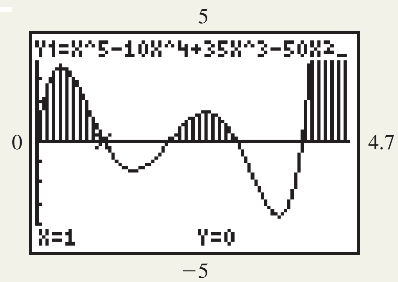


Figure 3.74



Exercise 1: Use window size $x \in [-4.7, 4.7]$; $y \in [-10, 20]$, **(DRAW) 7:Shade**, and TRACE to solve $P(x) < 0$ for $P(x) = x^4 + 1.1x^3 - 9.37x^2 - 4.523x + 16.4424$.

3.7 EXERCISES

CONCEPTS AND VOCABULARY

Fill in each blank with the appropriate word or phrase. Carefully reread the section if needed.

1. To solve a polynomial or rational inequality, begin by plotting the location of all zeroes and ______ asymptotes (if they exist), then consider the ______ of each.

2. For strict inequalities, the zeroes are ______ from the solution set. For nonstrict inequalities, zeroes are ______. The values at which vertical asymptotes occur are always ______.

3. If the graph of a quadratic function $g(x)$ opens downward with a vertex at $(5, -1)$, the solution set for $g(x) > 0$ is ______.

4. To solve a polynomial/rational inequality, it helps to find the sign of f in some interval. This can quickly be done using the ______ ______ or ______ of the function.

5. Compare/Contrast the process for solving $x^2 - 3x - 4 \geq 0$ with $\dfrac{1}{x^2 - 3x - 4} \geq 0$. Are there similarities? What are the differences?

6. Compare/Contrast the process for solving $(x + 1)(x - 3)(x^2 + 1) > 0$ with $(x + 1)(x - 3) > 0$. Are there similarities? What are the differences?

DEVELOPING YOUR SKILLS

Solve each quadratic inequality by locating the x-intercept(s) (if they exist), and noting the end behavior of the graph. Begin by writing the inequality in function form as needed.

7. $f(x) = -x^2 + 4x$; $f(x) > 0$

8. $g(x) = x^2 - 5x$; $g(x) < 0$

9. $h(x) = x^2 + 4x - 5$; $h(x) \geq 0$

10. $p(x) = -x^2 + 3x + 10$; $p(x) \leq 0$

11. $q(x) = 2x^2 - 5x - 7$; $q(x) < 0$

12. $r(x) = -2x^2 - 3x + 5$; $r(x) > 0$

13. $7 \geq x^2$

14. $x^2 \leq 13$

15. $x^2 + 3x \leq 6$

16. $x^2 - 2 \leq 5x$

17. $3x^2 \geq -2x + 5$

18. $4x^2 \geq 3x + 7$

19. $s(x) = x^2 - 8x + 16; s(x) \geq 0$

20. $t(x) = x^2 - 6x + 9; t(x) \geq 0$

21. $r(x) = 4x^2 + 12x + 9; r(x) < 0$

22. $f(x) = 9x^2 - 6x + 1; f(x) < 0$

23. $g(x) = -x^2 + 10x - 25; g(x) < 0$

24. $h(x) = -x^2 + 14x - 49; h(x) < 0$

25. $-x^2 > 2$ **26.** $x^2 < -4$

27. $x^2 - 2x > -5$ **28.** $-x^2 + 3x < 3$

29. $2x^2 \geq 6x - 9$ **30.** $5x^2 \geq 4x - 4$

Recall that for a square root expression to represent a real number, the radicand must be greater than or equal to zero. Applying this idea results in an inequality that can be solved using the skills from this section. Determine the domain of the following radical functions.

31. $h(x) = \sqrt{x^2 - 25}$ **32.** $p(x) = \sqrt{25 - x^2}$

33. $q(x) = \sqrt{x^2 - 5x}$ **34.** $r(x) = \sqrt{6x - x^2}$

35. $t(x) = \sqrt{-x^2 + 3x - 4}$

36. $Y_1 = \sqrt{x^2 - 6x + 9}$

Solve the inequality indicated using a number line and the behavior of the graph at each zero. Write all answers in interval notation.

37. $(x + 3)(x - 5) < 0$ **38.** $(x - 2)(x + 7) < 0$

39. $(x + 1)^2(x - 4) \geq 0$ **40.** $(x + 6)(x - 1)^2 \leq 0$

41. $(x + 2)^3(x - 2)^2(x - 4) \geq 0$

42. $(x - 1)^3(x + 2)^2(x - 3) \leq 0$

43. $x^2 + 4x + 1 < 0$ **44.** $x^2 - 6x + 4 > 0$

45. $x^3 + x^2 - 5x + 3 \leq 0$

46. $x^3 + x^2 - 8x - 12 \geq 0$

47. $x^3 - 7x + 6 > 0$ **48.** $x^3 - 13x + 12 > 0$

49. $x^4 - 10x^2 > -9$ **50.** $x^4 + 36 < 13x^2$

51. $x^4 - 9x^2 > 4x - 12$ **52.** $x^4 - 16 > 5x^3 - 20x$

53. $x^4 - 6x^3 \leq -8x^2 - 6x + 9$

54. $x^4 - 3x^2 + 8 \leq 4x^3 - 10x$

55. $f(x) = \dfrac{x + 3}{x - 2}; f(x) \leq 0$

56. $F(x) = \dfrac{x - 4}{x + 1}; F(x) \geq 0$

57. $g(x) = \dfrac{x + 1}{x^2 + 4x + 4}; g(x) < 0$

58. $G(x) = \dfrac{x - 3}{x^2 - 2x + 1}; G(x) > 0$

59. $\dfrac{2 - x}{x^2 - x - 6} \geq 0$ **60.** $\dfrac{1 - x}{x^2 - 2x - 8} \leq 0$

61. $\dfrac{2x - x^2}{x^2 + 4x - 5} < 0$ **62.** $\dfrac{x^2 + 3x}{x^2 - 2x - 3} > 0$

63. $\dfrac{x^2 - 4}{x^3 - 13x + 12} \geq 0$ **64.** $\dfrac{x^2 + x - 6}{x^3 - 7x + 6} \leq 0$

65. $\dfrac{x^2 + 5x - 14}{x^3 + x^2 - 5x + 3} > 0$

66. $\dfrac{x^2 + 2x - 8}{x^3 + 5x^2 + 3x - 9} < 0$

67. $\dfrac{2}{x - 2} \leq \dfrac{1}{x}$ **68.** $\dfrac{5}{x + 3} \geq \dfrac{3}{x}$

69. $\dfrac{x - 3}{x + 17} > \dfrac{1}{x - 1}$ **70.** $\dfrac{1}{x + 5} < \dfrac{x - 2}{x - 7}$

71. $\dfrac{x + 1}{x - 2} \geq \dfrac{x + 2}{x + 3}$ **72.** $\dfrac{x - 3}{x - 6} \leq \dfrac{x + 1}{x + 4}$

73. $\dfrac{x + 2}{x^2 + 9} > 0$ **74.** $\dfrac{x^2 + 4}{x - 3} < 0$

75. $\dfrac{x^3 + 1}{x^2 + 1} > 0$ **76.** $\dfrac{x^2 + 4}{x^3 - 8} < 0$

77. $\dfrac{x^4 - 5x^2 - 36}{x^2 - 2x + 1} > 0$ **78.** $\dfrac{x^4 - 3x^2 - 4}{x^2 - x - 20} < 0$

79. $x^2 - 2x \geq 15$ **80.** $x^2 + 3x \geq 18$

81. $x^3 \geq 9x$ **82.** $x^3 \leq 4x$

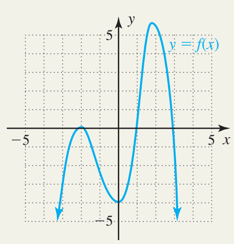
83. $-4x + 12 < -x^3 + 3x^2$

84. $x^3 + 8 < 5x^2 - 2x$ **85.** $\dfrac{x^2 - x - 6}{x^2 - 1} \geq 0$

86. $\dfrac{x^2 - 4x - 21}{x - 3} < 0$

Match the correct solution with the inequality and graph given.

87. $f(x) < 0$



a. $x \in (-5, -2) \cup (3, 5)$

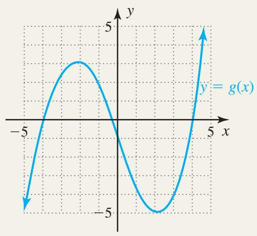
b. $x \in (-\infty, -2) \cup (-2, 1) \cup (3, \infty)$

c. $x \in (-\infty, -2) \cup (3, \infty)$

d. $x \in (-\infty, -2) \cup (-2, 1] \cup [3, \infty)$

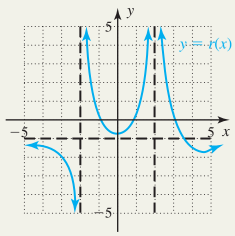
e. none of these

88. $g(x) \geq 0$



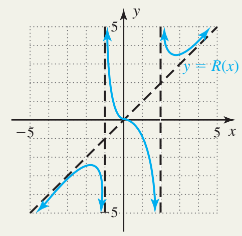
a. $x \in (-4, -0.5) \cup (4, \infty)$
b. $x \in [-0.5, 4] \cup [4, 5]$
c. $x \in (-\infty, -4) \cup (-0.5, 4)$
d. $x \in [-4, -0.5] \cup [4, \infty)$
e. none of these

89. $r(x) \geq 0$



a. $x \in (-\infty, -2) \cup [-1, 1] \cup [3, \infty)$
b. $x \in (-2, -1] \cup [1, 2) \cup (2, 3]$
c. $x \in (-\infty, -2) \cup (2, \infty)$
d. $x \in (-2, -1) \cup (1, 2) \cup (2, 3]$
e. none of these

90. $R(x) \leq 0$



a. $x \in (-\infty, -1) \cup (0, 2)$
b. $x \in [0, 1] \cup (2, \infty)$
c. $x \in [-5, -1] \cup [2, 5]$
d. $x \in (-\infty, -1) \cup [0, 2)$
e. none of these

WORKING WITH FORMULAS

91. Discriminant of the reduced cubic $x^3 + px + q = 0$**:** $D = -(4p^3 + 27q^2)$

The discriminant of a cubic equation is less well known than that of the quadratic, but serves the same purpose. The discriminant of the reduced cubic is given by the formula shown, where p is the linear coefficient and q is the constant term. If $D > 0$, there will be three real and distinct roots. If $D = 0$, there are still three real roots, but one is a repeated root (multiplicity two). If $D < 0$, there are one real and two complex roots. Suppose we wish to study the family of cubic equations where $q = p + 1$.

a. Verify the resulting discriminant is $D = -(4p^3 + 27p^2 + 54p + 27)$.

b. Determine the values of p and q for which this family of equations has a repeated real root. In other words, solve the equation $-(4p^3 + 27p^2 + 54p + 27) = 0$ using the rational zeroes theorem and synthetic division to write D in completely factored form.

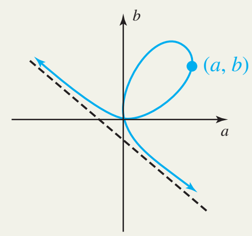
c. Use the factored form from part (b) to determine the values of p and q for which this family of equations has three real and distinct roots. In other words, solve $D > 0$.

d. Verify the results of parts (b) and (c) on a graphing calculator.

92. Coordinates for the folium of Descartes:

$$\begin{cases} a = \dfrac{3kx}{1 + x^3} \\ b = \dfrac{3kx^2}{1 + x^3} \end{cases}$$

The interesting relation shown here is called the folium (leaf) of Descartes. The folium is most often graphed using what are called *parametric equations,* in which the coordinates a and b are expressed in terms of the parameter x ("k" is a constant that affects the size of the leaf). Since each is an individual function, the x- and y-coordinates can be investigated individually in rectangular coordinates using $F(x) = \dfrac{3x}{1 + x^3}$ and $G(x) = \dfrac{3x^2}{1 + x^3}$ (assume $k = 1$ for now).



Folium of Descartes

a. Graph each function using the techniques from this section.

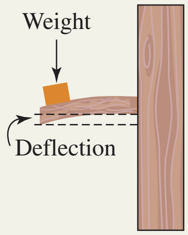
b. According to your graph, for what values of x will the *x-coordinate* of the folium be positive? In other words, solve $F(x) = \dfrac{3x}{1 + x^3} > 0$.

c. For what values of x will the *y-coordinate* of the folium be positive? Solve $G(x) = \dfrac{3x^2}{1 + x^3} > 0$.

d. Will $F(x)$ ever be equal to $G(x)$? If so, for what values of x?

▶ APPLICATIONS

Deflection of a beam: The amount of deflection in a rectangular wooden beam of length L ft can be approximated by $d(x) = k(x^3 - 3L^2x + 2L^3)$, where k is a constant that depends on the characteristics of the wood and the force applied, and x is the *distance from the unsupported end* of the beam ($x < L$).



93. Find the equation for a beam 8 ft long and use it for the following:

a. For what distances x is the quantity $\frac{d(x)}{k}$ less than 189 units?

b. What is the amount of deflection 4 ft from the unsupported end ($x = 4$)?

c. For what distances x is the quantity $\frac{d(x)}{k}$ greater than 475 units?

d. If safety concerns prohibit a deflection of more than 648 units, what is the shortest distance from the end of the beam that the force can be applied?

94. Find the equation for a beam 9 ft long and use it for the following:

a. For what distances x is the quantity $\frac{d(x)}{k}$ less than 216 units?

b. What is the amount of deflection 4 ft from the unsupported end ($x = 4$)?

c. For what distances x is the quantity $\frac{d(x)}{k}$ greater than 550 units?

d. Compare the answer to 93b with the answer to 94b. What can you conclude?

Average speed for a round-trip: Surprisingly, the average speed of a round-trip is *not* the sum of the average speed in each direction divided by two. For a fixed distance D, consider rate r_1 in time t_1 for one direction, and rate r_2 in time t_2 for the other, giving $r_1 = \frac{D}{t_1}$ and $r_2 = \frac{D}{t_2}$. The average speed for the round-trip is $R = \frac{2D}{t_1 + t_2}$.

95. The distance from St. Louis, Missouri, to Springfield, Illinois, is approximately 80 mi. Suppose that Sione, due to the age of his vehicle, made the round-trip with an average speed of 40 mph.

a. Use the relationships stated to verify that $r_2 = \frac{20r_1}{r_1 - 20}$.

b. Discuss the meaning of the horizontal and vertical asymptotes in this context.

c. Verify algebraically the speed returning would be greater than the speed going for $20 < r_1 < 40$. In other words, solve the inequality $\frac{20r_1}{r_1 - 20} > r_1$ using the ideas from this section.

96. The distance from Boston, Massachusetts, to Hartford, Connecticut, is approximately 100 mi. Suppose that Stella, due to excellent driving conditions, made the round-trip with an average speed of 60 mph.

a. Use the relationships above to verify that $r_2 = \frac{30r_1}{r_1 - 30}$.

b. Discuss the meaning of the horizontal and vertical asymptotes in this context.

c. Verify algebraically the speed returning would be greater than the speed going for $30 < r_1 < 60$. In other words, solve the inequality $\frac{30r_1}{r_1 - 30} > r_1$ using the ideas from this section.

Electrical resistance and temperature: The amount of electrical resistance $\boldsymbol{R}$ in a medium depends on the temperature, and for certain materials can be modeled by the equation $R(t) = 0.01t^2 + 0.1t + k$, where $R(t)$ is the resistance (in ohms Ω) at temperature t ($t \geq 0°$) in degrees Celsius, and k is the resistance at $t = 0°$C.

97. Suppose $k = 30$ for a certain medium. Write the resistance equation and use it to answer the following.

a. For what temperatures is the resistance less than 42 Ω?

b. For what temperatures is the resistance greater than 36 Ω?

c. If it becomes uneconomical to run electricity through the medium for resistances greater than 60 Ω, for what temperatures should the electricity generator be shut down?

98. Suppose Write the resistance equation and solve the following.

a. For what temperatures is the resistance less than 26 Ω?

b. For what temperatures is the resistance greater than 40 Ω?

c. If it becomes uneconomical to run electricity through the medium for resistances greater than 50 Ω, for what temperatures should the electricity generator be shut down?

99. Sum of consecutive squares: The sum of the first n squares $1^2 + 2^2 + 3^2 + \cdots + n^2$ is given by the formula $S(n) = \frac{2n^3 + 3n^2 + n}{6}$. Use the equation to solve the following inequalities.

a. For what number of consecutive squares is $S(n) \geq 30$?

b. For what number of consecutive squares is $S(n) \leq 285$?

c. What is the maximum number of consecutive squares that can be summed without the result exceeding three digits?

100. Sum of consecutive cubes: The sum of the first n cubes $1^3 + 2^3 + 3^3 + \cdots + n^3$ is given by the formula $S(n) = \frac{n^4 + 2n^3 + n^2}{4}$. Use the equation to solve the following inequalities.

a. For what number of consecutive cubes is $S(n) \geq 100$?

b. For what number of consecutive cubes is $S(n) \leq 784$?

c. What is the maximum number of consecutive cubes that can be summed without the result exceeding three digits?

▶ EXTENDING THE CONCEPT

101. (a) Is it possible for the solution set of a polynomial inequality to be all real numbers? If not, discuss why. If so, provide an example. (b) Is it possible for the solution set of a rational inequality to be all real numbers? If not, discuss why. If so, provide an example.

102. The domain of radical functions: As in Exercises 31–36, if n is an even number, the expression $\sqrt[n]{A}$ represents a real number only if $A \geq 0$. Use this idea to find the domain of the following functions.

a. $f(x) = \sqrt{2x^3 - x^2 - 16x + 15}$

b. $g(x) = \sqrt[4]{2x^3 + x^2 - 22x + 24}$

c. $p(x) = \sqrt[4]{\frac{x + 2}{x^2 - 2x - 35}}$

d. $q(x) = \sqrt{\frac{x^2 - 1}{x^2 - x - 6}}$

103. Find one polynomial inequality and one rational inequality that have the solution $x \in (-\infty, -2) \cup (0, 1) \cup (1, \infty)$.

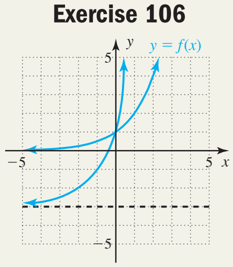
104. Using the tools of calculus, it can be shown that $f(x) = x^4 - 4x^3 - 12x^2 + 32x + 39$ is increasing in the intervals where $F(x) = x^3 - 3x^2 - 6x + 8$ is positive. Solve the inequality $F(x) > 0$ using the ideas from this section, then verify $f(x)\uparrow$ in these intervals by graphing f on a graphing calculator and using the TRACE feature.

105. Using the tools of calculus, it can be shown that $r(x) = \frac{x^2 - 3x - 4}{x - 8}$ is decreasing in the intervals where $R(x) = \frac{x^2 - 16x + 28}{(x - 8)^2}$ is negative. Solve the inequality $R(x) < 0$ using the ideas from this section, then verify $r(x)\downarrow$ in these intervals by graphing r on a graphing calculator and using the TRACE feature.

▶ MAINTAINING YOUR SKILLS

106. (2.5) Use the graph of $f(x)$ given to sketch the graph of $y = f(x + 2) - 3$.

Exercise 106



107. (3.5) Graph the function $f(x) = \frac{x^2 + 2x - 8}{x + 4}$. If there is a removable discontinuity, repair the break using an appropriate piecewise-defined function.

108. (1.6) Solve the equation $\frac{1}{2}\sqrt{16 - x} - \frac{x}{2} = 2$. Check solutions in the original equation.

109. (1.2/3.7) Graph the solution set for the relation: $3x + 1 < 10$ *and* $x^2 - 3 < 1$.

3.8 Variation: Function Models in Action

Learning Objectives

In Section 3.8 you will learn how to:

- **A.** Solve direct variations
- **B.** Solve inverse variations
- **C.** Solve joint variations

A study of direct and inverse variation offers perhaps our clearest view of how mathematics is used to model real-world phenomena. While the basis of our study is elementary, involving only the toolbox functions, the applications are at the same time elegant, powerful, and far reaching. In addition, these applications unite some of the most important ideas in algebra, including functions, transformations, rates of change, and graphical analysis, to name a few.

A. Toolbox Functions and Direct Variation

If a car gets 24 miles per gallon (mpg) of gas, we could express the distance d it could travel as $d = 24g$. Table 3.4 verifies the distance traveled by the car changes in *direct* or *constant proportion* to the number of gallons used, and here we say, "distance traveled *varies directly* with gallons used." The equation $d = 24g$ is called a **direct variation,** and the coefficient 24 is called the **constant of variation.** Using the rate of change notation, $\frac{\Delta \text{distance}}{\Delta \text{gallons}} = \frac{\Delta d}{\Delta g} = \frac{24}{1}$, and we note this is actually a *linear equation* with slope $m = 24$. When working with variations, the constant k is preferred over m, and in general we have the following:

Table 3.4

g	d
1	24
2	48
3	72
4	96

Direct Variation

y varies directly with x, or *y is directly proportional to x*,
if there is a nonzero constant k such that

$$y = kx.$$

k is called the *constant of variation.*

EXAMPLE 1 ▶ Writing a Variation Equation

Write the variation equation for these statements:

a. Wages earned varies directly with the number of hours worked.
b. The value of an office machine varies directly with time.
c. The circumference of a circle varies directly with the length of the diameter.

Solution ▶
a. **W**ages varies directly with **h**ours worked: $W = kh$
b. The **V**alue of an office machine varies directly with **t**ime: $V = kt$
c. The **C**ircumference varies directly with the **d**iameter: $C = kd$

Now try Exercises 7 through 10 ▶

Once we determine the relationship between two variables is a direct variation, we try to find the value of k and develop a general equation model for the relationship indicated. Note that "varies directly" indicates that one value is a constant multiple of the other. In Example 1(c), you may have realized that for $C = kd$, $k = \pi$ since $\pi = \frac{C}{d}$ and the formula for a circle's circumference is $C = \pi d$. The connection helps illustrate the procedure for finding k, as it shows that only *one known relationship is needed!* This suggests the following procedure:

Solving Applications of Variation

1. Write the information given as an equation, using k as the constant multiple.
2. Substitute the first relationship (pair of values) given and solve for k.
3. Substitute this value for k in the original equation to obtain the variation equation.
4. Use the variation equation to complete the application.

EXAMPLE 2 ▶ Solving an Application of Direct Variation

The weight of an astronaut on the surface of another planet **varies directly** with their weight on Earth. An astronaut weighing 140 lb on Earth weighs only 53.2 lb on Mars. How much would a 170-lb astronaut weigh on Mars?

Solution ▶

1. $M = kE$ — "**M**ars weight **varies directly** with **E**arth weight"
2. $53.2 = k(140)$ — substitute 53.2 for M and 140 for E

$k = 0.38$ — solve for k (constant of variation)

Substitute this value of k in the original equation to obtain the variation equation, then find the weight of a 170-lb astronaut that landed on Mars.

3. $M = 0.38E$ — variation equation
4. $= 0.38(170)$ — substitute 170 for E

$= 64.6$ — result

An astronaut weighing 170 lb on Earth weighs only 64.6 lb on Mars.

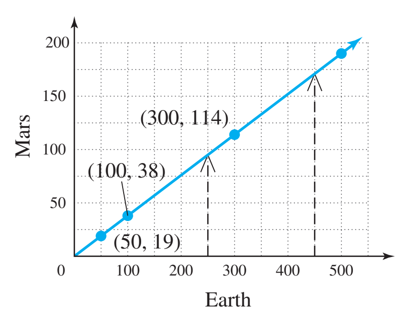
Now try Exercises 11 through 14 ▶

The toolbox function from Example 2 was a line with slope $k = 0.38$, or $k = \frac{19}{50}$ as a fraction in simplest form. As a rate of change, $k = \frac{\Delta M}{\Delta E} = \frac{19}{50}$, and we see that for every 50 additional pounds on Earth, the weight of an astronaut would increase by only 19 lb on Mars.

EXAMPLE 3 ▶ Making Estimates from the Graph of a Variation

The scientists at NASA are planning to send additional probes to the red planet (Mars), that will weigh from 250 to 450 lb. Graph the variation equation from Example 2, then *use the graph* to estimate the corresponding range of weights on Mars. Check your estimate using the variation equation.

Solution ▶ After selecting an appropriate scale, begin at (0, 0) and count off the slope $k = \frac{\Delta M}{\Delta E} = \frac{19}{50}$. This gives the points (50, 19), (100, 38), (200, 76), and so on. From the graph (see dashed arrows), it appears the weights corresponding to 250 lb and 450 lb on Earth are near 95 lb and 170 lb on Mars. Using the equation gives



$M = 0.38E$ — variation equation

$= 0.38(250)$ — substitute 250 for E

$= 95$, and

$M = 0.38E$ — variation equation

$= 0.38(450)$ — substitute 450 for E

$= 171$, very close to our estimate from the graph.

Now try Exercises 15 and 16 ▶

When a toolbox function is used to model a variation, our knowledge of their graphs and defining characteristics strengthens a contextual understanding of the application. Consider Examples 4 and 5, where the squaring function is used.

EXAMPLE 4 ▶ **Writing Variation Equations**

Write the variation equation for these statements:

a. In free fall, the distance traveled by an object varies directly with the square of the time.

b. The area of a circle varies directly with the square of its radius.

Solution ▶ **a.** Distance varies directly with the square of the time: $D = kt^2$.

b. Area varies directly with the square of the radius: $A = kr^2$.

Now try Exercises 17 through 20 ▶

Both variations in Example 4 use the squaring function, where k represents the amount of stretch or compression applied, and whether the graph will open upward or downward. However, regardless of the function used, the four-step solution process remains the same.

EXAMPLE 5 ▶ **Solving an Application of Direct Variation**

The range of a projectile varies directly with the square of its initial velocity. As part of a circus act, Bailey the Human Bullet is shot out of a cannon with an initial velocity of 80 feet per second (ft/sec), into a net 200 ft away.

a. Find the constant of variation and write the variation equation.

b. Graph the equation and *use the graph* to estimate how far away the net should be placed if initial velocity is increased to 95 ft/sec.

c. Determine the accuracy of the estimate from (b) using the variation equation.

Solution ▶ **a. 1.** $R = kv^2$ — "Range varies directly with the square of the velocity"

2. $200 = k(80)^2$ — substitute 200 for R and 80 for v

$k = 0.03125$ — solve for k (constant of variation)

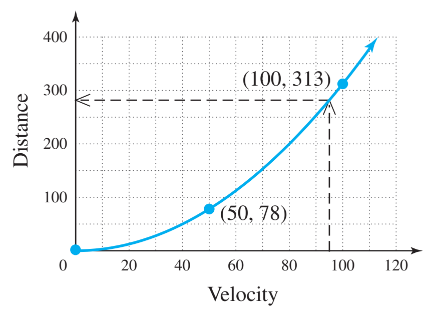
3. $R = 0.03125v^2$ — variation equation (substitute 0.03125 for k)

b. Since velocity and distance are positive, we again use only QI. The graph is a parabola that opens upward, with the vertex at (0, 0). Selecting velocities from 50 to 100 ft/s, we have:

$R = 0.03125v^2$ — variation equation

$= 0.03125(50)^2$ — substitute 50 for v

$= 78.125$ — result



Likewise substituting 100 for v gives $R = 312.5$ ft. Scaling the axes and using (0, 0), (50, 78), and (100, 313) produces the graph shown. At 95 ft/s (dashed lines), it appears the net should be placed about 280 ft away.

c. Using the variation equation gives:

4. $R = 0.03125v^2$ — variation equation

$= 0.03125(95)^2$ — substitute 95 for v

$R = 282.03125$ — result

Our estimate was off by about 2 ft. The net should be placed about 282 ft away.

Now try Exercises 21 through 26 ▶

☑ **A.** You've just learned how to solve direct variations

Note: For Examples 6 to 8, the four steps of the solution process are used in sequence, but are not numbered.

B. Inverse Variation

Table 3.5

Price (dollars)	Demand (1000s)
8	288
9	144
10	96
11	72
12	57.6

Numerous studies have been done that relate the price of a commodity to the demand—the willingness of a consumer to pay that price. For instance, if there is a sudden increase in the price of a popular tool, hardware stores know there will be a corresponding decrease in the demand for that tool. The question remains, "What is this rate of decrease?" Can it be modeled by a linear function with a negative slope? A parabola that opens downward? Some other function? Table 3.5 shows some (simulated) data regarding price versus demand. It appears that a linear function is not appropriate because the rate of change in the number of tools sold is not constant. Likewise a quadratic model seems inappropriate, since we don't expect demand to suddenly start rising again as the price continues to increase. This phenomenon is actually an example of an **inverse variation,** modeled by a transformation of the reciprocal function $y = \frac{k}{x}$. We will often rewrite the equation as $y = k(\frac{1}{x})$ to clearly see the inverse relationship. In the case at hand, we might write $D = k(\frac{1}{P})$, where k is the constant of variation, D represents the demand for the product, and P the price of the product. In words, we say that "demand *varies inversely* as the price." In other applications of inverse variation, one quantity may vary inversely as the *square* of another, and in general we have

Inverse Variation

y varies inversely with x, or *y is inversely proportional to x*,
if there is a nonzero constant k such that

$$y = k\left(\frac{1}{x}\right).$$

k is called the *constant of variation.*

EXAMPLE 6 ▶ Writing Inverse Variation Equations

Write the variation equation for these statements:

a. In a closed container, pressure varies inversely with the volume of gas.

b. The intensity of light varies inversely with the square of the distance from the source.

Solution ▶

a. Pressure varies inversely with the *V*olume of gas: $P = k(\frac{1}{V})$.

b. Intensity of light varies inversely with the square of the distance: $I = k\frac{1}{(d^2)}$.

Now try Exercises 27 through 30 ▶

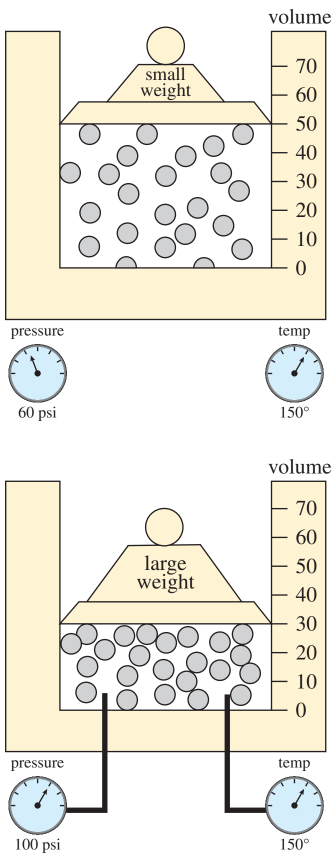
EXAMPLE 7 ▶ Solving an Application of Inverse Variation

Boyle's law tells us that in a closed container with constant temperature, the pressure of a gas varies inversely with its volume (see illustration on page 393). Suppose the air pressure in a closed cylinder is 60 pounds per square inch (psi) when the volume of the cylinder is 50 in^3.

a. Find the constant of variation and write the variation equation.

b. Use the equation to find the pressure, if volume is compressed to 30 in^3.

Illustration of Boyle's Law



Solution ▶ **a.** $P = k\left(\frac{1}{V}\right)$ "Pressure varies inversely with the volume"

$60 = k\left(\frac{1}{50}\right)$ substitute 60 for *P* and 50 for *V*.

$k = 3000$ constant of variation

$P = 3000\left(\frac{1}{V}\right)$ variation equation (substitute 3000 for *k*)

b. Using the variation equation we have:

$P = 3000\left(\frac{1}{V}\right)$ variation equation

$= 3000\left(\frac{1}{30}\right)$ substitute 30 for *V*

$= 100$ result

When the volume is decreased to 30 in^3, the pressure increases to 100 psi.

Now try Exercises 31 through 34 ▶

☑ **B.** You've just learned how to solve inverse variations

C. Joint or Combined Variations

Just as some decisions might be based on many considerations, often the relationship between two variables depends on a combination of factors. Imagine a wooden plank laid across the banks of a stream for hikers to cross the streambed (see Figure 3.75). The amount of weight the plank will support depends on the type of wood, the width and height of the plank's cross section, and the distance between the supported ends **(see Exercises 59 and 60).** This is an example of a **joint variation,** which can combine any number of variables in different ways. Two general possibilities are: (1) *y varies jointly with the product of x and p:* $y = kxp$; and (2) *y varies jointly with the product of x and p, and inversely with the square of q:* $y = kxp\left(\frac{1}{q^2}\right)$. For practice writing joint variations as an equation model, **see Exercises 35 through 40.**

Figure 3.75

EXAMPLE 8 ▶ **Solving an Application of Joint Variation**

The amount of fuel used by a ship traveling at a uniform speed varies jointly with the distance it travels and the square of the velocity. If 200 barrels of fuel are used to travel 10 mi at 20 nautical miles per hour, how far does the ship travel on 500 barrels of fuel at 30 nautical miles per hour?

Solution ▶

$F = kdv^2$ "fuel use *varies jointly* with distance and velocity squared"

$200 = k(10)(20)^2$ substitute 200 for *F*, 10 for *d*, and 20 for *v*

$200 = 4000k$ simplify and solve for *k*

$0.05 = k$ constant of variation

$F = 0.05dv^2$ equation of variation

To find the distance traveled at 30 nautical miles per hour using 500 barrels of fuel, substitute 500 for F and 30 for v:

$$\begin{aligned} F &= 0.05dv^2 && \text{equation of variation} \\ 500 &= 0.05d(30)^2 && \text{substitute 500 for } F \text{ and 30 for } v \\ 500 &= 45d && \text{simplify} \\ 11.\overline{1} &= d && \text{result} \end{aligned}$$

If 500 barrels of fuel are consumed while traveling 30 nautical miles per hour, the ship covers a distance of just over 11 mi.

Now try Exercises 41 through 44 ▶

It's interesting to note that the ship covers just over one additional mile, but consumes over 2.5 times the amount of fuel. The additional speed requires a great deal more fuel.

There is a variety of additional applications in the Exercise Set. **See Exercises 47 through 55.**

☑ **C.** You've just learned how to solve joint variations

3.8 EXERCISES

▶ CONCEPTS AND VOCABULARY

Fill in each blank with the appropriate word or phrase. Carefully reread the section if needed.

1. The phrase "y varies directly with x" is written $y = kx$, where k is called the ________ of variation.

2. If more than two quantities are related in a variation equation, the result is called a ________ variation.

3. The statement "y varies inversely with the square of x" is written ________.

4. For $y = kx$, $y = kx^2$, $y = kx^3$, and $y = k\sqrt{x}$, it is true that as $x \to \infty$, $y \to \infty$ (functions increase). One important difference among the functions is $\frac{\Delta y}{\Delta x}$, or their ________ of ________.

5. Discuss/Explain the general procedure for solving applications of variation. Include references to keywords, and illustrate using an example.

6. The basic percent formula is *amount equals percent times base,* or $A = PB$. In words, write this out as a direct variation with B as the constant of variation, then as an inverse variation with the amount A as the constant of variation.

▶ DEVELOPING YOUR SKILLS

Write the variation equation for each statement.

7. distance traveled varies directly with rate of speed

8. cost varies directly with the quantity purchased

9. force varies directly with acceleration

10. length of a spring varies directly with attached weight

For Exercises 11 and 12, find the constant of variation and write the variation equation. Then use the equation to complete the table.

11. y varies directly with x; $y = 0.6$ when $x = 24$.

x	y
500	
	16.25
750	

12. w varies directly with v; $w = \frac{1}{3}$ when $v = 5$.

v	w
291	
	21.8
339	

13. Wages earned varies directly with the number of hours worked. Last week I worked 37.5 hr and my gross pay was $344.25. Write the variation equation and determine how much I will gross this week if I work 35 hr. What does the value of k represent in this case?

14. The thickness of a paperback book varies directly as the number of pages. A book 3.2 cm thick has 750 pages. Write the variation equation and approximate the thickness of *Roget's 21st Century Thesaurus* (paperback—2nd edition), which has 957 pages.

15. The number of stairs in the stairwell of tall buildings and other structures varies directly as the height of the structure. The base and pedestal for the Statue of Liberty are 47 m tall, with 192 stairs from ground level to the observation deck at the top of the pedestal (at the statue's feet). (a) Find the constant of variation and write the variation equation, (b) graph the variation equation, (c) use the graph to estimate the number of stairs from ground level to the observation deck in the statue's crown 81 m above ground level, and (d) use the equation to check this estimate. Was it close?

16. The height of a projected image varies directly as the distance of the projector from the screen. At a distance of 48 in., the image on the screen is 16 in. high. (a) Find the constant of variation and write the variation equation, (b) graph the variation equation, (c) use the graph to estimate the height of the image if the projector is placed at a distance of 5 ft 3 in., and (d) use the equation to check this estimate. Was it close?

Write the variation equation for each statement.

17. Surface area of a cube varies directly with the square of a side.

18. Potential energy in a spring varies directly with the square of the distance the spring is compressed.

19. Electric power varies directly with the square of the current (amperes).

20. Manufacturing cost varies directly as the square of the number of items made.

For Exercises 21 through 26, find the constant of variation and write the variation equation. Then use the equation to complete the table or solve the application.

21. p varies directly with the square of q; $p = 280$ when $q = 50$

q	p
45	
	338.8
70	

22. n varies directly with m squared; $n = 24.75$ when $m = 30$

m	n
40	
	99
88	

23. The surface area of a cube varies directly as the square of one edge. A cube with edges of $14\sqrt{3}$ cm has a surface area of 3528 cm^2. Find the surface area in square meters of the spaceships used by the Borg Collective in *Star Trek—The Next Generation,* cubical spacecraft with edges of 3036 m.

24. The area of an equilateral triangle varies directly as the square of one side. A triangle with sides of 50 yd has an area of 1082.5 yd^2. Find the area in mi^2 of the region bounded by straight lines connecting the cities of Cincinnati, Ohio, Washington, D.C., and Columbia, South Carolina, which are each approximately 400 mi apart.

25. The distance an object falls varies directly as the square of the time it has been falling. The cannonballs dropped by Galileo from the Leaning Tower of Pisa fell about 169 ft in 3.25 sec. (a) Find the constant of variation and write the variation equation, (b) graph the variation equation, (c) use the graph to estimate how long it would take a hammer, accidentally dropped from a height of 196 ft by a bridge repair crew, to splash into the water below, and (d) use the equation to check this estimate. Was it close? (e) According to the equation, if a camera accidentally fell out of the *News 4 Eye-in-the-Sky* helicopter from a height of 121 ft, how long until it strikes the ground?

26. When a child blows small soap bubbles, they come out in the form of a sphere because the surface tension in the soap seeks to minimize the surface area. The surface area of any sphere varies directly with the square of its radius. A soap bubble with a $\frac{3}{4}$ in. radius has a surface area of approximately 7.07 in^2. (a) Find the constant of variation and write the variation equation, (b) graph the variation equation, (c) use the graph to estimate the radius of a seventeenth-century cannonball that has a surface area of 113.1 in^2, and (d) use the equation to check this estimate. Was it close? (e) According to the equation, what is the surface area of an orange with a radius of $1\frac{1}{2}$ in.?

Write the variation equation for each statement.

27. The force of gravity varies inversely as the square of the distance between objects.

28. Pressure varies inversely as the area over which it is applied.

29. The safe load of a beam supported at both ends varies inversely as its length.

30. The intensity of sound varies inversely as the square of its distance from the source.

For Exercises 31 through 34, find the constant of variation and write the variation equation. Then use the equation to complete the table or solve the application.

31. Y varies inversely as the square of Z; $Y = 1369$ when $Z = 3$

Z	*Y*
37	
	2.25
111	

32. A varies inversely with B; $A = 2450$ when $B = 0.8$

B	*A*
140	
	6.125
560	

33. The effect of Earth's gravity on an object (its weight) varies inversely as the square of its distance from the center of the planet (assume the Earth's radius is 6400 km). If the weight of an astronaut is 75 kg on Earth (when $r = 6400$), what would this weight be at an altitude of 1600 km *above the surface* of the Earth?

34. The demand for a popular new running shoe varies inversely with the cost of the shoes. When the wholesale price is set at $45, the manufacturer ships 5500 orders per week to retail outlets. Based on this information, how many orders would be shipped if the wholesale price rose to $55?

Write the variation equation for each statement.

35. Interest earned varies jointly with the rate of interest and the length of time on deposit.

36. Horsepower varies jointly as the number of cylinders in the engine and the square of the cylinder's diameter.

37. The area of a trapezoid varies jointly with its height and the sum of the bases.

38. The area of a triangle varies jointly with its height and the length of the base.

39. The volume of metal in a circular coin varies directly with the thickness of the coin and the square of its radius.

40. The electrical resistance in a wire varies directly with its length and inversely as the cross-sectional area of the wire.

For Exercises 41–44, find the constant of variation and write the related variation equation. Then use the equation to complete the table or solve the application.

41. C varies directly with R and inversely with S squared, and $C = 21$ when $R = 7$ and $S = 1.5$.

R	*S*	*C*
120		22.5
200	12.5	
	15	10.5

42. J varies directly with P and inversely with the square root of Q, and $J = 19$ when $P = 4$ and $Q = 25$.

P	*Q*	*J*
47.5		118.75
112	31.36	
	44.89	66.5

43. Kinetic energy: Kinetic energy (energy attributed to motion) varies jointly with the mass of the object and the square of its velocity. Assuming a unit mass of $m = 1$, an object with a velocity of 20 m per sec (m/s) has kinetic energy of 200 J. How much energy is produced if the velocity is increased to 35 m/s?

44. Safe load: The load that a horizontal beam can support varies jointly as the width of the beam, the square of its height, and inversely as the length of the beam. A beam 4 in. wide and 8 in. tall can safely support a load of 1 ton when the beam has a length of 12 ft. How much could a similar beam 10 in. tall safely support?

▶ WORKING WITH FORMULAS

45. Required interest rate: $R(A) = \sqrt[3]{A} - 1$
To determine the simple interest rate R that would be required for each dollar ($1) left on deposit for 3 yr to grow to an amount A, the formula $R(A) = \sqrt[3]{A} - 1$ can be applied. To what function family does this formula belong? Complete the table using a calculator, then use the table to estimate the interest rate required for each $1 to grow to $1.17. Compare your estimate to the value you get by evaluating $R(1.17)$.

Amount A	Rate R
1.0	
1.05	
1.10	
1.15	
1.20	
1.25	

46. Force between charged particles: $F = k\frac{Q_1Q_2}{d^2}$
The force between two charged particles is given by the formula shown, where F is the force (in joules—J), Q_1 and Q_2 represent the electrical charge on each particle (in coulombs—C), and d is the distance between them (in meters). If the particles have a like charge, the force is repulsive; if the charges are unlike, the force is attractive. (a) Write the variation equation in words. (b) Solve for k and use the formula to find the electrical constant k, given $F = 0.36$ J, $Q_1 = 2 \times 10^{-6}$ C, $Q_2 = 4 \times 10^{-6}$ C, and $d = 0.2$ m. Express the result in scientific notation.

▶ APPLICATIONS

Find the constant of variation "*k*" and write the variation equation, then use the equation to solve.

47. Cleanup time: The time required to pick up the trash along a stretch of highway varies inversely as the number of volunteers who are working. If 12 volunteers can do the cleanup in 4 hr, how many volunteers are needed to complete the cleanup in just 1.5 hr?

48. Wind power: The wind farms in southern California contain wind generators whose power production varies directly with the cube of the wind's speed. If one such generator produces 1000 W of power in a 25 mph wind, find the power it generates in a 35 mph wind.

49. Pull of gravity: The weight of an object on the moon varies directly with the weight of the object on Earth. A 96-kg object on Earth would weigh only 16 kg on the moon. How much would a fully suited 250-kg astronaut weigh on the moon?

50. Period of a pendulum: The time that it takes for a simple pendulum to complete one period (swing over and back) varies directly as the square root of its length. If a pendulum 20 ft long has a period of 5 sec, find the period of a pendulum 30 ft long.

51. Stopping distance: The stopping distance of an automobile varies directly as the square root of its speed when the brakes are applied. If a car requires 108 ft to stop from a speed of 25 mph, estimate the stopping distance if the brakes were applied when the car was traveling 45 mph.

52. Supply and demand: A chain of hardware stores finds that the demand for a special power tool varies inversely with the advertised price of the tool. If the price is advertised at $85, there is a monthly demand for 10,000 units at all participating stores. Find the projected demand if the price were lowered to $70.83.

53. Cost of copper tubing: The cost of copper tubing varies jointly with the length and the diameter of the tube. If a 36-ft spool of $\frac{1}{4}$-in.-diameter tubing costs $76.50, how much does a 24-ft spool of $\frac{3}{8}$-in.-diameter tubing cost?

54. Electrical resistance: The electrical resistance of a copper wire varies directly with its length and inversely with the square of the diameter of the wire. If a wire 30 m long with a diameter of 3 mm has a resistance of 25 Ω, find the resistance of a wire 40 m long with a diameter of 3.5 mm.

55. Volume of phone calls: The number of phone calls per day between two cities varies directly as the product of their populations and inversely as the square of the distance between them. The city of Tampa, Florida (pop. 300,000), is 430 mi from the city of Atlanta, Georgia (pop. 420,000).

Telecommunications experts estimate there are about 300 calls per day between the two cities. Use this information to estimate the number of daily phone calls between Amarillo, Texas (pop. 170,000), and Denver, Colorado (pop. 550,000), which are also separated by a distance of about 430 mi. Note: Population figures are for the year 2000 and rounded to the nearest ten-thousand.

Source: 2005 World Almanac, p. 626.

56. Internet commerce: The likelihood of an eBay® item being sold for its "Buy it Now®" price P, varies directly with the feedback rating of the seller, and inversely with the cube of $\frac{P}{MSRP}$, where MSRP represents the manufacturer's suggested retail price. A power eBay® seller with a feedback rating of 99.6%, knows she has a 60% likelihood of selling an item at 90% of the MSRP. What is the likelihood a seller with a 95.3% feedback rating can sell the same item at 95% of the MSRP?

57. Volume of an egg: The volume of an egg laid by an average chicken varies jointly with its length and the square of its width. An egg measuring 2.50 cm wide and 3.75 cm long has a volume of 12.27 cm^3. A Barret's Blue Ribbon hen can lay an egg measuring 3.10 cm wide and 4.65 cm long. (a) What is the volume of this egg? (b) As a percentage, how much greater is this volume than that of an average chicken?

58. Athletic performance: Researchers have estimated that a sprinter's time in the 100-m dash varies directly as the square root of her age and inversely as the number of hours spent training each week. At 20 yr old, Gail trains 10 hr per week (hr/wk) and has an average time of 11 sec. Assuming she continues to train 10 hr/wk, (a) what will her average time be at 30 yr old? (b) If she wants to keep her average time at 11 sec, how many hours per week should she train?

59. Maximum safe-load: The maximum safe load M that can be placed on a uniform horizontal beam supported at both ends varies directly as the width w and the square of the height h of the beam's cross section, and inversely as its length L (width and height are assumed to be in inches, and length in feet). (a) Write the variation equation. (b) If a beam 18 in. wide, 2 in. high, and 8 ft long can safely support 270 lb, what is the safe load for a beam of like dimensions with a length of 12 ft?

60. Maximum safe load: Suppose a 10-ft wooden beam with dimensions 4 in. by 6 in. is made from the same material as the beam in Exercise 59 (the same k value can be used). (a) What is the maximum safe load if the beam is placed so that width is 6 in. and height is 4 in.? (b) What is the maximum safe load if the beam is placed so that width is 4 in. and height is 6 in.?

▶ EXTENDING THE CONCEPT

61. In function form, the variations $Y_1 = k\frac{1}{x}$ and $Y_2 = k\frac{1}{x^2}$ become $f(x) = k\frac{1}{x}$ and $g(x) = k\frac{1}{x^2}$. Both graphs appear similar in Quadrant I and both may "fit" a scatter-plot fairly well, but there is a big difference between them—they decrease as x gets larger, but *they decrease at very different rates.* Assume $k = 1$ and use the ideas from Section 2.5 to compute the rate of change for f and g for the interval from $x = 0.5$ to $x = 0.6$. Were you surprised? In the interval $x = 0.7$ to $x = 0.8$, will the rate of decrease for each function be greater or less than in the interval $x = 0.5$ to $x = 0.6$? Why?

62. The gravitational force F between two celestial bodies varies jointly as the product of their masses and inversely as the square of the distance d between them. The relationship is modeled by Newton's law of universal gravitation: $F = k\frac{m_1 m_2}{d^2}$. Given that $k = 6.67 \times 10^{-11}$, what is the gravitational force exerted by a 1000-kg sphere on another identical sphere that is 10 m away?

63. The intensity of light and sound both vary inversely as the square of their distance from the source.

a. Suppose you're relaxing one evening with a copy of *Twelfth Night* (Shakespeare), and the reading light is placed 5 ft from the surface of the book. At what distance would the intensity of the light be twice as great?

b. *Tamino's Aria* (*The Magic Flute*—Mozart) is playing in the background, with the speakers 12 ft away. At what distance from the speakers would the intensity of sound be three times as great?

▶ MAINTAINING YOUR SKILLS

64. (R.3) Evaluate: $\left(\dfrac{2x^4}{3x^3y}\right)^{-2}$

65. (1.5) Find all zeroes, real and complex: $x^3 + 4x^2 + 8x = 0$.

66. (2.2) State the domains of f and g given here:

a. $f(x) = \dfrac{x - 3}{x^2 - 16}$ **b.** $g(x) = \dfrac{x - 3}{\sqrt{x^2 - 16}}$

67. (2.5) Graph by using transformations of the parent function and plotting a minimum number of points: $f(x) = -2|x - 3| + 5$.

SUMMARY AND CONCEPT REVIEW

SECTION 3.1 Quadratic Functions and Applications

KEY CONCEPTS

- A quadratic function is one of the form $f(x) = ax^2 + bx + c$; $a \neq 0$. The simplest quadratic is the squaring function $f(x) = x^2$, where $a = 1$ and $b, c = 0$.
- The graph of a quadratic function is a parabola. Parabolas have three distinctive features: (1) like end behavior on the left and right, (2) an axis of symmetry, (3) a highest or lowest point called the vertex.
- For a quadratic function in the standard form $y = ax^2 + bx + c$,
  - End behavior: graph opens upward if $a > 0$, opens downward if $a < 0$
  - Zeroes/x-intercepts: substitute 0 for y and solve for x (if they exist)
  - y-intercept: substitute 0 for $x \to (0, c)$
  - Vertex: (h, k), where $h = \dfrac{-b}{2a}$, $k = f\left(\dfrac{-b}{2a}\right)$
  - Maximum value: If the parabola opens downward, $y = k$ is the maximum value of f.
  - Minimum value: If the parabola opens upward, $y = k$ is the minimum value of f.
  - Line of symmetry: $x = h$ is the line (or axis) of symmetry [if $(h + c, y)$ is on the graph, then $(h - c, y)$ is also on the graph].
- By completing the square, $f(x) = ax^2 + bx + c$ can be written as the transformation $f(x) = a(x + h)^2 \pm k$, and graphed using transformations of $y = x^2$.

EXERCISES

Graph $f(x)$ by completing the square and using transformations of the parent function. Graph $g(x)$ and $h(x)$ using the vertex formula and y-intercept. Find the x-intercepts (if they exist) for all functions.

1. $f(x) = x^2 + 8x + 15$ **2.** $g(x) = -x^2 + 4x - 5$ **3.** $h(x) = 4x^2 - 12x + 3$

4. Height of a superball: A teenager tries to see how high she can bounce her superball by throwing it downward on her driveway. The height of the ball (in feet) at time t (in seconds) is given by $h(t) = -16t^2 + 96t$. (a) How high is the ball at $t = 0$? (b) How high is the ball after 1.5 sec? (c) How long until the ball is 135 ft high? (d) What is the maximum height attained by the ball? At what time t did this occur?

SECTION 3.2 Synthetic Division; the Remainder and Factor Theorems

KEY CONCEPTS

- Synthetic division is an abbreviated form of long division. Only the coefficients of the dividend are used, since "standard form" ensures like place values are aligned. Zero placeholders are used for "missing" terms. The "divisor" must be linear with leading coefficient 1.
- To divide a polynomial by $x - c$, use c in the synthetic division; to divide by $x + c$, use $-c$.
- After setting up the synthetic division template, drop the leading coefficient of the dividend into place, then multiply in the diagonal direction, place the product in the next column, and add in the vertical direction, continuing to the last column.
- The final sum is the remainder r, the numbers preceding it are the coefficients of $q(x)$.
- Remainder theorem: If $p(x)$ is divided by $x - c$, the remainder is equal to $p(c)$. The theorem can be used to evaluate polynomials at $x = c$.
- Factor theorem: If $p(c) = 0$, then c is a zero of p and $(x - c)$ is a factor. Conversely, if $(x - c)$ is a factor of p, then $p(c) = 0$. The theorem can be used to factor a polynomial or build a polynomial from its zeroes.
- The remainder and factor theorems also apply when c is a complex number.

EXERCISES

Divide using long division and clearly identify the quotient and remainder:

5. $\dfrac{x^3 + 4x^2 - 5x - 6}{x - 2}$

6. $\dfrac{x^3 + 2x - 4}{x^2 - x}$

7. Use the factor theorem to show that $x + 7$ is a factor of $2x^4 + 13x^3 - 6x^2 + 9x + 14$.

8. Complete the division and write $h(x)$ as $h(x) = d(x)q(x) + r(x)$, given $\dfrac{h(x)}{d(x)} = \dfrac{x^3 - 4x + 5}{x - 2}$.

9. Use the factor theorem to help factor $p(x) = x^3 + 2x^2 - 11x - 12$ completely.

10. Use the factor and remainder theorems to factor h, given $x = 4$ is a zero: $h(x) = x^4 - 3x^3 - 4x^2 - 2x + 8$.

Use the remainder theorem:

11. Show $x = \frac{1}{2}$ is a zero of V: $V(x) = 4x^3 + 8x^2 - 3x - 1$.

12. Show $x = 3i$ is a zero of W: $W(x) = x^3 - 2x^2 + 9x - 18$.

13. Find $h(-7)$ given $h(x) = x^3 + 9x^2 + 13x - 10$.

Use the factor theorem:

14. Find a degree 3 polynomial in standard form with zeroes $x = 1$, $x = -\sqrt{5}$, and $x = \sqrt{5}$.

15. Find a fourth-degree polynomial in standard form with one real zero, given $x = 1$ and $x = -2i$ are zeroes.

16. Use synthetic division and the remainder theorem to answer: At a busy shopping mall, customers are constantly coming and going. One summer afternoon during the hours from 12 o'clock noon to 6 in the evening, the number of customers in the mall could be modeled by $C(t) = 3t^3 - 28t^2 + 66t + 35$, where $C(t)$ is the number of customers (in tens), t hours after 12 noon. (a) How many customers were in the mall at noon? (b) Were more customers in the mall at 2:00 or at 3:00 P.M.? How many more? (c) Was the mall busier at 1:00 P.M. (after lunch) or 6:00 P.M. (around dinner time)?

SECTION 3.3 Zeroes of Polynomial Functions

KEY CONCEPTS

- Fundamental theorem of algebra: Every complex polynomial of degree $n \geq 1$ has at least one complex zero.
- Linear factorization theorem: Every complex polynomial of degree $n \geq 1$ has exactly n linear factors, and can be written in the form $p(x) = a(x - c_1)(x - c_2) \ldots (x - c_n)$, where $a \neq 0$ and $c_1, c_2, \ldots, c_n$ are (not necessarily distinct) complex numbers.

- For a polynomial p in factored form with repeated factors $(x - c)^m$, c is a zero of multiplicity m. If m is odd, c is a zero of odd multiplicity; if m is even, c is a zero of even multiplicity.
- Corollaries to the linear factorization theorem:
  I. If p is a polynomial with real coefficients, p can be factored into linear factors (not necessarily distinct) and irreducible quadratic factors having real coefficients.
  II. If p is a polynomial with real coefficients, the complex zeroes of p must occur in conjugate pairs. If $a + bi$ $(b \neq 0)$, is a zero, then $a - bi$ is also a zero.
  III. If p is a polynomial with degree $n \geq 1$, then p will have exactly n zeroes (real or complex), where zeroes of multiplicity m are counted m times.
- Intermediate value theorem: If p is a polynomial with real coefficients where $p(a)$ and $p(b)$ have opposite signs, then there is at least one c between a and b such that $p(c) = 0$.
- Rational zeroes theorem: If a real polynomial has integer coefficients, rational zeroes must be of the form $\frac{p}{q}$, where p is a factor of the constant term and q is a factor of the leading coefficient.
- Descartes' rule of signs, upper and lower bounds property, tests for -1 and 1, and graphing technology can all be used with the rational zeroes theorem to factor, solve, and graph polynomial functions.

EXERCISES

Using the tools from this section,

17. List all possible rational zeroes of $p(x) = 4x^3 - 16x^2 + 11x + 10$.

18. Find all rational zeroes of $p(x) = 4x^3 - 16x^2 + 11x + 10$.

19. Write $P(x) = 2x^3 - 3x^2 - 17x - 12$ in completely factored form.

20. Prove that $h(x) = x^4 - 7x^2 - 2x + 3$ has no rational zeroes.

21. Identify two intervals (of those given) that contain a zero of $P(x) = x^4 - 3x^3 - 8x^2 + 12x + 6$: $[-2, -1]$, $[1, 2]$, $[2, 3]$, $[4, 5]$. Then verify your answer using a graphing calculator.

22. Discuss the number of possible positive, negative, and complex zeroes for $g(x) = x^4 + 3x^3 - 2x^2 - x - 30$. Then identify which combination is correct using a graphing calculator.

SECTION 3.4 Graphing Polynomial Functions

KEY CONCEPTS

- All polynomial graphs are smooth, continuous curves.
- A polynomial of degree n has *at most* $n - 1$ turning points. The precise location of these turning points are the local maximums or local minimums of the function.
- If the degree of a polynomial is odd, the ends of its graph will point in opposite directions (like $y = mx$). If the degree is even, the ends will point in the same direction (like $y = ax^2$). The sign of the lead coefficient determines the actual behavior.
- The "behavior" of a polynomial graph near its zeroes is determined by the multiplicity of the zero. For any factor $(x - c)^m$, the graph will "cross through" the x-axis if m is odd and "bounce off" the x-axis (touching at just one point) if m is even. The larger the value of m, the flatter (more compressed) the graph will be near c.
- To "round-out" a graph, additional *midinterval points* can be found between known zeroes.
- These ideas help to establish the *Guidelines for Graphing Polynomial Functions*. See page 327.

EXERCISES

State the degree, end behavior, and y-intercept, but do not graph.

23. $f(x) = -3x^5 + 2x^4 + 9x - 4$

24. $g(x) = (x - 1)(x + 2)^2(x - 2)$

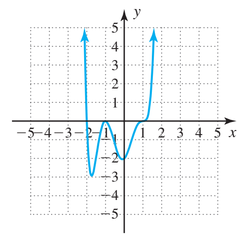
Graph using the *Guidelines for Graphing Polynomials.*

25. $p(x) = (x + 1)^3(x - 2)^2$

26. $q(x) = 2x^3 - 3x^2 - 9x + 10$

27. $h(x) = x^4 - 6x^3 + 8x^2 + 6x - 9$

28. For the graph of $P(x)$ shown, (a) state whether the degree of P is even or odd, (b) use the graph to locate the zeroes of P and state whether their multiplicity is even or odd, and (c) find the minimum possible degree of P and write it in factored form. Assume all zeroes are real.



SECTION 3.5 Graphing Rational Functions

KEY CONCEPTS

- A rational function is one of the form $V(x) = \frac{p(x)}{d(x)}$, where p and d are polynomials and $d(x) \neq 0$.
- The domain of V is all real numbers, except the zeroes of d.
- If zero is in the domain of V, substitute 0 for x to find the y-intercept.
- The zeroes of V (if they exist), are solutions to $p(x) = 0$.
- The line $y = k$ is a horizontal asymptote of V if as $|x|$ increases without bound, $V(x)$ approaches k.
- If $\frac{p(x)}{d(x)}$ is in simplest form, vertical asymptotes will occur at the zeroes of d.
- The line $x = h$ is a vertical asymptote of V if as x approaches h, $V(x)$ increases/decreases without bound.
- If the degree of p is less than the degree of d, $y = 0$ (the x-axis) is a horizontal asymptote. If the degree of p is equal to the degree of d, $y = \frac{a}{b}$ is a horizontal asymptote, where a is the leading coefficient of p, and b is the leading coefficient of d.
- The *Guidelines for Graphing Rational Functions* can be found on page 342.

EXERCISES

29. For the function $V(x) = \frac{x^2 - 9}{x^2 - 3x - 4}$, state the following but do not graph: (a) domain (in set notation), (b) equations of the horizontal and vertical asymptotes, (c) the x- and y-intercept(s), and (d) the value of $V(1)$.

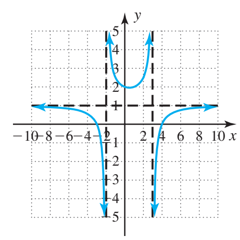
30. For $v(x) = \frac{(x + 1)^2}{x + 2}$, will the function change sign at $x = -1$? Will the function change sign at $x = -2$? Justify your responses.

Graph using the *Guidelines for Graphing Rational Functions.*

31. $v(x) = \frac{x^2 - 4x}{x^2 - 4}$

32. $t(x) = \frac{2x^2}{x^2 - 5}$

33. Use the vertical asymptotes, x-intercepts, and their multiplicities to construct an equation that corresponds to the given graph. Be sure the y-intercept on the graph matches the value given by your equation. Assume these features are integer-valued. Check your work on a graphing calculator.



34. The average cost of producing a popular board game is given by the function $A(x) = \frac{5000 + 15x}{x}$; $x \geq 1000$. (a) Identify the horizontal asymptote of the function and explain its meaning in this context. (b) To be profitable, management believes the average cost must be below \$17.50. What levels of production will make the company profitable?

SECTION 3.6 Additional Insights into Rational Functions

KEY CONCEPTS

- If $V = \frac{p(x)}{d(x)}$ is not in simplest form, with p and d sharing factors of the form $x - c$, the graph will have a removable discontinuity (a hole or gap) at $x = c$. The discontinuity can be "removed" (repaired) by redefining V using a piecewise-defined function.
- If $V = \frac{p(x)}{d(x)}$ is in simplest form, and the degree of p is greater than the degree of d, the graph will have an oblique or nonlinear asymptote, as determined by the quotient polynomial after division. If the degree of p is greater by 1, the result is a slant (oblique) asymptote. If the degree of p is greater by 2, the result is a parabolic asymptote.

EXERCISES

35. Determine if the graph of h will have a vertical asymptote or a removable discontinuity, then graph the function $h(x) = \frac{x^3 - 2x^2 - 9x + 18}{x - 2}$.

36. Sketch the graph of $h(x) = \frac{x^2 - 3x - 4}{x + 1}$. If there is a removable discontinuity, repair the break by redefining h using an appropriate piecewise-defined function.

Graph the functions using the *Guidelines for Graphing Rational Functions.*

37. $h(x) = \frac{x^2 - 2x}{x - 3}$

38. $t(x) = \frac{x^3 - 7x + 6}{x^2}$

39. The cost to make x thousand party favors is given by $C(x) = x^2 - 2x + 6$, where $x \geq 1$ and C is in thousands of dollars. For the average cost of production $A(x) = \frac{x^2 - 2x + 6}{x}$, (a) graph the function, (b) use the graph to estimate the level of production that will make average cost a minimum, and (c) state the average cost of a single party favor at this level of production.

SECTION 3.7 Polynomial and Rational Inequalities

KEY CONCEPTS

- To solve polynomial inequalities, write $P(x)$ in factored form and note the multiplicity of each real zero.
- Plot real zeroes on a number line. The graph will cross the x-axis at zeroes of odd multiplicity (P will change sign), and bounce off the axis at zeroes of even multiplicity (P will not change sign).
- Use the end behavior, y-intercept, or a test point to determine the sign of P in a given interval, then label all other intervals as $P(x) > 0$ or $P(x) < 0$ by analyzing the multiplicity of neighboring zeroes. Use the resulting diagram to state the solution.
- The solution process for rational inequalities and polynomial inequalities is virtually identical, considering that vertical asymptotes also create intervals where function values may change sign, depending on their multiplicity.
- Polynomial and rational inequalities can also be solved using an interval test method. Since polynomials and rational functions are continuous on their domains, the sign of the function at any one point in an interval will be the same as for all other points in that interval.

EXERCISES

Solve each inequality indicated using a number line and the behavior of the graph at each zero.

40. $x^3 + x^2 > 10x - 8$

41. $\frac{x^2 - 3x - 10}{x - 2} \geq 0$

42. $\frac{x}{x - 2} \leq \frac{-1}{x}$

SECTION 3.8 Variation: Function Models in Action

KEY CONCEPTS

- *Direct variation:* If there is a nonzero constant k such that $y = kx$, we say, "y varies directly with x" or "y is directly proportional to x" (k is called the constant of variation).
- *Inverse variation:* If there is a nonzero constant k such that $y = k\left(\frac{1}{x}\right)$ we say, "y varies inversely with x" or y is inversely proportional to x.
- In some cases, direct and inverse variations work simultaneously to form a *joint variation.*
- The process for solving variation equations can be found on page 380.

EXERCISES

Find the constant of variation and write the equation model, then use this model to complete the table.

43. y varies directly as the cube root of x; $y = 52.5$ when $x = 27$.

x	y
216	
	12.25
729	

44. z varies directly as v and inversely as the square of w; $z = 1.62$ when $w = 8$ and $v = 144$.

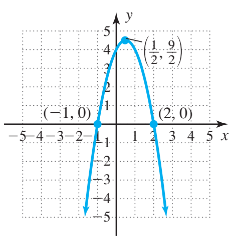
v	w	z
196	7	
	1.25	17.856
24		48

45. Given t varies jointly with u and v, and inversely as w, if $t = 30$ when $u = 2$, $v = 3$, and $w = 5$, find t when $u = 8$, $v = 12$, and $w = 15$.

46. The time that it takes for a simple pendulum to complete one period (swing over and back) is directly proportional to the square root of its length. If a pendulum 16 ft long has a period of 3 sec, find the time it takes for a 36-ft pendulum to complete one period.

MIXED REVIEW

1. Find the equation of the function whose graph is shown.



2. Complete the square to write each function as a transformation. Then graph each function, clearly labeling the vertex and all intercepts (if they exist).

a. $f(x) = 2x^2 + 8x + 3$ **b.** $g(x) = -x^2 - 4x$

3. A computer components manufacturer produces external 2.5″ hard drives. Their sizes range from 20 GB to 200 GB. The cost of producing a hard drive can be modeled by the function $C(s) = \frac{1}{180}s^2 - \frac{8}{9}s + \frac{680}{9}$, where s is the size of the hard drive, in gigabytes. Find the hard drive size that has the lowest cost of production. What is the cost of production?

4. Divide using long division and name the quotient and remainder: $\frac{x^3 + 3x^2 - 5x - 7}{x + 3}$.

5. Divide using synthetic division and name the quotient and remainder: $\frac{x^4 - 3x^2 + 5x - 1}{x + 2}$.

Use synthetic division and the remainder theorem to complete Exercises 6 and 7.

6. State which of the following *are not factors* of $x^3 - 9x^2 + 2x + 48$: (a) $(x + 6)$, (b) $(x - 8)$, (c) $(x - 12)$, (d) $(x - 4)$, (e) $(x + 2)$.

7. Given $P(x) = 6x^3 - 23x^2 - 40x + 31$, find (a) $P(-1)$, (b) $P(1)$, and (c) $P(5)$.

8. Use the factor theorem.

a. Find a real polynomial of degree 3 with roots $x = 3$ and $x = -5i$.

b. Find a real polynomial of degree 2 with $x = 2 - 3i$ as one of the roots.

9. Use the rational zeroes theorem.

a. Which of the following *cannot be* roots of $6x^3 + x^2 - 20x - 12 = 0$?
$x = 9 \quad x = -3 \quad x = \frac{3}{2} \quad x = \frac{8}{3} \quad x = -\frac{2}{3}$

b. Write P in completely factored form. Then state all zeroes of P, real and complex.
$P(x) = x^4 - x^3 + 7x^2 - 9x - 18$.

10. Graph using the *Guidelines for Graphing Polynomials*.

a. $f(x) = x^3 - 13x + 12$

b. $g(x) = x^4 - 10x^2 + 9$

c. $h(x) = (x - 1)^3(x + 2)^2(x + 1)$

Graph using the *Guidelines for Graphing Rational Functions.*

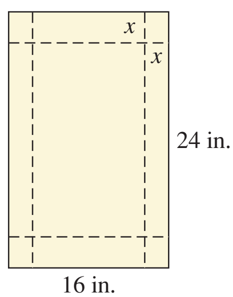
11. $p(x) = \dfrac{x^2 - 2x}{x^2 - 2x + 1}$ **12.** $q(x) = \dfrac{x^2 - 4}{x^2 - 3x - 4}$

13. $r(x) = \dfrac{x^3 - 13x + 12}{x^2}$ (see Exercise 10a) **14.** $y = \dfrac{x^2 - 4x}{x - 3}$

Solve each inequality.

15. $x^3 - 4x < 12 - 3x^2$ **16.** $\dfrac{4}{x + 2} \geq \dfrac{3}{x}$

17. An open, rectangular box is to be made from a 24-in. by 16-in. piece of sheet metal, by cutting a square from each corner and folding up the sides.



a. Show that the resulting volume is given by $V(x) = 4x^3 - 80x^2 + 384x$.

b. Show that for a desired volume of 512 in^3, the height "x" of the box can be found by solving $x^3 - 20x^2 + 96x - 128 = 0$.

c. According to the rational roots theorem *and the context of this application,* what are the possible rational zeroes for this equation?

d. Find the *rational zero x* (the height) that gives the box a volume of 512 in^3.

e. Use the zero from part (d) and synthetic division to help find the *irrational zero x* that also gives the box a volume of 512 in^3. Round the solution to hundredths.

Write the variation equation for each statement.

18. The volume of metal in a circular coin varies directly with the thickness of the coin and the square of its radius.

19. The electrical resistance in a wire varies directly with its length and inversely as the cross-sectional area of the wire.

20. **Cost of copper tubing:** The cost of copper tubing varies jointly with the length and the diameter of the tube. If a 36-ft spool of $\frac{1}{4}$-in. diameter tubing costs \$76.50, how much does a 24-ft spool of $\frac{3}{8}$-in. diameter tubing cost?

PRACTICE TEST

1. Complete the square to write each function as a transformation. Then graph each function and label the vertex and all intercepts (if they exist).

a. $f(x) = -x^2 + 10x - 16$

b. $g(x) = \frac{1}{2}x^2 + 4x + 16$

2. The graph of a quadratic function has a vertex of $(-1, -2)$, and passes through the origin. Find the other intercept, and the equation of the graph in standard form.

3. Suppose the function $d(t) = t^2 - 14t$ models the depth of a scuba diver at time t, as she dives underwater from a steep shoreline, reaches a certain depth, and swims back to the surface.

a. What is her depth after 4 sec? After 6 sec?

b. What was the maximum depth of the dive?

c. How many seconds was the diver beneath the surface?

4. Compute the quotient using long division:
$\dfrac{x^3 - 3x^2 + 5x - 2}{x^2 + 2x + 1}$.

5. Find the quotient and remainder using synthetic division: $\dfrac{x^3 + 4x^2 - 5x - 20}{x + 2}$.

6. Use the remainder theorem to show $(x + 3)$ is a factor of $x^4 - 15x^2 - 10x + 24$.

7. Given $f(x) = 2x^3 + 4x^2 - 5x + 2$, find the value of $f(-3)$ using synthetic division and the remainder theorem.

8. Given $x = 2$ and $x = 3i$ are two zeroes of a real polynomial $P(x)$ with degree 3. Use the factor theorem to find $P(x)$.

9. Factor the polynomial and state the multiplicity of each zero: $Q(x) = (x^2 - 3x + 2)(x^3 - 2x^2 - x + 2)$.

10. Given $C(x) = x^4 + x^3 + 7x^2 + 9x - 18$, (a) use the rational zeroes theorem to list all possible rational zeroes; (b) apply Descartes' rule of signs to count the number of possible positive, negative, and complex zeroes; and (c) use this information along with the tests for 1 and -1, synthetic division, and the factor theorem to factor C completely.

11. Over a 10-yr period, the balance of payments (deficit versus surplus) for a small county was modeled by the function $f(x) = \frac{1}{2}x^3 - 7x^2 + 28x - 32$, where $x = 0$ corresponds to 1990 and $f(x)$ is the deficit or surplus in millions of dollars. (a) Use the rational roots theorem and synthetic division to find the years the county "broke even" (debt = surplus = 0) from 1990 to 2000. (b) How many years did the county run a surplus during this period? (c) What was the surplus/deficit in 1993?

12. Sketch the graph of $f(x) = (x - 3)(x + 1)^3(x + 2)^2$ using the degree, end behavior, x- and y-intercepts, zeroes of multiplicity, and a few "midinterval" points.

13. Use the *Guidelines for Graphing Polynomials* to graph $g(x) = x^4 - 9x^2 - 4x + 12$.

14. Use the *Guidelines for Graphing Rational Functions* to graph $h(x) = \dfrac{x - 2}{x^2 - 3x - 4}$.

15. Suppose the cost of cleaning contaminated soil from a dump site is modeled by $C(x) = \dfrac{300x}{100 - x}$, where $C(x)$ is the cost (in \$1000s) to remove x% of the contaminants. Graph using $x \in [0, 100]$, and use the graph to answer the following questions.

a. What is the significance of the *vertical asymptote* (what does it mean in this context)?

b. If EPA regulations are changed so that 85% of the contaminants must be removed, instead of the 80% previously required, how much additional cost will the new regulations add? Compare the cost of the 5% increase from 80% to 85% with the cost of the 5% increase from 90% to 95%. What do you notice?

c. What percent of the pollutants can be removed if the company budgets \$2,200,000?

16. Graph using the *Guidelines for Graphing Rational Functions*.

a. $r(x) = \dfrac{x^3 - x^2 - 9x + 9}{x^2}$

b. $R(x) = \dfrac{x^3 + 7x - 6}{x^2 - 4}$

17. Find the level of production that will minimize the average cost of an item, if production costs are modeled by $C(x) = 2x^2 + 25x + 128$, where $C(x)$ is the cost to manufacture x hundred items.

18. Solve each inequality

a. $x^3 - 13x \leq 12$ **b.** $\dfrac{3}{x - 2} < \dfrac{2}{x}$

19. Suppose the concentration of a chemical in the bloodstream of a large animal h hr after injection into muscle tissue is modeled by the formula $C(h) = \dfrac{2h^2 + 5h}{h^3 + 55}$.

a. Sketch a graph of the function for the intervals $x \in [-5, 20]$, $y \in [0, 1]$.

b. Where is the vertical asymptote? Does it play a role in this context?

c. What is the concentration after 2 hr? After 8 hr?

d. How long does it take the concentration to fall below 20% $[C(h) < 0.2]$?

e. When does the maximum concentration of the chemical occur? What is this maximum?

f. Describe the significance of the horizontal asymptote in this context.

20. The maximum load that can be supported by a rectangular beam varies jointly with its width and its height squared and inversely with its length. If a beam 10 ft long, 3 in. wide, and 4 in. high can support 624 lb, how many pounds could a beam support with the same dimensions but 12 ft long?

CALCULATOR EXPLORATION AND DISCOVERY

Complex Zeroes, Repeated Zeroes, and Inequalities

This *Calculator Exploration and Discovery* will explore the relationship between the solution of a polynomial (or rational) inequality and the complex zeroes and repeated zeroes of the related function. After all, if complex zeroes can never create an x-intercept, how do they affect the function? And if a zero of even multiplicity never crosses the x-axis (always bounces), can it still affect a nonstrict (*less than or equal to* or *greater than or equal to*) inequality? These are interesting and important questions, with numerous avenues of exploration. To begin, consider the function $Y_1 = (x + 3)^2(x^3 - 1)$. In completely factored form $Y_1 = (x + 3)^2(x - 1)(x^2 + x + 1)$. This is a polynomial function of degree 5 with two real zeroes (one repeated), two complex zeroes (the quadratic factor is irreducible), and after viewing the graph on Figure 3.76, four turning points. From the graph (or by analysis), we have $Y_1 \le 0$ for $x \le 1$. Now let's consider $Y_2 = (x + 3)^2(x - 1)$, the same function as Y_1, less the quadratic factor. Since complex zeroes never "cross the x-axis" anyway, the removal of this factor *cannot affect the solution set of the inequality!* But how does it affect the function? Y_2 is now a function of degree three, with three real zeroes (one repeated) and only two turning points (Figure 3.77). But even so, the solution to $Y_2 \le 0$ is the same as for $Y_1 \le 0$: $x \le 1$. Finally, let's look at $Y_3 = x - 1$, the same function as Y_2 but with the repeated zero removed. The key here is to notice that since $(x - 3)^2$ will be nonnegative for any value of x, it too does not change the solution set of the "less than or equal to inequality," only the shape of the graph. Y_3 is a function of degree 1, with one real zero and no turning points, *but the solution interval for* $Y_3 \le 0$ *is the same solution interval as* Y_2 *and* Y_1: $x \le 1$ (see Figure 3.78).

Explore these relationships further using the following exercises and a "greater than or equal to" inequality. Begin by writing Y_1 in completely factored form.

Exercise 1: $Y_1 = (x^3 - 6x^2 + 32)(x^2 + 1)$
$Y_2 = x^3 - 6x^2 + 32$
$Y_3 = x + 2$

Exercise 2: $Y_1 = (x + 3)^2(x^3 - 2x^2 + x - 2)$
$Y_2 = (x + 3)^2(x - 2)$
$Y_3 = x - 2$

Exercise 3: Based on what you've noticed, comment on how the irreducible quadratic factors of a polynomial affect its graph. What role do they play in the solution of inequalities?

Exercise 4: How do zeroes of even multiplicity affect the solution set of nonstrict inequalities (less/greater than or equal to)?

For more on these ideas, see the *Strengthening Core Skills* feature from this chapter.

Figure 3.76

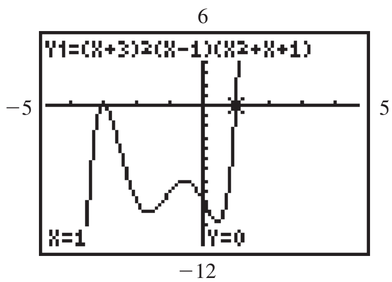


Figure 3.77

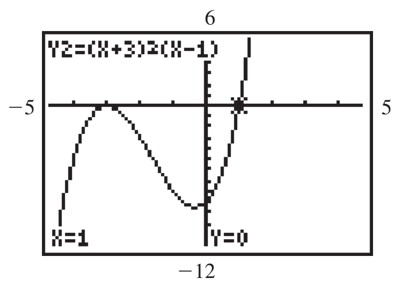
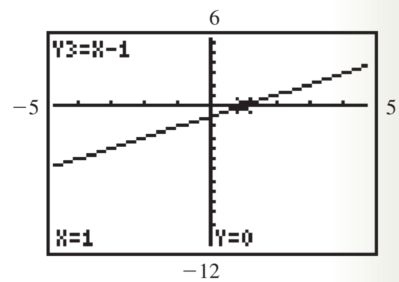


Figure 3.78

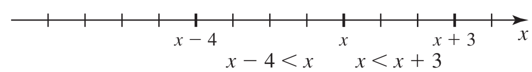


STRENGTHENING CORE SKILLS

Solving Inequalities Using the Push Principle

The most common method for solving polynomial inequalities involves finding the zeroes of the function and checking the sign of the function in the intervals between these zeroes. In Section 3.7, we relied on the end behavior of the graph, the sign of the function at the y-intercept, and the multiplicity of the zeroes to determine the solution. There is a third method that is more conceptual in nature, but in many cases highly efficient. It is based on two very simple ideas, the first involving only order relations and the number line:

A. Given any number x and constant $k > 0$: $x > x - k$ and $x < x + k$.



This statement simply reinforces the idea that if a is left of b on the number line, then $a < b$. As shown in the diagram, $x - 4 < x$ and $x < x + 3$, from which $x - 4 < x + 3$ for any x.

B. The second idea reiterates well-known ideas regarding the multiplication of signed numbers. For any number of factors:

if there are an even number of negative factors, the result is positive;

if there are an odd number of negative factors, the result is negative.

These two ideas work together to solve inequalities using what we'll call the *push principle*. Consider the inequality $x^2 - x - 12 > 0$. The factored form is $(x - 4)(x + 3) > 0$ and we want the product of these two factors to be positive. From (A), both factors will be positive if $(x - 4)$ is positive, since it's the smaller of the two; and both factors will be negative if $x + 3 < 0$, since it's the larger. The solution set is found by solving these two simple inequalities: $x - 4 > 0$ gives $x > 4$ and $x + 3 < 0$ gives $x < -3$. If the inequality were $(x - 4)(x + 3) < 0$ instead, we require one negative factor and one positive factor. Due to order relations and the number line, the larger factor must be the positive one: $x + 3 > 0$ so $x > -3$. The smaller factor must be the negative one: $x - 4 < 0$ and $x < 4$. This gives the solution $-3 < x < 4$ as can be verified using any alternative method. Solutions to all other polynomial and rational inequalities are an extension of these two cases.

Illustration 1 ▶ Solve $x^3 - 7x + 6 < 0$ using the push principle.

Solution ▶ The polynomial can be factored using the tests for 1 and -1 and synthetic division. The factors are $(x - 2)(x - 1)(x + 3) < 0$, which we've conveniently written in increasing order. For the product of three factors to be negative we require: (1) three negative factors or (2) one negative and two positive factors. The first condition is met by simply making the largest factor negative, as it will ensure the smaller factors are also negative: $x + 3 < 0$ so $x < -3$. The second condition is met by making the smaller factor negative and the "middle" factor positive: $x - 2 < 0$ *and* $x - 1 > 0$. The second solution interval is $x < 2$ and $x > 1$, or $1 < x < 2$.

Note the push principle does not require the testing of intervals between the zeroes, nor the "cross/bounce" analysis at the zeroes and vertical asymptotes (of rational functions). In addition, irreducible quadratic factors can still be ignored as they contribute nothing to the solution of real inequalities, and factors of even multiplicity can be overlooked precisely because there is no sign change at these roots.

Illustration 2 ▶ Solve $(x^2 + 1)(x - 2)^2(x + 3) \geq 0$ using the push principle.

Solution ▶ Since the factor $(x^2 + 1)$ does not affect the solution set, this inequality will have the same solution as $(x - 2)^2(x + 3) \geq 0$. Further, since $(x - 2)^2$ will be nonnegative for all x, the original inequality *has the same solution set as* $(x + 3) \geq 0$! The solution is $x \geq -3$.

With some practice, the push principle can be a very effective tool. Use it to solve the following exercises. Check all solutions by graphing the function on a graphing calculator.

Exercise 1: $x^3 - 3x - 18 \leq 0$

Exercise 2: $\dfrac{x + 1}{x^2 - 4} > 0$

Exercise 3: $x^3 - 13x + 12 < 0$

Exercise 4: $x^3 - 3x + 2 \geq 0$

Exercise 5: $x^4 - x^2 - 12 > 0$

Exercise 6: $(x^2 + 5)(x^2 - 9)(x + 2)^2(x - 1) \geq 0$

CUMULATIVE REVIEW CHAPTERS R–3

1. Solve for R: $\dfrac{1}{R} = \dfrac{1}{R_1} + \dfrac{1}{R_2}$

2. Solve for x: $\dfrac{2}{x + 1} + 1 = \dfrac{5}{x^2 - 1}$

3. Factor the expressions:

   a. $x^3 - 1$ **b.** $x^3 - 3x^2 - 4x + 12$

4. Solve using the quadratic formula. Write answers in both exact and approximate form: $2x^2 + 4x + 1 = 0$.

5. Solve the following inequality: $x + 3 < 5$ *or* $5 - x < 4$.

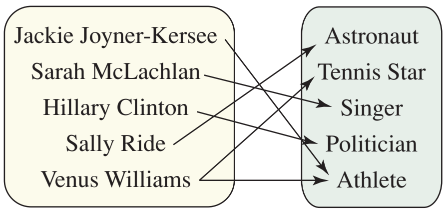
6. Name the eight toolbox functions, give their equations, then draw a sketch of each.

7. Use substitution to verify that $x = 2 - 3i$ is a solution to $x^2 - 4x + 13 = 0$.

8. Solve the rational inequality: $\dfrac{x + 4}{x - 2} < 3$.

9. As part of a study on traffic conditions, the mayor of a small city tracks her driving time to work each day for six months and finds a linear and increasing relationship. On day 1, her drive time was 17 min. By day 61 the drive time had increased to 28 min. Find a linear function that models the drive time and use it to estimate the drive time on day 121, if the trend continues. Explain what the slope of the line means in this context.

10. Does the relation shown represent a function? If not, discuss/explain why not.



11. The data given shows the profit of a new company for the first 6 months of business, and is closely modeled by the function $p(m) = 1.18x^2 - 10.99x + 4.6$; where $p(m)$ is the profit earned in month m. Assuming this trend continues, use this function to find the first month a profit will be earned $(p > 0)$.

Exercise 11

Month	Profit (1000s)
1	−5
2	−13
3	−18
4	−20
5	−21
6	−19

12. Graph the function $g(x) = \dfrac{-1}{(x+2)^2} + 3$ using transformations of a basic function.

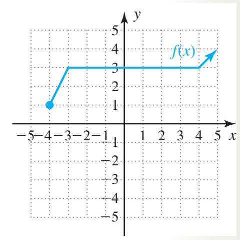
13. Find $f^{-1}(x)$, given $f(x) = \sqrt[3]{2x - 3}$, then use composition to verify your inverse is correct.

14. Graph $f(x) = x^2 - 4x + 7$ by completing the square, then state intervals where:

a. $f(x) \geq 0$ **b.** $f(x)\uparrow$

15. Given the graph of a general function $f(x)$, graph $F(x) = -f(x + 1) + 2$.

Exercise 15



16. Graph the piecewise-defined function given:

$$f(x) = \begin{cases} -3 & x < -1 \\ x & -1 \leq x \leq 1 \\ 3x & x > 1 \end{cases}$$

17. Y varies directly with X and inversely with the square of Z. If $Y = 10$ when $X = 32$ and $Z = 4$, find X when $Z = 15$ and $Y = 1.4$.

18. Use the rational zeroes theorem and synthetic division to find all zeroes (real and complex) of $f(x) = x^4 - 2x^2 + 16x - 15$.

19. Sketch the graph of $f(x) = x^3 - 3x^2 - 6x + 8$.

20. Sketch the graph of $h(x) = \dfrac{x - 1}{x^2 - 4}$ and use the zeroes and vertical asymptotes to solve $h(x) \geq 0$.