11.8 Conditional Probability and Expected Value

LEARNING OBJECTIVES

In Section 11.8 you will learn how to:

A. Compute conditional probabilities
B. Identify independent and dependent events
C. Compute expected values and decide the "fairness" of a game of chance

INTRODUCTION

In the movie *Apollo 13* (1995, Tom Hanks, Kevin Bacon), the Saturn rocket loses one of its four boosters shortly after takeoff and a decision must be made. Should the flight be aborted or can it safely continue? Essentially, the people in flight control had to decide on the probability of successfully making orbit, given that one of the boosters had failed (intuitively we know the probability must be lower than if all four engines were functioning properly). When we compute the probability of an event, knowing a related event has already occurred, we are using *conditional probability*.

POINT OF INTEREST

Although many believe it to be a modern phenomenon, lotteries have existed for over 2000 years. The emperors of ancient Rome often raised public funds by sponsoring a lottery, as did other civilizations. Lotteries were even used in the colonial days of the United States, as authorized by the colonial congress to raise funds for improving public works such as roads, buildings, harbors, and churches. In 1776, the Continental Congress authorized a lottery to support the revolutionary army. Using the concept of expected value, we will see that lotteries are by their nature an "unfair" game, since the state makes a profit only if the game is slanted in its favor.

A. Computing Conditional Probabilities

To help understand the concept of **conditional probability,** consider the simple experiment of drawing two colored balls from a bag containing 5 red, 2 green, and 3 blue balls. If we draw one ball, replace it, then draw a second time (this is called **drawing with replacement**), the probability both are red is $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$. If we do not replace the first ball, the probability both are red drops to $\frac{1}{2} \cdot \frac{4}{9} = \frac{2}{9}$, since one red was removed, leaving only 4 of 9 in the bag. When the probability of a second event E_2, depends on some first event E_1, we say they are **dependent events.** We will soon define this relationship more formally, but for now we use the idea as an indication the probability of E_2 is no longer simply $\dfrac{n(E_2)}{n(S)}$. This dependency places certain conditions on the outcomes, hence the name conditional probability. For convenience, we will use the notation $P(E_2|E_1)$, which is understood to mean, "the probability of E_2, given that E_1 has occurred." Many applications of conditional probability can be solved using tree diagrams, or counting outcomes and using the basic definition of probability.

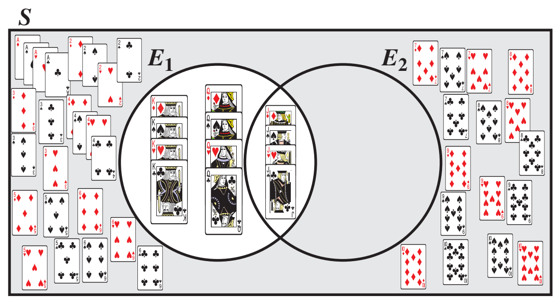
EXAMPLE 1 One card is randomly selected from a standard deck of 52 cards. What is the probability it is a Jack, given that it is a face card?

Solution: For this exercise we have E_1:(a face card is drawn) and E_2:(a Jack is drawn). The value of $P(E_2)$ alone is $\frac{4}{52}$ since there are four Jacks in the deck. However, we already know a face card was drawn and there are only 12 of these. This means $P(E_2|E_1) = \frac{4}{12} = \frac{1}{3}$, a much higher probability.

NOW TRY EXERCISES 7 THROUGH 11

Conditional probabilities can also be illustrated using Venn diagrams, as shown in Figure 11.28 where we see that since E_1:(face card) has already occurred, the chances of E_2:(Jack) are greatly increased.

Figure 11.28



EXAMPLE 2 Again consider the experiment of drawing two colored balls, without replacement, from a bag containing 5 red, 2 green, and 3 blue balls. What is the probability the second ball is green if the first ball drawn is green?

Solution: Define E_1: *a green ball is drawn* and E_2: *a second green ball is drawn*. Because we're not replacing the first ball, the probability of E_2 depends on E_1. To find $P(E_2|E_1)$, we reason that the value of $P(E_1)$ alone is $\frac{2}{10} = \frac{1}{5}$, since 2 of the 10 balls are green. However, there is one fewer green ball as we draw the second time, leaving 1 out of 9, so $P(E_2|E_1) = \frac{1}{9}$.

NOW TRY EXERCISES 12 THROUGH 20

For complex events or large sample spaces, computing $P(E_2|E_1)$ through direct reasoning becomes more difficult. However, by making some observations regarding the first two examples we can construct a formula for $P(E_2|E_1)$. Using Example 1 and the related diagram, we found $P(E_2|E_1)$ was $\frac{4}{12}$ by counting the number of elements in $E_2 \cap E_1$ and dividing by the number of elements in E_1. In other words, $P(E_2|E_1) = \dfrac{n(E_2 \cap E_1)}{n(E_1)}$ from the elementary definition of probability. Dividing the numerator and denominator of the right-hand expression by $n(S)$ yields this formula for conditional probability: $P(E_2|E_1) = \dfrac{P(E_2 \cap E_1)}{P(E_1)}$.

CONDITIONAL PROBABILITY

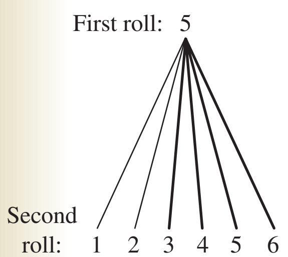
Given sample space S and events E_1 and E_2 defined relative to S. The probability of E_2, given that E_1 has already occurred is

$$P(E_2|E_1) = \frac{P(E_2 \cap E_1)}{P(E_1)}$$

EXAMPLE 3 Consider the experiment of rolling one die twice in succession. What is the probability the sum of both rolls is greater than 7, given the first roll is a five?

WORTHY OF NOTE

As suggested by the discussion prior to Example 3, many conditional probabilities can be reasoned out using the elementary definition of probability and a tree diagram or organized list. For Example 3, the sample space is $S = \{(5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6)\}$ and four of the six yield a sum greater than 7: $P(E_2) = \frac{4}{6} = \frac{2}{3}$. This probability is also illustrated by the tree diagram shown in the figure.



Solution: Define E_1: *first die is 5* and E_2: *sum* > 7. We have $P(E_1) = \frac{1}{6}$ directly and $P(E_2 \cap E_1) = \frac{4}{36} = \frac{1}{9}$, since there are four ways to obtain a sum greater than 7, with a first roll of 5: (5, 3), (5, 4), (5, 5), and (5, 6). Therefore, $P(E_2|E_1) = \dfrac{P(E_2 \cap E_1)}{P(E_1)} = \dfrac{\frac{1}{9}}{\frac{1}{6}} = \dfrac{2}{3}$.

NOW TRY EXERCISES 21 THROUGH 26

As illustrated by Examples 4 and 5, simple counting, using tables or data, and the quick-counting methods studied previously can all be used to help compute conditional probabilities.

EXAMPLE 4 A survey is taken to gauge public opinion regarding government spending on defense. Part of the results are shown in the table, which show a distinct difference of opinion by age. Use the table to answer the questions that follow.

Age Group (years)	Favor an Increase	Oppose an Increase	Total
18–45	80	270	350
46–80	120	30	150
Total	200	300	500

If one person from the survey were randomly selected, what is the probability the voter felt (a) defense spending should be increased; (b) defense spending should be increased, given she is between 18 and 45 years of age; and (c) defense spending should be increased, given he is between 46 and 80?

Solution:

a. Since 200 of the 500 voters surveyed felt spending should be increased, $P(\text{increase spending}) = \dfrac{200}{500} = 0.4$.

b. Define E_1:$(18 \le \text{age} \le 45)$ and E_3:(increase spending). Using the table, $P(E_1) = \dfrac{350}{500} \approx 0.7$ and $P(E_3 \cap E_1) = \dfrac{80}{500} = 0.16$, so $P(E_3|E_1) = \dfrac{0.16}{0.7} \approx 0.23$.

c. Define E_2:$(46 \le \text{age} \le 80)$, giving $P(E_2) = 0.3$. Again from the table, $P(E_3 \cap E_2) = \dfrac{120}{500} = 0.24$, so $P(E_3|E_2) = \dfrac{0.24}{0.3} = 0.8$. The support for increased defense spending is dramatically higher among the older group.

NOW TRY EXERCISES 27 AND 28

Although the calculations can become unwieldy, the formula for conditional probability can be extended to include any number of dependent events, as when drawing cards from a standard deck without replacement.

EXAMPLE 5 Four cards are drawn without replacement from a well-shuffled deck of 52 cards. What is the probability that four Aces are drawn?

Solution: If we let A_i represent "an Ace is drawn," we can compute this probability using the FPC as $P(A_1) \cdot P(A_2|A_1) \cdot P(A_3|A_2 \cap A_1) \cdot P(A_4|A_3 \cap A_2 \cap A_1)$. This means $P(\text{four Aces are drawn}) = \frac{4}{52} \cdot \frac{3}{51} \cdot \frac{2}{50} \cdot \frac{1}{49}$, or about 0.0000037. This probability can actually by computed more efficiently using quick-counting: $P(\text{four Aces are drawn}) = \frac{{}_4C_4}{{}_{52}C_4}$ or about 0.0000037.

NOW TRY EXERCISES 29 AND 30

B. Dependent and Independent Events

Although the notion of dependency and dependent events is often intuitive (we *sense* that one event depends on another), a more formal test can be helpful. Consider the experiment of rolling one die twice in succession, and define E_1:(a 5 is rolled) and E_2:(a 2 is rolled). From our previous work we know $P(E_1) = \frac{1}{6} = P(E_2)$. To compute $P(E_2|E_1)$, we can use the chart for all 36 possibilities of two dice, which shows 1 occurrence of (5, 2) (5 rolled first, 2 rolled second) so $P(E_2 \cap E_1) = \frac{1}{36}$. The computation for $P(E_2|E_1)$ is $\frac{\frac{1}{36}}{\frac{1}{6}} = \frac{1}{6}$, *which is the same as* $P(E_2)$! In other words, $P(E_2|E_1) = P(E_2)$. Upon reflection, we realize that the number rolled second does not depend on the first, and that E_1 and E_2 are **independent events.** In fact, these observations lead to our formal definition.

INDEPENDENT EVENTS

Given a sample space S and events E_1 and E_2 defined relative to S,

If $P(E_2|E_1) = P(E_2)$,

then E_1 and E_2 are independent events, otherwise E_1 and E_2 are dependent.

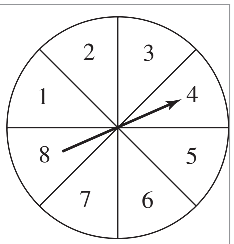
The formula for conditional probability can be used to find an alternative test for independence that is often useful. If $P(E_2|E_1) = P(E_2)$, then $P(E_2|E_1) = \frac{P(E_2 \cap E_1)}{P(E_1)}$ gives $P(E_2) = \frac{P(E_2 \cap E_1)}{P(E_1)}$ by substitution, and leads to $P(E_1)P(E_2) = P(E_2 \cap E_1)$ after multiplying both sides by $P(E_1)$. We conclude that if $P(E_2 \cap E_1) = P(E_1)P(E_2)$, then E_1 and E_2 are independent events.

The formula for conditional probability can also be used to find an alternative form for $P(E_2 \cap E_1)$. Multiplying both sides by $P(E_1)$ yields $P(E_2 \cap E_1) = P(E_1) \cdot P(E_2|E_1)$, which for Example 2 gives the probability that *both balls drawn are green* (instead of the probability the second ball drawn is green). With $P(E_1) = \frac{2}{10}$, and $P(E_2|E_1) = \frac{1}{9}$, we have $P(E_2 \cap E_1) = P(E_1)P(E_2|E_1) = \left(\frac{2}{10}\right)\left(\frac{1}{9}\right) = \frac{1}{45}$.

WORTHY OF NOTE

No matter how many times a coin is flipped, each flip is independent of the other. As we've seen, the probability of flipping two heads in a row is $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$. This is a simple illustration of the formula for independent events, since $P(E_2 \cap E_1) = P(E_2)P(E_1)$ can be written as $P(H_2 \cap H_1) = P(H_2)P(H_1)$ and extended to include any number of flips (or any independent events).

EXAMPLE 6A Consider the spinner shown in the figure. Define E_1:(first spin is 3) and E_2:(second spin is 7). Are E_1 and E_2 independent?



Solution: Using the FPC, the sample space for this experiment has 64 outcomes, 8 possibilities for the first spin and 8 for the second. E_1 has the potential outcomes (3, 1), (3, 2), . . . , (3, 7), (3, 8), so $P(E_1) = \frac{1}{8}$. For E_2 we have (1, 7), (2, 7), . . . , (7, 7), (8, 7), so $P(E_2) = \frac{1}{8}$. Since $E_2 \cap E_1$ has the one common element (3, 7), $P(E_2 \cap E_1) = \frac{1}{64}$. Using the test for independence we have $P(E_2)P(E_1) = P(E_2 \cap E_1)$, indicating that E_1 and E_2 are indeed independent events.

EXAMPLE 6B Consider the spinner from Example 6A, and define the events E_1: *first spin is greater than* 2, and E_2: *sum of two spins is* 5. Are E_1 and E_2 independent?

Solution: On the spinner, six of the eight numbers are greater than 2 so $P(E_1) = \frac{3}{4}$. The possibilities for E_2 are (1, 4), (2, 3), (3, 2), and (4, 1), and with $8 \cdot 8 = 64$ elements in the sample space, $P(E_2) = \frac{4}{64} = \frac{1}{16}$. Since only (3, 2) and (4, 1) satisfy $(E_2 \cap E_1)$, we have $P(E_2 \cap E_1) = \frac{2}{64} = \frac{1}{32}$. Noting $\frac{3}{4} \cdot \frac{1}{16} = \frac{3}{64} \neq \frac{1}{32}$, $P(E_2 \cap E_1) \neq P(E_1)P(E_2)$, so E_1 and E_2 are dependent.

NOW TRY EXERCISES 33 THROUGH 45

EXAMPLE 7 The Graduate Department at Cahokia University is trying to establish a policy regarding the number of times a student should be allowed to take the entrance exam. Over a long period of time, data on pass/fail rates are collected and the results are shown in the table. (a) Find the probability of a student failing the first attempt and passing on the second, and (b) find the probability of a student failing the first two attempts and passing on the third.

First Attempt		Second Attempt		Third Attempt	
Pass	Fail	Pass	Fail	Pass	Fail
64%	36%	70%	30%	25%	75%

Solution:

a. Define E_1:(fails on first attempt) and E_2:(passes on second attempt). Since E_2 depends on E_1, $P(E_1 \cap E_2) = P(E_1) \cdot P(E_2|E_1)$. From the table, $P(E_1) = 0.36$ and $P(E_2|E_1) = 0.70 \Rightarrow P(E_1 \cap E_2) = 0.36(0.70) = 0.252$.

b. Define E_1:(fails on first attempt), E_2:(fails on second attempt), and E_3:(passes on third attempt). The events are once again dependent meaning $P(E_1 \cap E_2 \cap E_3) = P(E_1) \cdot P(E_2|E_1) \cdot P(E_3|E_2 \cap E_1)$. From the table, $P(E_3|E_2 \cap E_1) = 0.25$, so $P(E_1 \cap E_2 \cap E_3) = 0.36 \cdot 0.30 \cdot 0.25$, or about 0.027. With so small a probability of passing on the third attempt, the

Graduate Department may decide to limit applicants to two attempts on the entrance exam.

NOW TRY EXERCISES 46 AND 47

C. Expected Values

Although people who play games of chance know the odds of winning favor the establishment, there is still a small hope of "winning the big one." But most are realistic enough to know that over a long period of time and repeated trials, there is a high expectation of losing. The concept of **expected value,** also called **mathematical expectation,** helps to quantify the expected return on a game of chance *assuming it is played a large number of times.*

Consider a simple game where you pay \$0.50 to roll one die, and win \$0.85 if you roll a 1 or 2, but lose your money otherwise. If you play this game many, many times, how much would expect to lose (or win)? The probability of losing on any one roll is $\frac{2}{3}$, while the probability of winning is $\frac{1}{3}$. So if you play this game 900 times (a large, arbitrarily chosen number), you could expect to lose an average of $\frac{2}{3} \cdot 900 = 600$ times and win an average of $\frac{1}{3} \cdot 900 = 300$ times. Your "winnings" at this point would be $\$0.85(300) + (-\$0.50)(600) = -\$45$ and if you average this loss over the 900 games, your expected loss (expected value) per game is $\frac{-45}{900} = 0.05$ cents per roll. Using this example as a model, the basic idea can be extended to include other payoffs and other outcomes, or generalized to fit many different situations.

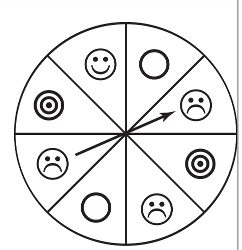
EXPECTED VALUE

If an experiment has two defined outcomes that occur with probabilities p_1 and p_2, and if the value of each is v_1 and v_2, respectively, then the expected value E of the experiment is given by

$$E = p_1v_1 + p_2v_2$$

If $E = 0$, the experiment is said to be fair (favoring neither player nor the establishment). The formula can be extended to cover any number of outcomes.

EXAMPLE 8A A school board comes up with a novel idea to attract volunteers, and invites all parents to step up and spin a specially made spinner. If the spinner lands on an ○, there is no penalty or reward, the player simply spins again. If it lands on a target ◎, the player must work one volunteer hour. If it lands on a frown, the player must commit to double-time (2 hr), but if it lands on a smile, the player wins 6 hr of domestic help from the board members themselves. From the prospective volunteer's point of view, what is the expected value of this game?



Solution: First we consider the respective probabilities: $P(○) = \frac{1}{4}$, $P(◎) = \frac{1}{4}$, $P(☹) = \frac{3}{8}$, and $P(☺) = \frac{1}{8}$. The value of each is 0, −1, −2, and 6 volunteer hours, respectively. This means the expected value of the game is $E = (0)\left(\frac{1}{4}\right) + (-1)\left(\frac{1}{4}\right) + (-2)\left(\frac{3}{8}\right) + 6\left(\frac{1}{8}\right) = -0.25$. Players can expect to commit to 0.25 hr = 15 min of volunteer work for each spin. From the board's point of view, if the game is played 1000 times, they will gain 0.25(1000) = 250 hr of volunteer work.

If the expected value is 0, neither the player nor the establishment (the one sponsoring the game or experiment) has an advantage, so the game is considered **fair.** Gaming houses generally operate on very small expected values, because the volume of people playing the games still makes them very profitable.

EXAMPLE 8B With regard to Example 8A, what value should be assigned to the smiley face "☺" if the board wanted the game to be fair?

Solution: Since we want a factor (or value) other than 6 for $\frac{1}{8}$, replace 6 with the variable V, set $E = 0$ and solve: $(0)\left(\frac{1}{4}\right) + (-1)\left(\frac{1}{4}\right) + (-2)\left(\frac{3}{8}\right) + V\left(\frac{1}{8}\right) = 0$ gives $V = 8$. The game will be fair if players win 8 hr for spinning a ☺.

Many amusement parks and game rooms have a "Claw Machine" that allows players to move a claw front and back and left and right until it is directly above a desired toy or stuffed animal. The player then drops the claw in hopes it will latch onto the object, lift it, and drop it into a chute that delivers the object to the player. In contrast to Example 8A, there is a cost to play the game *that must be deducted from the value of any prize(s) won.*

EXAMPLE 8C For the Claw Machine described, the game delivers a \$6 prize 2.5% of the time, a \$3 prize 5% of the time and a \$1.50 bag of candy 20% of the time.

a. If it costs \$1 to play the game, what is the expected value for the player?

b. If the vendor wanted an expected value of \$0.60 profit, what should a patron be charged to play the game?

Solution:

a. Pairing the value of each prize (minus the cost) with the likelihood of winning that prize yields $E = \$5(0.025) + \$2(0.05) + \$0.50(0.20) - \$1(0.725) = -\$0.40$. If the game is played a large number of times, a player can expect to lose (or the vendor can expect to gain) 40¢ each time it's played.

b. To find the necessary charge for an expected value of $-\$0.60$ (a vendor profit of 60¢), we let V represent what the vendor should charge, set $E = -\$0.60$, then solve. This gives $-\$0.60 = \$5(0.025) + \$2(0.05) + \$0.50(0.20) - V(\$0.725)$, and after simplifying we obtain $-0.925 = -0.725V$, and find V is approximately \$1.28. For an expected value (profit) of 60¢, about \$1.28 should be charged to play the game. However, it is more likely the vendor will charge \$1.25 and be satisfied with an expected value of 58.125¢ (Why?).

NOW TRY EXERCISES 48 THROUGH 59

There are other ways the vendor could increase the expected profit. One is by offering cheaper prizes. Can you think of another? This idea is explored in the *Technology Highlight* that follows.

As mentioned in Section 11.7, the formal study of probability began with questions regarding gambling and games of chance. The game of roulette originated in late seventeenth-century France, and is typically played on a wheel with 38 slots numbered 00, 0, and 1 through 36, although not in sequence. The 00 and 0 slots are green, and all other slots alternate in color, black/red/black (and so on), enabling players to place wagers many different ways. The wheel is spun, then a ball is dropped onto the wheel and is equally likely to end up in any one of the 38 slots. To make a "single-number" wager, a bet is placed for any slot except 0 and 00, and if the ball ends up in that slot, the player wins \$35 for every \$1 wagered. Also, unlike Example 8A, where there was no cost associated with playing, and Example 8C where the cost was \$1 whether you won a prize or not, the game of roulette *returns your initial bet to you if you win.*

EXAMPLE 9 Max decides to play roulette for the rest of the evening and repeatedly places a \$1 wager on the number 22. (a) What is the probability the ball lands on 22? (b) What is the probability the ball does not land on 22? (c) What is the expected value of this game?

Solution:

a. Since the are 38 slots and Max has bet on only one of them, $P(22) = \frac{1}{38}$.

b. This is the complement of part (a), so $P(\sim 22) = \frac{37}{38}$.

c. Max expects to gain \$35 if he wins and lose \$1 if he loses, so expected value is $E = 35\left(\frac{1}{38}\right) + (-1)\left(\frac{37}{38}\right)$ or about -0.053. If Max continues to play this game for a long period of time, he can expect to lose about 5.3¢ on every dollar wagered. Note the value of winning (\$35) was not decreased by the cost to play since the initial wager is returned if you win.

NOW TRY EXERCISES 60 AND 61

TECHNOLOGY HIGHLIGHT

Using the Storage and Recall Abilities of a Graphing Calculator

The keystrokes shown apply to a TI-84 Plus model. Please consult your manual or our Internet site for other models.

To explore relationships that have more than one variable, the temporary storage locations on a graphing calculator are a valuable aid. As mentioned, the vendor in Example 8C could also increase the expected value by changing some of the probabilities assigned to each prize, making them harder or easier to win. To preclude the need of repeatedly entering a new expression on the home screen or the Y= screen, the equation can be built using the storage locations A through Z. On the TI-84 Plus, they are printed in green directly above and to the right of various keys. To understand how this is done, first note that if we let *V* represent value and *P* the probability a prize is won, the expected value equation from Example 8C could be written in general terms as

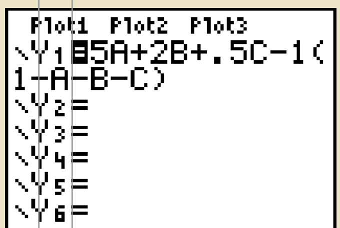
$E = (V_1)(P_1) + (V_2)(P_2) + (V_3)(P_3) - (\text{cost of playing}) \times (1 - P_1 - P_2 - P_3)$.

Let's assume the vendor wants to keep the charge to play the game at $1, and increases the expected profit (lowers the expected value) by adjusting the probability a prize is won, instead offering less expensive prizes. This yields the equation

$E = (5)(P_1) + (2)(P_2) + (0.5)(P_3) - (1)(1 - P_1 - P_2 - P_3)$.

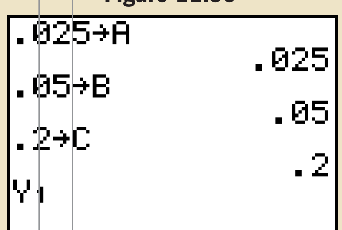
To investigate how various probabilities affect the expected value, we enter $Y_1 = 5A + 2B + 0.50C - 1(1 - A - B - C)$ on the **Y =** screen, using the variables A (**ALPHA** **MATH**), B (**ALPHA** **MATRX**), and C (**ALPHA** **PRGM**). See Figure 11.29.

Figure 11.29



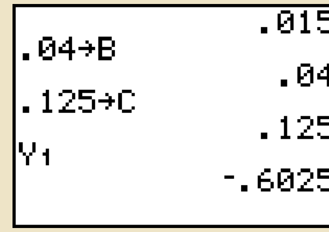
Note that since A, B, and C are constants (although we will be changing their values), Y_1 is a constant function. Let's begin by storing the probabilities from the original problem in A, B, and C (Figure 11.30). To find the expected value, we need only call up the current value of Y_1. As in previous *Technology Highlights*, this is done using **VARS** **▶** **Y-VARS** **ENTER** (since **1:Function** is already highlighted) and **ENTER** (since **1:Y_1** is already highlighted).

Figure 11.30



For Example 8C, the calculator returns an expected value of −0.40. To change any one (or all three) of the probabilities, simply store a new value in that location and recall Y_1 using **2nd** **ENTER** to find the new expected value. Note that changing $P_1 \rightarrow A$ to 0.015, $P_2 \rightarrow B$ to 0.04, and $P_3 \rightarrow C$ to 0.125 comes very close to the vendor's goal of a 60¢ expected profit (see Figure 11.31). Use these ideas to work the following exercises.

Figure 11.31



Exercise 1: Discuss how the equation $Y_1 = 5A + 2B + 0.5C - 1(1 - A - B - C)$ works. In particular, why do we subtract $1(1 - A - B - C)$?

Exercise 2: If $P_1 \rightarrow A$ must be fixed at 0.025, find other (reasonable) possibilities for $P_2 \rightarrow B$ and $P_3 \rightarrow C$ that will result in an expected value of approximately 25¢ for the vendor.

Exercise 3: If the vendor were to charge customers $1.50 to play the game, what is the expected value using:

a. the costs and probabilities from Example 8C?

b. the costs and probabilities used in Exercise 2?

c. prizes costing $7.50, $5.50 and $3.50 with probabilities of 0.03, 0.06 and 0.25 respectively?

11.8 EXERCISES

CONCEPTS AND VOCABULARY

Fill in the blank with the appropriate word or phrase. Carefully reread the section if needed.

1. When the probability of an event E_2, depends on some first event E_1, we say E_2 and E_1 are ______________ events.

2. "The probability of E_2 occurring, given that E_1 has already occurred" is written notationally as ______________.

3. E_1 and E_2 are independent if either $P(E_1 \cap E_2) =$ ________ or $P(E_2|E_1) =$ ______.

4. If an experiment has two outcomes with values V_1 and V_2, and probabilities P_1 and P_2, the expected value is $E =$ ______ + ______.

5. In your own words, discuss how the formula for conditional probability was developed and explain why/how it works.

6. Discuss/explain how an *independent event* differs from a *dependent event*. Include two examples that illustrate the difference.

DEVELOPING YOUR SKILLS

Determine the following probabilities using direct reasoning, tree diagrams, or an organized list.

7. Two colored balls are drawn from a bag containing four blue, three red, and two orange balls. What is the probability both are blue if the two balls are drawn
   a. with replacement
   b. without replacement

8. Two cards are drawn from a standard deck of 52 playing cards. What is the probability they are both face cards if the cards are drawn
   a. with replacement
   b. without replacement

9. At a company party, two door prizes are to be awarded by drawing names from a hat. Every employee can enter twice and writes their name on two slips of paper for the drawing. If the group consists of three males and five females, what is the probability that females win both awards if the drawing is
   a. done with replacement
   b. done without replacement

10. A card is randomly selected from a standard deck. What is the probability the card is
    a. a 7, given that it is a numbered card
    b. a heart, given that it is a red card
    c. a King, given that it is a face card
    d. an even number, given it is a numbered card

11. One pool ball (see Exercise 36 from Section 11.6) is randomly drawn from a bag. Compute the probability the ball is
    a. a multiple of 2, given it is a striped ball
    b. a 10, given it is a striped ball
    c. a 3, given it is a solid color
    d. the cue ball, given it is white

12. Three colored balls are drawn from a bag containing four blue, three red and two orange balls. What is the probability the balls drawn are blue first, then red, then orange if the three balls are drawn
    a. with replacement
    b. without replacement

13. Three cards are drawn from a standard deck of 52 playing cards. What is the probability the cards drawn are face card first, numbered card second, then an Ace if the three cards are drawn
    a. with replacement
    b. without replacement

14. At a company party, three door prizes are to be awarded by drawing names from a hat. Every employee can enter three times and writes their name on three slips of paper for the drawing. If the group consists of three males and five females, what is the probability that females win all three awards if the drawing is
    a. done with replacement
    b. done without replacement

15. Two colored balls are drawn without replacement from a bag containing six green, four maroon, and two yellow balls. What is the probability the second ball drawn is
    a. green, given the first ball was green
    b. maroon, given the first ball was maroon
    c. yellow, given the first ball was not yellow
    d. green, given the first ball was not green

16. Chess is a game played with 16 pieces per player, a light-colored set for one player and a dark-colored set for the other. Each set contains eight pawns, which are front-line pieces, and two castles, two knights, two bishops, one Queen, and one King, which are all back-line pieces. All 32 pieces are placed in a bag and mixed, then two pieces are drawn without replacement. What is the probability the second piece drawn is a

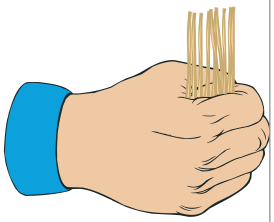
a. pawn, given the first piece was a castle

b. knight, given the first piece was also a knight

c. dark piece, given the first piece was light

d. bishop, given the first piece was front-line

17. The practice of drawing straws was often used in ancient times to assign a task that was usually distasteful or dangerous. One person would take a number of straws (or some other like object) equal to the number of people, make sure that one of the straws was significantly shorter than the others, and grasp them in such a way that they all appeared to be the same length. One was then drawn by each member of the group. If there are 10 people,



a. What is the probability the second person draws a short straw given the first person drew a long straw?

b. What is the probability the third person draws a short straw given the first two people drew long straws?

18. In an alternative form of "drawing straws," an army captain needs to assign latrine duty to someone in her 10-soldier squad. She writes a number between 1 and 10 inclusive on a legal pad, has her soldiers pick a number (with no repetitions), and will assign the duty to whomever picks her number.

a. What is the probability the second person picks the number given the first person did not?

b. What is the probability the fourth person picks the number given the first three people did not?

The game of Black Jack (also called "21") is played with a standard deck of cards. Aces are worth either 1 or 11 points (player's option), face cards are worth 10 points, and numbered cards are worth face value. For instance, King + 7 = 17 points, Ace + 5 = 6 or 16 points, and Ace + (10 or face card) = 11 or 21 points. The object of the game is to get exactly 21 (automatic winner) or as close to 21 as you can, *without going over*. Suppose a person is playing the game alone with a standard deck.

19. What is the probability the player will have

a. 20 points after the second card, given the first card is a 10?

b. 21 points after the second card, given the first card is a 10?

20. What is the probability the player will have

a. 21 points after the second card, given the first card is an Ace?

b. 16 to 20 points (inclusive) after the second card, given the first card is a 10?

Compute the value of $P(E_2|E_1)$ for the values of $P(E_2 \cap E_1)$ and $P(E_1)$ given.

21. $P(E_2 \cap E_1) = 0.12$ and $P(E_1) = 0.4$

22. $P(E_2 \cap E_1) = 0.15$ and $P(E_1) = 0.6$

23. $P(E_2 \cap E_1) = 0.08$ and $P(E_1) = 0.25$

24. $P(E_2 \cap E_1) = 0.10$ and $P(E_1) = 0.3$

25. An experiment consists of rolling a die two times. What is the probability

a. the sum of both rolls is less than 8, given the first roll was a 4.

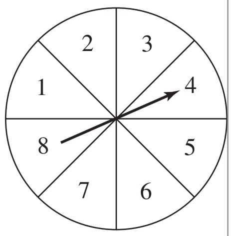
b. the sum of both rolls is more than 9, given the first roll was a 6.

26. The spinner shown is spun twice in succession. What is the probability

a. the sum of both spins less than 12, given the first spin was an 8.

b. the sum of both spins is more than 6, given a 2 was spun first

Exercise 26



27. A poll is taken to measure public opinion concerning America's decision to return to the moon and develop a permanent presence there. The results are given in the table.

Age Group (years)	Favor a Lunar Presence	Oppose a Lunar Presence	Total
18–29	258	102	360
30–49	155	130	285
50–89	107	248	355
Total	520	480	1000

If one person from the survey were selected randomly, what is the probability he/she felt (a) America should not build a permanent presence on the moon; (b) America should not build a permanent presence, given the person is between 50 and 89 years of age; and (c) America should not build a permanent presence, given the person is between ages 18 and 29.

28. A local dealership asks every customer that comes into their Customer Care Center to fill out a survey regarding the quality of service. The results are given in the table.

Value of Car	Excellent Service	Generally Good Service	Generally Poor Service	Total
$5,000–$10,000	76	74	30	180
$10,001–$20,000	52	93	75	221
$20,001–$45,000	22	23	55	100
Total	150	190	160	500

If one person from the survey were selected randomly, what is the probability he/she felt (a) service was excellent; (b) service was excellent, if their car was valued between $10,001 and $20,000; and (c) service was excellent, if their car was valued between $20,001 and $45,000.

29. The 16 balls from a game of pool are placed in a large bag and mixed thoroughly, then 4 balls are drawn without replacement. What is the probability that all 4 balls are striped balls?

30. The chess pieces from Exercise 16 are placed in a large bag and mixed, then five pieces are drawn without replacement. What is the probability of the following sequence: a dark piece is drawn, a dark piece is drawn, a dark piece is drawn, a light piece is drawn, a light piece is drawn.

WORKING WITH FORMULAS

Bayes' theorem: $P(E_2|E_1) = \dfrac{P(E_2) \cdot P(E_1|E_2)}{P(E_2) \cdot P(E_1|E_2) + P(\overline{E}_2) \cdot P(E_1|\overline{E}_2)}$

If the probabilities represented by $P(E_1|E_2)$ and $P(E_1|E_2)$ are known (E_2 represents $1 - E_2$ or that the event that E_2 does not occur), the conditional probability $P(E_2|E_1)$ can be found using Bayes' theorem. For example, consider the experiment of rolling two dice. If E_1 represents *doubles are rolled* and E_2 represents the *sum is greater than or equal to 9*, we then have $P(E_1) = \frac{1}{6}$, $P(E_2) = \frac{10}{36}$, $P(\overline{E}_2) = \frac{26}{36}$, and $P(E_1|E_2) = \frac{2}{10}$, since 2 of the 10 possibilities for E_2 are doubles. In addition we know $P(E_1|\overline{E}_2) = \frac{4}{26}$ since there are four possibilities for doubles in the 26 rolls where the sum is less than 9.

31. Use the values given in Bayes' theorem to calculate $P(E_2|E_1)$, the probability that the sum of the dice is greater than nine, given that doubles have been rolled.

32. Again considering the roll of two dice, let E_1 represent *the sum is a prime number* and E_2 the *sum is 5 or less*. Develop the values required for Bayes' theorem and use it to calculate $P(E_2|E_1)$.

APPLICATIONS

Use the values indicated to determine if E_1 and E_2 are dependent or independent.

33. $P(E_2 \cap E_1) = 0.24$, $P(E_1) = 0.48$, and $P(E_2) = 0.52$

34. $P(E_2 \cap E_1) = 0.36$, $P(E_1) = 0.8$, and $P(E_2) = 0.45$

35. $P(E_2 \cap E_1) = 0.52$, $P(E_1) = 0.8$, and $P(E_2) = 0.65$

36. $P(E_2 \cap E_1) = 0.38$, $P(E_1) = 0.62$, and $P(E_2) = 0.61$

Find the value of E_2 that will ensure E_1 and E_2 are independent.

37. $P(E_2 \cap E_1) = 0.35$, $P(E_1) = 0.7$, and $P(E_2) =$ _____

38. $P(E_2 \cap E_1) = 0.144$, $P(E_1) = 0.8$, and $P(E_2) =$ _____

39. $P(E_2 \cap E_1) = 0.096$, $P(E_1) = 0.16$, and $P(E_2) =$ _____

40. $P(E_2 \cap E_1) = 0.42$, $P(E_1) = 0.8$, and $P(E_2) =$ _____

State whether the events E_1 and E_2 given are dependent or independent. Justify your answer.

41. One die is rolled twice in succession.

a. E_1: first roll is a 5
E_2: second roll is a 3

b. E_1: first roll is greater than 3
E_2: sum of two rolls is less than 5

c. E_1: first roll is greater than 4
E_2: second roll is less than 5

d. E_1: first roll is less than 3
E_2: difference $E_2 - E_1$ is positive

42. A 300-page book is opened to a random page number, then closed and opened randomly once again.

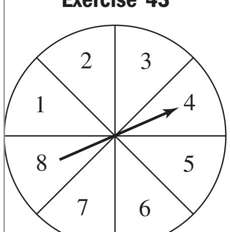
a. E_1: first page number is less than 100
E_2: sum $E_1 + E_2$ is greater than 250

b. E_1: first page number is odd
E_2: second page number is even

c. E_1: first page number is less than 25
E_2: second page number is more than 25

d. E_1: first page number is prime
E_2: second page number is divisible by three

Exercise 43



43. The spinner shown to the left is spun twice in succession.

a. E_1: first spin is a 2
E_2: second spin is a 2

b. E_1: first spin is more than 2
E_2: sum of spins is less than 5

c. E_1: first spin is less than 6
E_2: difference $E_2 - E_1$ is negative

d. E_1: first spin is even
E_2: second spin is a six

44. Dependent events—weather: Data was collected to study the relationship between weather conditions and accident rates. Use the data and the formula for conditional probability to verify that the events E_2: *an accident occurs* and E_1: *the weather is bad* are dependent events.

	Good Weather	Bad Weather	Total
Accident	0.01	0.05	0.06
No accident	0.72	0.22	0.94
Totals	0.73	0.27	1.00

45. Dependent events—options: Data were collected to study the relationship between the cost of options ordered by car owners and the cost of options ordered by truck owners. Use the data and the formula for conditional probability to determine whether the events E_2: *options cost more than $1000* and E_1: *vehicle is a car* are dependent events.

	$C < 1000$	$C > 1000$	Total
Cars	150	250	400
Trucks	200	200	400
Totals	350	450	800

46. Probability of success: Neosho County is attempting to establish a policy regarding the number of times a student driver should be allowed to take/retake the written driver's test before being asked to wait 6 months prior to trying again. Data collected over the past 3 yr are shown in the table. (a) Find the probability of a student failing the first time and passing the second time; and (b) find the probability of a student failing the first two times, and passing the test on the third attempt.

First Attempt		Second Attempt		Third Attempt	
Pass	Fail	Pass	Fail	Pass	Fail
72%	28%	65%	35%	52%	48%

47. Probability of success: Officials are assessing the results of the state bar exam for the past 4 yr, to decide if the standard for passing should be increased or decreased. Data collected over the past 4 yr are shown in the table. One important factor is the number of times the test is failed after each attempt. (a) Find the probability of a prospective lawyer failing the bar exam the first two times, will also fail the third time; and (b) find the probability that a prospective lawyer fails all three attempts at the bar exam.

First Attempt		Second Attempt		Third Attempt	
Pass	Fail	Pass	Fail	Pass	Fail
58%	42%	64%	36%	28%	72%

Compute the expected value for the probabilities P given and the values V assigned to each outcome.

48. cost to play: \$6
$V_1 = \$25, V_2 = \$12$
$P_1 = 0.08, P_2 = 0.15$

49. cost to play: \$10
$V_1 = \$5, V_2 = \$8$
$P_1 = 0.22, P_2 = 0.35$

50. cost to play: 25¢
$V_1 = 50¢, V_2 = 75¢$
$P_1 = 0.15, P_2 = 0.12$

51. cost to play: \$2
$V_1 = \$8, V_2 = \$10, V_3 = \$12$
$P_1 = 0.15, P_2 = 0.12, P_3 = 0.04$

52. cost to play: \$0.50
$V_1 = \$1, V_2 = \$2, V_3 = \$3$
$P_1 = 0.08, P_2 = 0.05, P_3 = 0.03$

53. cost to play: 25¢
$V_1 = 25¢, V_2 = 50¢, V_3 = 75¢$
$P_1 = 0.14, P_2 = 0.08, P_3 = 0.02$

54. Rolling a die: A single die is rolled, and a player is paid 50¢ times the number showing. Find

a. the expected value for playing this game

b. what the house should charge for an expected value of 10¢ in their favor

55. Drawing a card: A single card is drawn. If a numbered card is drawn, the player is paid $1. If a face card, the player is paid $2. If an Ace is drawn, the player is paid $4. Find

a. the expected value for playing this game

b. the expected value for the house (rounded to the nearest penny) if they charge $1.50 to play

56. Expected value—lottery tickets: The mayor of a city decides to conduct a lottery to raise money for some local projects. The town will sell 5000 tickets at $1 each, and offer a $2500 prize, a $1000 prize, and a $500 prize.

a. If you buy one ticket, what is your mathematical expectation?

b. Determine what the town should charge for tickets if they desire an expected value of 70¢.

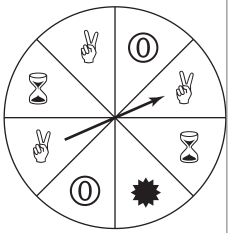
57. Expected value—raffle tickets: The local Outdoor Club wants to raise money for some camping equipment and decides to operate a raffle. The club will sell 1000 raffle tickets for $0.75 each, and offer a tent worth $350, hiking boots worth $120, and a backpack worth $75 as prizes.

a. If you buy one ticket, what is your mathematical expectation?

b. Find what the club should charge for a ticket if they want an expected value of 50¢.

58. Expected value—spinning a spinner: Homes for the Homeless is trying to attract volunteers to support their program of building homes for families in need, and they decide to place the spinner shown at the center of a busy shopping mall. Interested passers-by are invited to have a spin. If the spinner lands on an ⓪, there is no penalty or reward, the player simply spins again. If it lands on an hourglass ⌛, the player must work one volunteer hour. If it lands on a ✌, the player must commit to double the time (2 hr), but if it lands on the sun ✸, the player wins a full 6 hr of free labor for home repairs or improvements.

Exercise 58



a. From the volunteer's point of view, what is the expected value of this game?

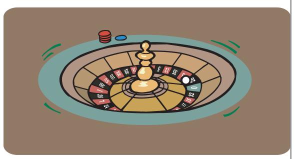
b. What value should be assigned to the sun ✸ for this to be a fair game?

59. Expected value—ring toss: Many carnivals offer a ring toss game where players attempt to toss a quoit (a hardened circular rope) over the necks of bottles of various sizes. Ringing the thin neck bottles wins a $2 prize, ringing the medium neck bottles wins a $5 prize and ringing the thick necked bottles wins a $15 stuffed animal. The carnival operators know the $15 stuffed animal is won only 2% of the time, the $5 prize is won 10% of the time, and the $2 prize is won 20% of the time.

a. If it costs $2 to play the game, what is the expected value for the player?

b. If the carnival operators need an expected value of 84¢, what should a player be charged to play?

Expected value—roulette: For Exercises 60 and 61, refer to Example 9 and the roulette table shown.



60. Max is back at the roulette table again and is continually wagering $1 on the various possibilities offered by the wheel. Find the expected value of each game if he wagers,

a. the ball will land in a red slot

b. the ball will land on an odd number

c. the number will be between 1 and 18 inclusive

d. the number will be between 19 and 36 inclusive

61. Maxine is more adventurous than Max, and is continually wagering \$1 on the possibilities described below. Find the expected value of each game if her wager is a

a. split: betting on two numbers, placing her chip on the line between them. A "split" win pays \$17.

b. corner: betting on four numbers, placing her chip on the corner of the four desired numbers. A "corner" win pays \$8.

c. street: betting on three numbers, placing her chip to the right of the chosen row. A "street" win pays \$11.

d. column: betting on any one column, placing her chip at the bottom of the chosen column. A "column" win pays \$2.

WRITING, RESEARCH, AND DECISION MAKING

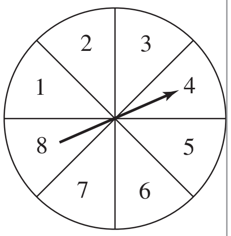
62. Using the Internet, a telephone interview, or the resources of your local library, gather information regarding the expected value of any lottery games sponsored by your state or a neighboring state. In this context, respond to the view held by some politicians that a lottery is just another tax on the poor.

63. Many amusement parks and state fairs offer a game where a player pays \$3 to shoot a free throw and wins a \$5 prize (usually a rubber basketball) if they make the shot (prizes are usually limited to three since the game obviously favors a good shooter). Use a trial-and-error approach to find what free throw average carnival goers must have if this is to be a fair game.

64. Using Exercises 58 and 59 for ideas, create your own expected value problem. You can be as creative and imaginative as you like, or simply use games you encounter at amusement parks, game rooms, or state fairs. Have another student solve the problem and make comments.

EXTENDING THE CONCEPT

65. Consider the spinner from Exercise 26, which is spun twice in succession. Carefully discuss/explain why the events in part (a) are *dependent,* while the events in Part (b) are *independent.*



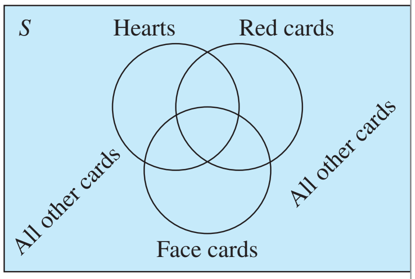
a. E_1: first spin is a 3

E_2: sum of both spins is greater than 6

b. E_1: first spin is a 3

E_2: sum of both spins is nine

66. Using a standard 52-card deck of playing cards, fill in the related portions of the Venn diagram as labeled. Suppose one card is drawn. Use the diagram and direct reasoning to find the probability that the card is a heart, given the card drawn is a red, face card.



MAINTAINING YOUR SKILLS

67. (8.5) Compute the value of ${}_8P_5$, 5!, and ${}_8C_5$ and state the relationship between them.

68. (8.1) Expand and evaluate the summation:

$$\sum_{k=0}^{6} \frac{(-1)^{k+1}}{k^2 + 1}$$

69. (2.1) Given the points $(-3, -4)$ and $(5, 2)$ find
- **a.** the distance between them
- **b.** the midpoint between them
- **c.** the slope of the line through them

70. (5.3) Use a calculator to find the value of each expression, then explain the results.
- **a.** $\log 2 + \log 5 =$ ________
- **b.** $\log 20 - \log 2 =$ ________

71. (9.4) Solve $2|x + 1| - 3 = 7$ two ways:
- **a.** using the definition of absolute value
- **b.** graphically using a system

72. (4.2) Use the rational roots theorem to solve the equation completely, given $x = -3$ is one root.
$x^4 + x^3 - 3x^2 + 3x - 18 = 0$

11.9 Probability and the Normal Curve—Applications for Today

LEARNING OBJECTIVES

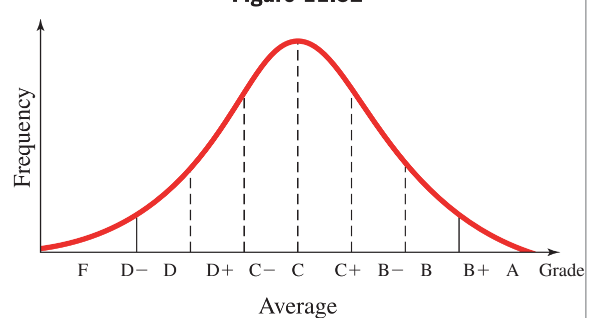
In Section 11.9 you will learn how to:

- **A. Find the mean and standard deviation for a set of data**
- **B. Apply standard deviations to a normal curve**
- **C. Use the normal curve to make probability statements**
- **D. Use *z*-scores to make probability statements**

INTRODUCTION

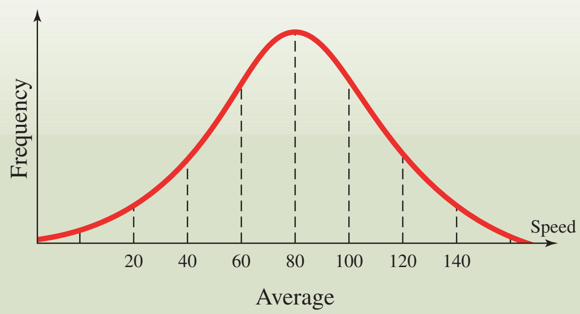
In previous sections, we've made probability statements using counting methods, simple games, formulas, information from tables, and other devices. In this section, we learn to make such statements using observations drawn from a large set of data. Specific characteristic of a large population tend to be **normally distributed,** meaning a large portion of the sample will be average, with decreasing portions tending to be below average and above average. One example might be the grade distribution for a large college, which might be represented by the graph shown in Figure 11.32.

Figure 11.32



POINT OF INTEREST

Many human characteristics and abilities have a normal distribution and the graph of any large sample would resemble that of Figure 11.32. For example, the typing speed of a human will have a like distribution, with a select few being extremely fast (150+ words per minute), and an equally small number being very slow.



Section 8.8 (shown in this document as 11.8) Student Solutions

1. dependent

3. $P(E_1) \cdot P(E_2)$

5. Answers will vary.

7. a) $\frac{16}{81}$ b) $\frac{1}{6}$

9. a) $\frac{25}{64}$ b) $\frac{3}{8}$

11. a) $\frac{3}{7}$ b) $\frac{1}{7}$ c) 0 d) 1

13. a) $\frac{27}{2197}$ b) $\frac{72}{5225}$

15. a) $\frac{5}{11}$ b) $\frac{3}{11}$ c) $\frac{2}{11}$ d) $\frac{6}{11}$

17. a) $\frac{1}{9}$ b) $\frac{1}{8}$

19. a) $\frac{5}{17}$ b) $\frac{4}{51}$

21. 0.3

23. 0.32

25. a) $\frac{1}{2}$ b) $\frac{1}{2}$

27. a) 0.48 b) ≈ 0.70 c) ≈ 0.28

29. $\frac{1}{52}$

31. $\frac{1}{3}$

33. dependent

35. independent

37. 0.5

39. 0.6

41. a) independent b) dependent c) independent d) dependent

43. a) independent b) dependent c) dependent d) independent

45. dependent: $P(E_1)P(E_2) \neq P(E_1) \cap P(E_2)$

47. a) 0.72 b) 0.109

49. –\$0.40

51. \$1.50

53. –\$0.10

55. a) \$1.46 b) – \$0.04

57. a) –\$0.21 b) ≈ \$1.05

59. a) –\$0.16 b) \$3.00

61. a) ≈ –\$0.053 b) ≈ –\$0.053 c) ≈ –\$0.053 d) ≈ –\$0.053

63. $\frac{3}{8}$ or 0.375

65. a) Answers will vary b) Answers will vary

67. ${}_8P_5 = 6720$; $5! = 120$; ${}_8C_5 = 56$; $\frac{{}_8P_5}{{}_8C_5} = 5!$

69. a) 10 units b) (1, -1) c) $m = \frac{3}{4}$

71. a) x = -6, x = 4 b) x = -6, x = 4